

On certain types of Kähler spaces

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(Dedicated to Prof. Ram Behari on his 80th birthday)

Introduction. A non-flat n -dimensional Riemannian space V_n in which the curvature tensor R_{kjih} satisfies a relation of the form

$$(1) \quad \nabla_m \nabla_l R_{kjih} = \beta_m \nabla_l R_{kjih} + a_{lm} R_{kjih}$$

where β_m and a_{lm} are not both zero, has been called a generalised 2-recurrent space or briefly a G 2-recurrent space [1] and has been studied in some details [2, 3, 4]. A V_n in which the conformal curvature tensor C_{kjih} satisfies a relation of the type (1) has been called a generalised conformally 2-recurrent space [1]. It may be briefly called a GC 2-recurrent space. The present paper deals with an n -dimensional ($n=2N$, $N \neq 1, 2$) Kähler space. In the first part of the paper, consisting of sections 2—5, canonical representations of the relevant tensors have been obtained for a G 2-recurrent Kähler space and as a consequence it has been shown that such a space is necessarily a recurrent space. The second part (Section 6) deals with a GC 2-recurrent Kähler space. It has been shown that such a space reduces to a G 2-recurrent space and is therefore a recurrent Kähler space (β_m and a_{lm} are called the vector and tensor of recurrence. It is assumed throughout that a_{lm} is different from zero)

1. *Preliminaries.* Let F_i^h be the structure tensor and g_{ij} the positive definite Riemannian metric of a Kähler space in real representation. Then

$$F_j^r F_r^i = -\delta_j^i \quad \text{and} \quad g_{rt} F_j^r F_i^t = g_{ji}$$

It is also known that

$$(1.1) \quad F_{ji} = g_{ri} F_j^r = -F_{ij}, \quad F^{ji} = g^{jr} F_r^i = -F^{ij}, \quad \nabla_k F_{ji} = 0.$$

Let R_{ij} be the Ricci tensor and $R = g^{ij} R_{ij}$ the scalar curvature. Also let $H_{ij} = \frac{1}{2} R_{ijkl} F^{kl}$. Then the following relations hold [5]:

$$(1.2) \quad H_{ij} = -H_{ji} \qquad (1.3) \quad R_{ks} F_j^s = H_{kj}$$

$$(1.4) \quad H_{ks} F_j^s = -R_{kj} \qquad (1.5) \quad H_{kj} F^{kj} = -R.$$

Part I. — G 2-recurrent Kähler space

2. *Some useful relations in G 2-recurrent Kähler space.* Let the defining relation of the space be

$$(2.1) \quad \nabla_m \nabla_l R_{kjih} = \beta_m \nabla_l R_{kjih} + a_{lm} R_{kjih}.$$

From the Bianchi identity:

$$\nabla_l R_{kjih} + \nabla_j R_{lkih} + \nabla_k R_{jlhi} = 0$$

we have

$$\nabla_m \nabla_l R_{kjih} + \nabla_m \nabla_j R_{lkih} + \nabla_m \nabla_k R_{jlhi} = 0.$$

In virtue of (2.1) this gives

$$(2.2) \quad a_{lm} R_{kjih} + a_{jm} R_{lkih} + a_{km} R_{jlhi} = 0.$$

(The β terms cancel out.) Replacing m by s and transvecting with $a_m^s = g^{rs} a_{mr}$ we get

$$(2.3) \quad b_{lm} R_{kjih} + b_{jm} R_{lkih} + b_{km} R_{jlhi} = 0,$$

where

$$b_{lm} = a_{ls} a_m^s = a_{ls} a_{mt} g^{st}.$$

It may be noted that b_{lm} is a symmetric tensor. Let $\Theta = g^{lm} b_{lm}$. Then $\Theta = a^{mt} a_{mt} \neq 0$, for otherwise a_{ij} would vanish identically. This shows that $b_{ij} \neq 0$.

We may now quickly deduce the following results:

$$(2.4) \quad R_{jli}^r b_{rm} = R_{il} b_{jm} - R_{ij} b_{lm}$$

$$(2.5) \quad R_{rj} b_m^r = \frac{1}{2} R b_{jm}$$

$$(2.6) \quad \Theta R_{kjih} = R_{hk} b_{ij} + R_{ij} b_{hk} - R_{hj} b_{ik} - R_{ik} b_{nj}.$$

In fact, (2.4) may be obtained from (2.3) by contracting with g^{hk} , (2.5) follows from (2.4) on contraction with g^{il} and (2.6) may be obtained by contracting (2.3) with g^{lm} and using (2.4). Now, we have

$$\begin{aligned} 2\Theta H_{kj} &= \Theta R_{kjih} F^{ih} = \\ &= R_{hk} F^{ih} b_{ij} + R_{ij} F^{ih} b_{hk} - R_{hj} F^{ih} b_{ik} - R_{ik} F^{ih} b_{hj} = \\ &= H_{ks} g^{is} b_{ij} - H_{js} g^{hs} b_{hk} - H_{js} g^{is} b_{ik} + H_{ks} g^{hs} b_{hj} = \\ &= 2(H_{ks} b_j^s - H_{js} b_k^s). \end{aligned}$$

Therefore

$$(2.7) \quad \Theta H_{kj} = H_{ks} b_j^s - H_{js} b_k^s.$$

Transvecting (2.5) with F_p^j we get $H_{rp} b_m^r = \frac{1}{2} R b_{jm} F_p^j = \frac{1}{2} R b_{ms} F_p^s$. Hence (2.7) gives

$$(2.8) \quad \Theta H_{kj} = \frac{1}{2} R (b_{ks} F_j^s - b_{js} F_k^s).$$

Transvecting this with F_p^j we get

$$(2.9) \quad \Theta R_{kp} = \frac{1}{2} R (b_{kp} + b_{kp}^*)$$

where

$$(2.10) \quad b_{kp}^* = b_{sj} F_k^s F_p^j.$$

It is to be noted that symmetry of b_{kp} implies that b_{kp}^* is also symmetric.

3. Form of R_{ij} in G 2-recurrent Kähler Space

From the relation (2.5) viz.,

$$(3.1) \quad R_{rj}b_m^r = \frac{1}{2}Rb_{jm}$$

we see that the columns and therefore the rows of b_{ij} define a set of vectors which are all eigenvectors of R_{ij} corresponding to the root $\frac{1}{2}R$ (Such a set depends upon the coordinate system. Under a different coordinate system we shall get a different set). If $R=0$ then from (2.9) we get $R_{ij}=0$ (because $\Theta \neq 0$) whence by (2.6) $R_{kji h}=0$. But our space is by definition non-flat. Hence R cannot be zero. Also the relation $\nabla_l \nabla_m R = \beta_m \nabla_l R + a_{lm} R$, obtained from (2.1), shows that if in a G 2-recurrent space R is nonzero, then it is non-constant. Hence we obtain

Lemma 1. *In a G 2-recurrent Kähler space the scalar curvature is non-zero and non-constant.*

Let m be the dimension of the eigensubspace of R_{ij} corresponding to the root $\frac{1}{2}R$ and let $u_1, \dots, u_m$ be an orthonormalised set of vectors spanning this subspace. Then evidently b_{ij} will be of the form

$$(*) \quad b_{ij} = \lambda_{i1}u_{j1} + \lambda_{i2}u_{j2} + \dots + \lambda_{im}u_{jm} = \sum_{p=1}^m \lambda_{ip}u_{jp}$$

where λ 's define another set of m vectors. Symmetry of b_{ij} requires

$$\sum_{p=1}^m (\lambda_{ip}u_{jp} - \lambda_{jp}u_{ip}) = \sum_{p=1}^m (\lambda \wedge u)_{ij} = 0,$$

where $\wedge$ denotes exterior product. By Cartan's lemma ([6], p. 18), we then have

$$(*)' \quad \lambda_{ip} = \sum_{q=1}^m d_{pq}u_{iq},$$

where d_{pq} is an m -ordered symmetric matrix. From $(*)$ and $(*)'$ we have

$$(3.2) \quad b_{ij} = \sum_{p=1}^m \sum_{q=1}^m d_{pq}u_{ip}u_{jq}.$$

It is clear that $\text{rank } d_{pq} = \text{rank } b_{ij} = l$, say. (Indices like p, q , denoting primarily collection of objects, have no tensorial significance. Summation over any such index will be explicitly denoted by $\sum$ notation). Now,

$$(*)'' \quad b_{ij}^* = b_{kl}F_i^k F_j^l = \sum_{p=1}^m \sum_{q=1}^m d_{pq}u_k u_l F_i^k F_j^l = \sum_{p=1}^m \sum_{q=1}^m d_{pq}V_i^p V_j^q$$

where

$$V_i^p = u_k F_i^k \quad (p = 1, 2, \dots, m).$$

Let for any vector u , the vector V given by $V_i = u_k F_i^k$ be called the associate vector of u with respect to the structure tensor F . Then

$$V_k = g_{it} u^t F_k^i = F_i^l F_t^m g_{lm} u^t F_k^i = -g_{km} u^t F_t^m u^t$$

whence

$$(*)''' \quad V^j = -F_t^j u^t.$$

We now prove a simple result in the form of

Lemma 2. *If u_i is an eigenvector of R_{ij} corresponding to a characteristic root ϱ , then V_i , the associate vector of u_i , is also an eigenvector corresponding to the same root.*

Let

$$(3.3) \quad R_{ij} u^j = \varrho u_i.$$

From (1.3) and (1.4) we have $R_{ij} = R_{kl} F_i^k F_j^l$. Therefore (3.3) can be written as $R_{kl} F_i^k F_j^l u^j = \varrho u_i$. In virtue of $(*)'''$ this gives $R_{kl} F_i^k (-V^l) = \varrho u_i$. Transvecting with F_s^i we get $R_{kl} (-\delta_s^k) (-V^l) = \varrho V_s$ whence $R_{sl} V^l = \varrho V_s$. This proves the lemma.

In virtue of this result, we see that the vectors $V_1, \dots, V_m$ also all belong to the eigensubspace of R_{ij} corresponding to the root $\frac{1}{2}R$ and therefore these vectors are all linear combinations of the u -vectors.

Hence b_{ij}^* is eventually of the form

$$(3.4) \quad \sum_{p=1}^m \sum_{q=1}^m d_{pq}^* u_p u_q$$

where d_{pq}^* is also symmetric. Since F is non-singular, it may be seen that $\text{rank } b_{ij} = \text{rank } b_{ij}^*$. Hence $\text{rank } d_{pq}^* = \text{rank } d_{pq} = l$. From (2.9), (3.2) and (3.4) we obtain

$$(3.5) \quad \Theta R_{ij} = \frac{1}{2} R \sum_{p=1}^m \sum_{q=1}^m D_{pq} u_p u_q$$

where

$$D_{pq} = d_{pq} + d_{pq}^*.$$

Thus D_{pq} is also symmetric in p and q .

Since every u -vector is an eigenvector of R_{ij} corresponding to the root $\frac{1}{2}R$, we have

$$(3.6) \quad R_{ir} u_s^r = \frac{1}{2} R u_i \quad (s = 1, \dots, m).$$

From (3.5) and (3.6) we get

$$\Theta R_{ir} u_s^r = \frac{1}{2} R \sum_{p=1}^m \sum_{q=1}^m D_{pq} u_p u_q u_s^r = \Theta \frac{1}{2} R u_i,$$

or

$$\langle u, u \rangle \left(\sum_{p=1}^m D_{pq} u_p \right) = \Theta u_i,$$

where

$$\langle u_s, u_r \rangle = u_s^r = \delta_{rs}$$

in virtue of the orthonormality of the vectors u_s . Hence

$$\sum_{p=1}^m D_{ps} u_p = \Theta u_s.$$

Since the u -vectors are independent, we obtain

$$(3.7) \quad \begin{cases} D_{ps} = 0 & \text{for } p \neq s \\ D_{ss} = \Theta & (s = 1, \dots, m). \end{cases}$$

This gives $\Theta R_{ij} = \frac{1}{2} R \Theta (u_i u_j + \dots + u_i u_j)$. Since $\Theta \neq 0$, we have

$$(3.8) \quad R_{ij} = \frac{1}{2} R (u_i u_j + \dots + u_i u_j).$$

Raising i and contracting with j , we get $R = \frac{m}{2} R$ showing that m is precisely equal to 2. Writing u_i and V_i for u_i and u_i we get

$$(3.9) \quad R_{ij} = \frac{1}{2} R (u_i u_j + V_i V_j).$$

The eigensubspace of R_{ij} corresponding to the root $\frac{1}{2} R$ is thus of dimension 2 and coincides with the row space (Column space) of R_{ij} . Hence $\text{rank } R_{ij} = 2$. In fact, from (3.9) we get $R_{ik} R_j^k = \frac{1}{2} R R_{ij}$. Hence the minimal equation of R_{ij} is $\varrho^2 - \frac{1}{2} R \varrho = 0$ implying that the characteristic roots are $\frac{1}{2} R, \frac{1}{2} R, 0, 0, \dots, 0$ ($n-2$ zeros). It may further be noted that u_i may be chosen arbitrarily in the eigensubspace of R_{ij} corresponding to the root $\frac{1}{2} R$ and since V_i is in the same eigensubspace and orthogonal to u_i , we have either $V_i = u_k F_i^k$ or $V_i = -u_k F_i^k$. For convenience we choose $V_i = u_k F_i^k$.

4. Forms of H_{ij} and R_{kji} .

In virtue of (3.9) we have

$$(4.1) \quad H_{ij} = R_{ik} F_j^k = \frac{1}{2} R (u_i u_k + V_i V_k) F_j^k = \frac{1}{2} R (u_i V_j - V_i u_j) = \frac{1}{2} R A_{ij}$$

where

$$A_{ij} = u_i V_j - V_i u_j.$$

We next observe that d_{pq} is a 2-ordered symmetric matrix. Therefore b_{ij} is of the form

$$(4.2) \quad b_{ij} = \lambda u_i u_j + \varrho (u_i V_j + V_i u_j) + \mu V_i V_j$$

where $\lambda + \mu = \Theta$ and the rank of the matrix $\begin{bmatrix} \lambda & \varrho \\ \varrho & \mu \end{bmatrix}$ is 2 or 1. Of course, in the latter case by choosing u_i appropriately we may have

$$(4.3) \quad b_{ij} = \Theta u_i u_j.$$

In either case by straightforward calculation we obtain from (2.6)

$$(4.4) \quad R_{kjih} = -\frac{1}{2} R A_{kj} A_{ih}$$

where

$$A_{ij} = u_i V_j - V_i u_j.$$

5. A G 2-recurrent Kähler space is a recurrent space

Since $A_{ih} A^{ih} = 2$ we see that

$$(5.1) \quad R^{kjih} R_{kjih} = \left(-\frac{1}{2} R A^{kj} A^{ih}\right) \left(-\frac{1}{2} R A_{kj} A_{ih}\right) = R^2.$$

As an immediate consequence of (5.1) and the defining relation (2.1) of the space we have

$$(5.2) \quad \nabla_l R^{kjih} \nabla_m R_{kjih} = \nabla_l R \nabla_m R.$$

Let

$$(5.3) \quad S_{kjihl} = \nabla_l R_{kjih} - \lambda_l R_{kjih}$$

where

$$(5.4) \quad \lambda_l = \frac{1}{R} \nabla_l R.$$

Then

$$(5.5) \quad \begin{aligned} S^{kjihl} S_{kjihl} &= \lambda^l \lambda_l R^{kjih} R_{kjih} - \lambda^p R_{kjih} \nabla_p R^{kjih} \\ &\quad - \lambda^l R^{kjih} \nabla_l R_{kjih} + g^{lp} \nabla_l R_{kjih} \nabla_p R^{kjih}. \end{aligned}$$

In virtue of (5.1) and (5.2) the righthand side of (5.5) can be expressed as

$$\lambda^l \lambda_l R^2 - 2R \lambda^p \nabla_p R + g^{lp} \nabla_l R \nabla_p R.$$

But this becomes zero because of (5.4). Hence $S^{kjihl} S_{kjihl} = 0$. Since the metric of the space is positive definite we get $S_{kjihl} = 0$. Hence $\nabla_l R_{kjih} = \lambda_l R_{kjih}$. That is, the space is recurrent. We summarise the main results in the form of

Theorem 1. *A G 2-recurrent Kähler space is a recurrent Kähler space and is necessarily of non-vanishing (therefore non-constant) scalar curvature. The tensors R_{ij} , H_{ij} and R_{kjih} may be simultaneously rendered the canonical forms given by (3.9), (4.1) and (4.4) where u_i and V_i are mutually orthogonal unit vectors spanning the eigen subspace of R_{ij} corresponding to the root $\frac{1}{2}R$.*

Putting $\beta = 0$ throughout, we obtain the following corollary: A 2-recurrent Kähler space is a recurrent Kähler space (Other statements of Theorem 1 are also equally valid for this space)

Part II — A GC 2-recurrent Kähler space

6. A GC 2-recurrent Kähler space is a G 2-recurrent Kähler space

We now consider a non-flat n -dimensional ($n=2N$, $N \neq 1, 2$) Kähler space in which Weyl's conformal curvature tensor C_{kjih} satisfies the relation

$$(6.1) \quad \nabla_m \nabla_l C_{kjih} = \beta_m \nabla_l C_{kjih} + a_{lm} C_{kjih}$$

where a_{lm} is not zero. Now,

$$C_{kjih} \stackrel{\text{def}}{=} R_{kjih} - \frac{1}{n-2} [g_{kh} R_{ji} + g_{ji} R_{kh} - g_{ki} R_{jh} - g_{jh} R_{ki}] + \frac{R}{(n-1)(n-2)} [g_{kh} g_{ji} - g_{ki} g_{jh}].$$

This gives

$$\begin{aligned} C_{kjih} F^{ih} &= 2H_{kj} + \frac{1}{n-2} [R_{ji} F_k^i - R_{kh} F_j^h + R_{jh} F_k^h - R_{ki} F_j^i] - \\ &\quad - \frac{R}{(n-1)(n-2)} [g_{ji} F_k^i + g_{jh} F_k^h] = \\ (6.2) \quad &= 2H_{kj} + \frac{1}{n-2} [-4H_{kj}] - \frac{R}{(n-1)(n-2)} [2F_{kj}] = \\ &= \frac{2(n-4)}{n-2} H_{kj} - \frac{2R}{(n-1)(n-2)} F_{kj}. \end{aligned}$$

Since covariant derivative of F_{kj} vanishes identically, from (6.1) and (6.2) we have

$$\begin{aligned} (6.3) \quad &\frac{2(n-4)}{n-2} [\nabla_m \nabla_l H_{kj} - \beta_m \nabla_l H_{kj} - a_{lm} H_{kj}] + \\ &- \frac{2R}{(n-1)(n-2)} [\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R] F_{kj} = 0. \end{aligned}$$

Transvecting this with F^{kj} we have

$$-\frac{2(n-4)}{n-2} [\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R] - \frac{2(n-2)}{n-1} [\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R] = 0$$

or,

$$-\frac{2(n-2)}{n-1} [\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R] = 0.$$

Hence (6.4)

$$\nabla_m \nabla_l R - \beta_m \nabla_l R - a_{lm} R = 0$$

whence from (6.3) we get

$$(6.5) \quad \nabla_m \nabla_l H_{kj} - \beta_m \nabla_l H_{kj} - a_{lm} H_{kj} = 0.$$

Transvecting this with F_p^j and then writing j in place of p we get

$$(6.6) \quad \nabla_m \nabla_l R_{kj} - \beta_m \nabla_l R_{kj} - a_{lm} R_{kj} = 0.$$

In virtue of (6.1), (6.4) and (6.6) we now obtain

$$(6.7) \quad \nabla_m \nabla_l R_{kjih} - \beta_m \nabla_l R_{kjih} - a_{lm} R_{kjih} = 0,$$

i.e. the space is a G 2-recurrent space. Thus we arrive at the following theorem:

Theorem 2. *A GC 2-recurrent Kähler space is a G 2-recurrent Kähler space and is therefore a recurrent Kähler space.*

Corollary: *A C 2-recurrent (conformally 2-recurrent) Kähler space is a 2-recurrent Kähler space and is therefore a recurrent Kähler space.*

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