

Almost additive functions on semigroups and a functional equation

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§ 1. J. ACZÉL has asked in [1] what can be said about functions g satisfying the conditional functional equation

$$(1) \quad g(x+y)g(x)g(y) \neq 0 \quad \text{implies} \quad \frac{1}{g(x+y)} = \frac{1}{g(x)} + \frac{1}{g(y)}.$$

Here g is assumed to be of the type: $S \rightarrow K$ where $(S, +)$ is a semigroup and $(K, +, \cdot)$ is a field (both not necessarily commutative). As was shown in [8], equation [1] may be reduced to the following one:

$$(2) \quad f(x+y) \neq 0 \quad \text{and} \quad f(x) \neq 0 \quad \text{and} \quad f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y).$$

This proves that, in genuine, Aczél's problem is of a (semi)group-theoretical nature. It was also pointed out in [8] that even in the case where f is a real-valued function defined on the real line $\mathbf{R}$ it may happen that equation (2) does not furnish any information whatever about nonzero values of f . This shows that, in general, some further assumptions concerning the greatness of $f^{-1}(\{0\})$ are rather natural. Under such type of assumptions equation (2) has been solved in the class H^G where $(G, +)$ and $(H, +)$ are two abelian groups. The commutativity assumptions were caused by the fact that DE BRUIJN's [3] result on almost additive functions had been used as a tool. With no essential changes Theorem 2 from [8] may be improved by avoiding the commutativity assumptions since the above quoted de Bruijn's result remains valid in the non-abelian case, too (see [7]). Our main purpose here is to extend the de Bruijn's result to the case of almost additive functions defined on semigroups being in a special case embeddable into groups (also without commutativity assumptions). This question is directly connected to the problem of extending of homomorphisms of subsemigroups to homomorphisms of groups (compare [2]). Theorem 1 below yields a joint generalization of de Bruijn's main result from [3] and some theorems from [2]. In section 4 we apply our theorem to Aczél's question regarding equation (1).

§ 2. Let $(G, +)$ be a group (not necessarily commutative) and let $\mathcal{I} \neq 2^G$ be a non-empty family of subsets of G closed under finite unions, hereditary with respect to descending inclusions and such that jointly with a set $U \subset G$ it contains the family $\{x - U : x \in G\}$. In the sequel every such family will be called a proper linearly invariant ideal (abbreviated to p.l.i. ideal). The notion of a p.l.i. ideal yields a generalization of null-sets in the theory of Haar-measure and allows to

introduce the notion "almost everywhere" in the usual manner (cf. also the section devoted to ideals of negligible sets in [9]). Namely, a property $\mathcal{P}(x)$, $x \in A \subset G$, is said to hold $\mathcal{I}$ -almost everywhere in A iff $A \setminus \{x: \mathcal{P}(x)\} \in \mathcal{I}$ (for more details and results as well as for examples see [5] and [6]).

Given a p.l.i. ideal $\mathcal{I}$ in $(G, +)$ we put

$$\Omega(\mathcal{I}) := \left\{ M \subset G^2: \bigvee_{U(M) \in \mathcal{I}} \bigwedge_{x \in G \setminus U(M)} M_x := \{y \in G: (x, y) \in M\} \in \mathcal{I} \right\}.$$

$\Omega(\mathcal{I})$ turns out to be a p.l.i. ideal in $(G^2, +)$.

Given a set $Z \subset G$ we denote by $\mathcal{I}(Z)$ the family of all sets of the form

$$\bigcup_{i=1}^n [x_i + Z \cup (-Z) - y_i],$$

where n is a positive integer, $x_i, y_i \in G$, $i = 1, \dots, n$, and all their subsets. It is readily seen from this definition that $\mathcal{I}(Z)$ is the smallest set family contained in 2^G such that $Z \in \mathcal{I}(Z)$ and all the conditions occurring in the definition of a p.l.i. ideal except, possibly, that $\mathcal{I}(Z) \neq 2^G$ are satisfied. For this reason $\mathcal{I}(Z)$ is called to be a linearly invariant set ideal generated by Z .

§ 3. Now, suppose that we are given a p.l.i. ideal $\mathcal{I}$ in $(G, +)$ and that $(S, +)$ is a subsemigroup of $(G, +)$ fulfilling the conditions

$$(3) \quad S - S = G$$

and

$$(4) \quad S \notin \mathcal{I}.$$

Remark 1. (3) does not imply (4). Take, for instance, $G = \mathbf{R}^2$ and $S = [0, \infty)^2$. Clearly $S - S = \mathbf{R}^2 = G$ whereas the linearly invariant set-ideal $\mathcal{I}(S)$ (generated by S) is proper, i.e. $\mathcal{I}(S) \neq 2^{\mathbf{R}^2}$ *).

We proceed with some lemmas:

Lemma 1. For every $s, t \in S$ we have $(s + S) \cap (t + S) \notin \mathcal{I}$.

PROOF. Suppose the contrary, i.e. $(s + S) \cap (t + S) \in \mathcal{I}$ for some $s, t \in S$. Then, for all $u, v \in S$, we have

$$(5) \quad (s + u + S) \cap (t + v + S) \in \mathcal{I},$$

because of $u + S \subset S$ and $v + S \subset S$. Since $G = S - S = \bigcup_{u \in S} (S - u)$, we may find an $u_0 \in S$ such that $v_0 := -s + t + u_0 \in S$ whence, in view of (5),

$$t + u_0 + S = (s + v_0 + S) \cap (t + u_0 + S) \in \mathcal{I}.$$

Consequently, $S \in \mathcal{I}$ which contradicts (4).

*) One may also give several further examples; this suprisingly simple one has been suggested to me by M. SABLIK.

Remark 2. A semigroup $(S, +)$ is called to be left reversible iff the intersection $(s+S) \cap (t+S)$ is non-void for any $s, t \in S$ (see e.g. [4]). Thus, Lemma 1 states, in particular, that a semigroup under considerations is left reversible.

Lemma 2. For every $s, t \in S$ we have $(-s+S) \cap (-t+S) \notin \mathcal{I}$.

PROOF. If we had $(-s+S) \cap (-t+S) \in \mathcal{I}$ for some $s, t \in S$ then, in view of the inclusion $s+S \subset S$, we would also get $S \cap (-t+s+S) \in \mathcal{I}$ and hence $(t+S) \cap (s+S) \in \mathcal{I}$, contrary to Lemma 1.

Lemma 3. For every sets $U_1, U_2 \in \mathcal{I}$ we have

$$G = (S \setminus U_1) - (S \setminus U_2).$$

PROOF. Take an $x \in G = S - S$ and sets $U_1, U_2 \in \mathcal{I}$. Then $x = s - t$, $s, t \in S$, and, on account of Lemma 2,

$$T := [-s + (S \setminus U_1)] \cap [-t + (S \setminus U_2)] \notin \mathcal{I}.$$

Thus, $T \neq \emptyset$ and we may find an α such that

$$s + \alpha \in S \setminus U_1 \quad \text{and} \quad t + \alpha \in S \setminus U_2.$$

Now,

$$x = s - t = (s + \alpha) - (t + \alpha) \in (S \setminus U_1) - (S \setminus U_2),$$

which was to be proved.

Lemma 4. Let $U \in \mathcal{I}$ and $u, s', t' \in S \setminus U$. There exists a pair $(s, t) \in (S \setminus U)^2$ such that $s' - t' = s - t$ and $t \in u + S$.

PROOF. In virtue of Lemma 3 applied to $U_1 := U$ and $U_2 := -u + U$ we get the equality

$$G = (S \setminus U) - [S \setminus (-u + U)]$$

whence

$$G = \bigcup_{x \in S \setminus (-u + U)} [(S \setminus U) - x].$$

This proves that for every $y \in G$ there exists an $x \in S \setminus (-u + U)$ such that $y + x \in S \setminus U$. Take $y := s' - t' + u$ and a corresponding x . Then

$$s := s' - t' + u + x \in S \setminus U$$

and

$$t := u + x \in (u + S) \setminus U \subset (S \setminus U) \cap (u + S).$$

Evidently, $s - t = s' - t'$ which ends the proof.

Now, assume that we are given two groups $(G, +)$ and $(H, +)$, a p.l.i. ideal $\mathcal{I}$ in $(G, +)$, a subsemigroup $(S, +)$ of $(G, +)$ fulfilling conditions (4) and (5) and a map $f: S \rightarrow H$ such that

$$(6) \quad f(x+y) = f(x) + f(y) \quad \text{for all } (x, y) \in S^2 \setminus M$$

for a certain set $M \in \Omega(\mathcal{I})$. By means of the definition of $\Omega(\mathcal{I})$, there exists a set $U(M) \in \mathcal{I}$ such that $M_x := \{y \in G: (x, y) \in M\} \in \mathcal{I}$ provided $x \in G \setminus U(M)$. We have the following

Lemma 5. For every $x, y, u, v \in S \setminus U(M)$ the equality $x - y = u - v$ implies $f(x) - f(y) = f(u) - f(v)$.

PROOF. Take $x, y, u, v \in S \setminus U(M)$ such that $x - y = u - v$. Lemma 1 ensures that $(y + S) \cap (v + S) \notin \mathcal{I}$. Consequently, $(-v + y + S) \cap S \notin \mathcal{I}$ and hence

$$[-v + y + (S \setminus M_x)] \cap (S \setminus M_u) \notin \mathcal{I}.$$

This enables one to find an

$$s \in ((-v + y) + (S \setminus M_x)) \cap (S \setminus M_u) \setminus [M_v \cup (-v + y + M_y)].$$

For such an s we have

$$s \in S, \quad (u, s) \notin M, \quad (v, s) \notin M,$$

$$z := -y + v + s \in S \setminus M_x \quad (\text{whence } (x, z) \notin M)$$

and

$$(y, z) \notin M.$$

On the other hand

$$x - y + v + s = u - v + v + s = u + s$$

i.e.

$$x + z = u + s$$

whence

$$(7) \quad f(x) + f(z) = f(u) + f(s).$$

The definition of z gives $y + z = v + s$ which implies the equality

$$f(z) = -f(y) + f(v) + f(s).$$

This compared with (7) gives our assertion.

Now, we are able to prove our main result:

Theorem 1. Let $(G, +)$ and $(H, +)$ be two groups (not necessarily commutative) and let $\mathcal{I}$ be a p.l.i. ideal in $(G, +)$. Suppose that $(S, +)$ is a subsemigroup of $(G, +)$ fulfilling (3) and (4) and $f: S \rightarrow H$ satisfies the additivity condition $\Omega(\mathcal{I})$ -almost everywhere in S^2 *). Then there exists exactly one additive function $F: G \rightarrow H$ such that $F|_S = f$ $\mathcal{I}$ -almost everywhere in S .

PROOF. Suppose that f satisfies (6) for a certain set $M \in \Omega(\mathcal{I})$. Take a $z \in G = [S \setminus U(M)] - [S \setminus U(M)]$ (see Lemma 3). Then $z = x - y$, $x, y \in S \setminus U(M)$. Put

$$F(z) := f(x) - f(y).$$

On account of Lemma 5, the latter formula defines a function $F: G \rightarrow H$. We shall show that F is additive. For, take $x, y \in G$. We have

$$x = s' - t', \quad y = u - v \quad \text{and} \quad x + y = p - q,$$

with $s', t', u, v, p, q \in S \setminus U(M)$. According to Lemma 4 applied for $U = U(M)$ we may write

$$x = s - t, \quad s, t \in S \setminus U(M)$$

*) In other words, f is $\Omega(\mathcal{I})$ -almost additive.

with

$$(8) \quad t \in u + S.$$

Obviously, we have $p - q = x + y = s - t + u - v$, i.e.

$$(9) \quad s - t + u = p - q + v.$$

Since $x + y + q$ and $x + t$ are members of S we infer, by Lemma 1, that $(x + t + S) \cap (x + y + q + S) \notin \mathcal{J}$ whence

$$(-y + t + S) \cap (q + S) \notin \mathcal{J}.$$

Therefore, because of $-y + t = v - u + t$,

$$(v - u + t + S) \cap (q + S) \notin \mathcal{J}$$

and, consequently,

$$(-u + t + S) \cap (-v + q + S) \notin \mathcal{J}.$$

Thus, we may find a

$$w \in [(-u + t + S) \cap (-v + q + S)] \setminus [(-v + q + M_p) \cup (-v + q + M_q) \cup (-u + t + M_s) \cup (-u + t + M_t) \cup M_u \cup M_v].$$

For such a w we have

$$w \in -u + t + S \subset S + S \subset S \quad (\text{cf. (8)}),$$

$$z_1 := -t + u + w \in S \setminus (M_s \cup M_t),$$

$$z_2 := -q + v + w \in S \setminus (M_p \cup M_q)$$

and

$$(10) \quad (u, w) \notin M, \quad (v, w) \notin M.$$

Relation (9) gives the equality

$$s + z_1 = p + z_2$$

whence, because of $s, p, z_1, z_2 \in S$ and $(s, z_1) \notin M$, $(p, z_2) \notin M$, we get

$$(11) \quad f(s) + f(z_1) = f(p) + f(z_2).$$

On the other hand, since

$$t + z_1 = u + w, \quad q + z_2 = v + w,$$

(10) is satisfied as well as $(t, z_1) \notin M$ and $(q, z_2) \notin M$, we may write

$$f(t) + f(z_1) = f(u) + f(w), \quad f(q) + f(z_2) = f(v) + f(w).$$

Therefore, by means of (11),

$$f(s) - f(t) + f(u) + f(w) = f(p) - f(q) + f(v) + f(w)$$

whence

$$[f(s) - f(t)] + [f(u) - f(v)] = f(p) - f(q)$$

i.e.

$$F(x) + F(y) = F(x + y).$$

To prove that $F|_S = f$ $\mathcal{J}$ -almost everywhere in S , take an $x \in S \setminus U(M)$. Thus $x = s - t$, $s, t \in S$. Since, in view of Lemma 2, the set

$$(-s + [S \setminus U(M)]) \cap (-t + S \setminus [U(M) \cup M_x])$$

is non-void (because it does not belong to $\mathcal{J}$) one may find a $y \in G$ such that

$$s + y \in S \setminus U(M), \quad t + y \in S \setminus (U(M) \cup M_x).$$

Obviously

$$x = s - t = (s + y) - (t + y)$$

which implies

$$(12) \quad F(x) = f(s + y) - f(t + y).$$

On the other hand

$$x + (t + y) = s + y$$

and $(x, t + y) \notin M$. Consequently

$$f(x) + f(t + y) = f(s + y)$$

which compared with (12) gives $F(x) = f(x)$.

To finish the proof it remains to show that F is unique. Suppose that F_1 and F_2 map additively G into H with

$$F_1(s) = F_2(s) = f(s) \quad \text{for } s \in S \setminus U(M)$$

and take an $x \in G$; we have $x = s - t$, $s, t \in S \setminus U(M)$. Thus

$$F_1(x) = F_1(s - t) = F_1(s) - F_1(t) = F_2(s) - F_2(t) = F_2(s - t) = F_2(x),$$

which means that $F_1 = F_2$. This completes the proof.

Corollary 1. Taking $S = G$ we obtain de Bruijn's result [3] in the non-abelian case (cf. also [7]).

Corollary 2. Taking $\mathcal{J} = \{\emptyset\}$ we obtain Theorem 3 (and hence also Theorems 1 and 2) from [2].

§ 4. We proceed with the following

Lemma 6. Let $(S, +)$ be a subsemigroup of a group $(G, +)$ such that $G = S - S$ and let $Z \subset S$ satisfy the condition

$$(C) \quad \begin{cases} \text{for every positive integer } k \text{ and for every } s, s_1, \dots, s_k, \\ t_1, \dots, t_k \in S \text{ there exists a } t \in S + s \text{ such that} \\ t_i + t \notin Z + s_i \text{ and } s_i \notin Z + t_i + t \text{ for } i = 1, \dots, k. \end{cases}$$

Then the linearly invariant set ideal $\mathcal{J}(Z)$ (generated by Z) does not include S ; in particular $\mathcal{J}(Z)$ is proper, i.e. $\mathcal{J}(Z) \neq 2^G$.

PROOF (indirect). Suppose that $S \in \mathcal{J}(Z)$, i.e. there exists a positive integer k and elements $x_1, \dots, x_k, y_1, \dots, y_k \in G$ such that

$$S \subset \bigcup_{i=1}^k [x_i + Z \cup (-Z) + y_i].$$

Since $G = S - S = \bigcup_{s \in S} (S - s)$, we claim that for every $y \in G$ there exists an $\tilde{s} \in S$ such that $y + \tilde{s} \in S$. Take an $\tilde{s}_1 \in S$ such that $y_1 + \tilde{s}_1 \in S$, an $\tilde{s}_2 \in S$ such that $y_2 + \tilde{s}_1 + \tilde{s}_2 \in S$ and so on up to $\tilde{s}_k \in S$ such that $y_k + \tilde{s}_1 + \dots + \tilde{s}_k \in S$. Put $s := \tilde{s}_1 + \dots + \tilde{s}_k$ and $s_i := y_i + s$, $i = 1, \dots, k$. Evidently, s and s_i belong to S , $i = 1, \dots, k$. We have

$$S + s \subset \bigcup_{i=1}^k [x_i + Z \cup (-Z) + s_i]$$

whence

$$-s - S \subset \bigcup_{i=1}^k [-s_i + Z \cup (-Z) - x_i].$$

Repeating the above construction, one can find elements $\tilde{t}, t_i \in S$, $i = 1, \dots, k$, such that

$$-s - S + \tilde{t} \subset \bigcup_{i=1}^k [-s_i + Z \cup (-Z) + t_i]$$

or, equivalently,

$$-\tilde{t} + S + s \subset \bigcup_{i=1}^k [-t_i + Z \cup (-Z) + s_i]$$

whence, in view of the inclusion $S + s \subset -\tilde{t} + S + s$, we get

$$S + s \subset \bigcup_{i=1}^k [-t_i + Z \cup (-Z) + s_i].$$

Therefore, for every $t \in S + s$ there exists an $i \in \{1, \dots, k\}$ such that

$$t \in -t_i + Z + s_i \quad \text{or} \quad t \in -t_i - Z - s_i$$

i.e.

$$t_i + t \in Z + s_i \quad \text{or} \quad s_i \in Z + t_i + t.$$

This contradicts (C) and ends the proof.

Remark 3. Note that in the case where $S \notin \mathcal{J}(Z)$ condition (C) is simply satisfied. In fact, take $s, s_1, \dots, s_k, t_1, \dots, t_k \in S$ and

$$t \in (S + s) \setminus \bigcup_{i=1}^k [-t_i + Z \cup (-Z) + s_i].$$

Such a t does exist, because $\bigcup_{i=1}^k [-t_i + Z \cup (-Z) + s_i]$ belongs to $\mathcal{J}(Z)$ whereas $S + s$ does not. Thus $t \in S + s$ and

$$t_i + t \notin Z + s_i \quad \text{and} \quad s_i \notin Z + t_i + t$$

for $i = 1, \dots, k$. Consequently, condition (C) is equivalent for S not to belong to $\mathcal{J}(Z)$. Observe, however, that (C) involves semigroup terms only.

The assumptions on a semigroup $(S, +)$ we have been doing up to Lemma 6 imply that $(S, +)$ is left reversible (see Remark 2) and cancellative (since $(S, +)$ was a subsemigroup of a given group). It is known (for details, see [4]) that a left

reversible semigroup with the cancellation law is embeddable into a group $(G, +)$ in such a manner that $S - S = G$ (*). This together with Lemma 6 enables one to state a theorem on functions $f: S \rightarrow H$ fulfilling equation (2) with no use of the corresponding group terms.

Theorem 2. *Let $(S, +)$ be a left reversible semigroup (not necessarily commutative) with the cancellation law and let $(H, +)$ be a group (not necessarily commutative). Assume that $f: S \rightarrow H$ is a solution of (2) such that $Z = f^{-1}(\{0\})$ satisfies condition (C). Then there exists exactly one additive function $F: S \rightarrow H$ such that $F(x) = f(x)$ for $x \in S \setminus Z$.*

PROOF. $(S, +)$ is embeddable into a group $(G, +)$ with $G = S - S$. Moreover, $\mathcal{J}(Z)$ is a p.l.i. ideal in $(G, +)$ and $S \not\subseteq \mathcal{J}(Z)$ (see Lemma 6). Consider the set

$$M := \{(x, y) \in S^2 : x \in Z \text{ or } y \in Z \text{ or } x + y \in Z\}.$$

On account of Lemmas 1 and 2 from [5], $M \in \Omega(\mathcal{J}(Z))$. Clearly, $f(x+y) = f(x) + f(y)$ for $(x, y) \in S^2 \setminus M$. Thus f is $\Omega(\mathcal{J}(Z))$ -almost additive. Making use of Theorem 1 we infer that there exists exactly one additive function $F: S \rightarrow H$ such that $E := \{x \in S : f(x) \neq F(x)\} \in \mathcal{J}$. To show that $E \subset Z$ it suffices to repeat the appropriate reasoning applied in the proof of Theorem 2 in [8].

As a corollary we get easily

Theorem 3. *Let $(S, +)$ be a left reversible and cancellative semigroup and let $(K, +, \cdot)$ be a field (both not necessarily commutative). Suppose that a function $g: S \rightarrow K$ is a solution of (1) such that $Z = g^{-1}(\{0\})$ satisfies condition (C). Then there exists exactly one additive function $F: S \rightarrow K$ such that $g(x) = \frac{1}{F(x)}$ for $x \in S \setminus Z$.*

Finally, we shall present an example in which we are going to visualize that, in our considerations, it was worth-while to handle semigroup terms only (omitting the embedding procedure).

Example. Put $\mathbf{N} := \{0, 1, 2, \dots\}$, $S := \mathbf{N} \times 2\mathbf{N}$ and consider a map $+: S \times S \rightarrow S$ given by the formula

$$(m, x) + (n, y) := (m + n, 2^n x + y), \quad (m, x), (n, y) \in S$$

(the sign $+$ on the right hand side denotes the usual addition in $\mathbf{N}$). It is not hard to check that the pair $(S, +)$ yields a cancellative and left reversible semigroup(**) with $(0, 0)$ as a neutral element. In spite of the fact that the set

$$Z_0 := \{(p, z) \in S : z \leq p\}$$

is rather "large" in S , we are able to determine all the solutions $g: S \rightarrow K$ (with $(K, +, \cdot)$ — an arbitrary field) of equation (1) which satisfy the condition $g^{-1}(\{0\}) \subset Z_0$. For, we shall show that Z_0 satisfies condition (C). In fact, take $s = (n, y)$, $s_i = (n_i, y_i) \in S$ and $t_i = (m_i, x_i) \in S$, $i = 1, \dots, k$. We have to find a pair

$$(m, x) = t \in S + s = \{(p + n, 2^n z + y) \in S : (p, z) \in S\}$$

(*) Obviously, every commutative semigroup is left reversible. Therefore, every commutative semigroup $(S, +)$ is embeddable into a group $(G, +)$ (with $G = S - S$) if and only if it is cancellative.

(**) It is not right reversible (compare [4], Exercise 1 for § 12.4).

such that

$$(13) \quad (m_i + m, 2^m x_i + x) = t_i + t \notin Z_0 + s_i = \{(p + n_i, 2^{n_i} z + y_i) \in S : z \equiv p\}$$

and

$$(14) \quad (n_i, y_i) = s_i \notin Z_0 + t_i + t = \{(p + m_i + m, 2^{m_i+m} z + 2^m x_i + x) \in S : z \equiv p\}$$

for $i=1, \dots, k$. In order to have $t=(m, x) \in S+s$, take $m=n$ and $x=2^n z_x + y$ with $z_x \in 2\mathbb{N}$ (unrestricted temporarily). To realize (13) and (14) (with $m=n$) for $i=1, \dots, k$, denote by P the set of all $p \in \mathbb{N}$ such that p is a solution of at least one of the equations

$$m_i + n = p + n_i, \quad n_i = p + m_i + n, \quad i = 1, \dots, k,$$

and put $p_0 := \max P$. Obviously, for all $p > p_0$, $p \in \mathbb{N}$, and all $z \in 2\mathbb{N}$ we have

$$(13') \quad (m_i + n, 2^n x_i + x) \neq (p + n_i, 2^{n_i} z + y_i)$$

and

$$(14') \quad (n_i, y_i) \neq (p + m_i + n, 2^{m_i+m} z + 2^n x_i + x)$$

for $i=1, \dots, k$ (independently of the choice of x). If one has $p \leq p_0$ and $(p, z) \in Z_0$, then necessarily $z \leq p_0$ whence, in order to get (13') and (14') for $i=1, \dots, k$, it suffices to take x large enough (which may be done by making z_x large enough).

Consequently, Z_0 satisfies condition (C) (and, obviously, so does an arbitrary subset of Z_0). According to Theorem 3, a function $g: S \rightarrow K$ fulfilling (1) and the condition $Z := g^{-1}(\{0\}) \subset Z_0$ is of the form

$$g(x) = \begin{cases} 0 & \text{for } x \in Z \\ \frac{1}{F(x)} & \text{for } x \in S \setminus Z \end{cases}$$

where F is an arbitrary homomorphism of S into K . Now, we have to find a representation of such homomorphisms, i.e. to solve the functional equation

$$(15) \quad F(m+n, 2^n x + y) = F(m, x) + F(n, y), \quad (m, x), (n, y) \in S.$$

Putting $\varphi(m) := F(m, 0)$, $m \in \mathbb{N}$, $\psi(y) := F(0, y)$, $y \in 2\mathbb{N}$, and setting $n=x=0$ in (15) we get

$$F(m, y) = \varphi(m) + \psi(y), \quad (m, y) \in S.$$

Setting $x=y=0$ and, subsequently, $m=n=0$ in (15) we obtain the relations

$$\varphi(m+n) = \varphi(m) + \varphi(n), \quad (m, n) \in \mathbb{N}^2,$$

$$\psi(x+y) = \psi(x) + \psi(y), \quad (x, y) \in (2\mathbb{N})^2,$$

which imply easily

$$\varphi(m) = m\alpha, \quad m \in \mathbb{N}, \quad \psi(y) = y\beta, \quad y \in 2\mathbb{N},$$

where α, β are certain constants from K . Thus $F(m, y) = m\alpha + y\beta$, $(m, y) \in S$, which inserted to (15) gives $\beta=0$. Therefore

$$g(m, x) = \begin{cases} 0 & \text{for } (m, x) \in Z \\ \frac{1}{m\alpha} & \text{for } (m, x) \in S \setminus Z. \end{cases}$$

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