

On the orthonormal frame bundle of a Riemannian manifold

By PÉTER T. NAGY (Szeged)

1 §. Introduction

There are more possibilities to introduce Riemannian metrics on the tangent bundle or on the tangent sphere bundle of a Riemannian manifold. One of the most natural metrics is introduced by S. SASAKI in [4]. In recent papers KLINGENBERG and SASAKI [1] and the author [2] investigated the tangent sphere bundle of a Riemannian 2-sphere and in generally of a 2-manifold, respectively.

However, the tangent sphere bundle of an oriented 2-manifold is isomorphic to its orthonormal frame bundle. So, it is a natural question: How could we define a Riemannian metric on the orthonormal frame bundle of a Riemannian manifold analogously to the Sasaki metric, and how could we generalize the results in [1] and [2]. The purpose of the present paper is to answer for these questions.

We shall characterize the geodesics on $O(M)$ in § 4.

In § 5, we consider the case when the basic manifold is locally symmetric.

In § 6, we investigate the geometry of the orthonormal frame bundle of an n -sphere.

Here I express my gratitude to Professor A. M. VASIL'EV (Moscow State University) for turning my attention to this problem and for many valuable suggestions.

2 §. Preliminaries

Let R^n be the linear euclidean space and V^n denote the manifold of orthonormal n -frames in R^n . The orthogonal group $O(n)$ acts simply transitively on V^n . It is possible to define a Riemannian metric on V^n , which is invariant with respect to the action of $O(n)$ on V^n , in a natural manner. The construction may be described as follows.

Let $z(t) = (e_1(t), \dots, e_n(t))$ be a curve on V^n . Let $\alpha(t)$ be a corresponding curve on $O(n)$:

$$z(t) = \alpha(t)z(t_0).$$

The Riemannian metric $\tilde{g}$ on V^n at $z(t_0)$ is defined by

$$\tilde{g}(\dot{z}, \dot{z}) := \frac{1}{2} \text{Trace}(\dot{\alpha}\dot{\alpha}^*),$$

where α^* is the transposed matrix and point denotes the derivation by t . Let E_{ik}

$(i > k)$ denote the usual basis of the orthogonal Lie algebra $o(n)$, and let $\dot{\alpha} = \sum_{i>k} \omega_{ik} E_{ik}$. Then

$$\dot{z}(t_0) = (\dots, \dot{e}_i(t_0), \dots) = d\alpha z(t_0) = (\dots, \sum_{i>k} \omega_{ik} e_k, \dots),$$

and

$$(1) \quad \tilde{g}(\dot{z}, \dot{z}) = \sum_{i>k} (\omega_{ik})^2.$$

Now, let M be a Riemannian manifold and g denote its metric tensor. We can define a family of natural Riemannian metrics on the orthonormal frame bundle $O(M)$ as follows.

Let X, Y be tangent vectors at $u \in O(M)$, and $x = p(u)$, where $p: O(M) \rightarrow M$ is the projection map. We denote by vX and vY the vertical components of X and Y , and by $\tilde{g}_x$ the Riemannian metric on the manifold $O_x(M)$, defined by the former construction. Then the metric $\tilde{g}$ on $O(M)$ is defined by

$$(2) \quad \tilde{g}(X, Y) = g(dp X, dp Y) + \varrho \tilde{g}_x(vX, vY),$$

where ϱ is an arbitrary positive constant.

3§. The Riemannian connection on $O(M)$

We denote by ω_i and ω_{ik} the components of the R^n -valued basic form and of the $o(n)$ -valued Riemannian connection form on $O(M)$ respectively. It is well known that they define a parallelization of the total manifold $O(M)$. The Riemannian metric (1) can be expressed with help of these forms:

$$(3) \quad dp^* ds^2 = \sum_{i=1}^n (\omega_i)^2 + \varrho \sum_{i>k} (\omega_{ik})^2.$$

It follows that the parallelization defined by the coframe consisting from the forms

$$\Theta_i = \omega_i, \quad \Theta_{ik} = \sqrt{\varrho} \omega_{ik}$$

is orthonormal, and

$$dp^* ds^2 = \sum_{i=1}^n (\Theta_i)^2 + \sum_{i>k} (\Theta_{ik})^2.$$

Theorem 1. *The Riemannian connection on $O(M)$ can be reduced to the bundle of orthonormal absolute parallelization defined by the forms Θ_i, Θ_{ij} . The components $\Theta_{i,k}, \Theta_{ij,k}, \Theta_{i,kl}, \Theta_{ij,kl}$ of this connection form can be expressed as follows:*

$$\Theta_{i,k} = \omega_{ik} - \sqrt{\varrho}/2 \sum_{l>m} R_{lmik} \Theta_{lm},$$

$$\Theta_{ij,k} = \sqrt{\varrho}/2 R_{ijkm} \Theta_m,$$

$$\Theta_{i,kl} = -\sqrt{\varrho}/2 R_{klim} \Theta_m,$$

$$\Theta_{ij,kl} = \frac{1}{2\sqrt{\varrho}} (\Theta_{ik} \delta_{jl} - \Theta_{il} \delta_{jk} + \Theta_{jk} \delta_{il} - \Theta_{lj} \delta_{ik}).$$

PROOF. Since the components of the metric tensor $\bar{g}$ are $\bar{g}_{ik} = \delta_{ik}$, $\bar{g}_{i,kl} = 0$, $\bar{g}_{ij,kl} = \delta_{ik}\delta_{jl}$, it is sufficient to prove that the components Θ_{IK} are antisymmetric: $\Theta_{IK} + \Theta_{KI} = 0$ and that the components of the torsion form $d\Theta_I + \Theta_{IK} \wedge \Theta_K$ vanish identically (the indices I, K run over $1, \dots, n$ and over the pairs (i, k) , $i > k$). But they are easily verifiable consequences of the structure equations

$$\begin{aligned} d\omega_i &= -\omega_{ik} \wedge \omega_k, \\ d\omega_{ik} &= -\omega_{im} \wedge \omega_{mk} + \frac{1}{2} R_{iklm} \omega_l \wedge \omega_m. \end{aligned}$$

4§. Geodesics on $O(M)$

In the following we denote by s the arc-length parameter of curves on the basic manifold M and by $\bar{s}$ the arc-length parameter of curves on $O(M)$. Let dash denote the derivation by s and point the derivation by $\bar{s}$.

Let $u = (x, e_1, \dots, e_n) \in O(M)$, $X \in T_u O(M)$. We denote by $\lambda(X)$ the following element of $T_x M \wedge T_x M$:

$$\lambda(X) := \sum_{i>k} \omega_{ik} e_i \wedge e_k.$$

It is clear that the tensor of curvature R at $x \in M$ can be regarded as a map $R_x: T_x M \wedge T_x M \otimes T_x M \rightarrow T_x M$, and so the expression $R(\lambda(X) \otimes dp(X))$ has meaning.

Theorem 2. *The curve $(x(s), e_1(s), \dots, e_n(s))$ is a geodesic on $O(M)$ with respect to the metric (1) if and only if*

a) *the first vector of curvature $\nabla_s x'$ of the curve $x(s)$ is*

$$\nabla_s x' = \varrho R(\lambda(u') \otimes x'),$$

where $x' = dp(u')$,

b) *the bivector field $\lambda(u')$ is parallel along $x(s)$,*

c) *the curve $\tau(s, s_0)e_1(s), \dots, \tau(s, s_0)e_n(s)$ is an affinely parametrized geodesic on $O_{x(s_0)}(M)$, where $\tau(s, s_0): T_{x(s_0)} M \rightarrow T_{x(s)} M$ denotes the operator of parallel translation along $x(s)$.*

PROOF. Let $u(\bar{s}) = (x(\bar{s}), e_1(\bar{s}), \dots, e_n(\bar{s}))$ be a curve on $O(M)$. It is a geodesic if and only if its tangent vector u satisfies the differential equations

$$\begin{aligned} \frac{d}{d\bar{s}} \Theta_i + \Theta_{i,k} \Theta_k + \sum_{k>l} \Theta_{i,kl} \Theta_{kl} &= 0, \\ \frac{d}{d\bar{s}} \Theta_{ij} + \Theta_{ij,k} \Theta_k + \sum_{k>l} \Theta_{ij,kl} \Theta_{kl} &= 0. \end{aligned}$$

Now, using Theorem 1 we can rewrite the above equations as follows:

$$\begin{aligned} \frac{d}{d\bar{s}} \omega_i + \omega_{ik} \omega_k - \varrho \sum_{k>l} R_{klim} \omega_{kl} \omega_m &= 0, \\ \frac{d}{d\bar{s}} \omega_{ij} &= 0, \end{aligned}$$

where $\dot{x} = \omega_i(\dot{u})e_i$, $\nabla_{\bar{s}}e_i = \omega_{ik}(\dot{u})e_k$. These equations are in coordinate free formulation

$$(4) \quad \nabla_{\bar{s}}\dot{x} = \varrho R(\lambda(\dot{u}) \otimes \dot{x}),$$

$$(5) \quad \nabla_{\bar{s}}(\lambda(\dot{u})) = 0.$$

The equation (5) means that the curve $(\tau(\bar{s}, \bar{s}_0)e_1, \dots, \tau(\bar{s}, \bar{s}_0)e_n)$ is an affinely parametrized geodesic on $O_{x(s_0)}(M)$. Thus the length of its tangent vector is constant:

$$\tilde{g}(v\dot{u}, v\dot{u}) = \sum_{i>k} (\omega_{ik}(\dot{u}))^2 = \text{constant}.$$

From this follows that $g(dp(\dot{u}), dp(\dot{u})) = \sum_{i=1}^n (\omega_i(\dot{u}))^2 = \text{constant}$, as well.

$$\text{Let } b = \left(\sum_{i=1}^n (\omega_i(\dot{u}))^2 \right)^{1/2}.$$

If $b=0$, the conditions of Theorem are proved.

If $b>0$ we can rewrite the equation (3):

$$\nabla_s x' = \varrho R(\lambda(u') \otimes x'),$$

and so get the condition (a).

Now, let $(x(s), e_1(s), \dots, e_n(s))$ be a curve on $O(M)$ satisfying conditions (a), (b), (c). Then

$$\nabla_s x' = R(\lambda(u') \otimes x'), \quad \nabla_s(\lambda(u')) = 0,$$

and so the equations of geodesics (4) and (5) are fulfilled.

5§. The locally symmetric case

Now we shall investigate the case, when the basic Riemannian manifold is locally symmetric, i.e. $\nabla R=0$.

Let a geodesic $(x(s), e_1(s), \dots, e_n(s))$ on $O(M)$ be given. Then by Theorem 2 $\nabla_s x' = \varrho R(\lambda(u') \otimes x')$. We denote the operator $R_x(\lambda(u')): T_x M \rightarrow T_x M$ along the curve $x(s)$ by K_x and the vector x' by v_1 . It is clear that the operator K_x is anti-symmetric with respect to the scalar product g . Moreover it follows from $\nabla R=0$ and from Theorem 2 that $\nabla_s K=0$.

The i^{th} curvature κ_i and the i^{th} vector of curvature $\kappa_i v_{i+1}$ of the curve $x(s)$ on M are defined by the generalized Frenet formulas

$$(6) \quad \begin{cases} \nabla_s v_1 = \kappa_1 v_2, \\ \nabla_s v_i = -\kappa_{i-1} v_{i-1} + \kappa_i v_{i+1}, \\ \nabla_s v_n = -\kappa_{n-1} v_{n-1}, \end{cases}$$

$i=2, \dots, n-1$.

Lemma. *The curvatures $\kappa_1, \dots, \kappa_{n-1}$ of the curve $x(s)$ are constant.*

PROOF. We shall prove the Lemma by induction on i . The vectorfields $w_1, \dots, w_n$ along $x(s)$ are defined by the recurrence

$$w_1 = v_1, \quad w_i = \nabla_s w_{i-1} = \varrho^{i-1} K^{i-1} v_1.$$

We assume $\kappa_1, \dots, \kappa_{i-1}$ are constant. Then the vectors of the Frenet frame can be expressed as linear combinations $v_i = \sum_{j=1}^i c_j w_j$, where the coefficients $c_1, \dots, c_i$ are constant.

It follows that $\nabla_s v_i = \sum_{j=1}^i c_j w_{j+1}$.

Now, we get that $g(\nabla_s v_i, \nabla_s v_i)$ is constant. In fact,

$$\nabla_s (g(\nabla_s v_i, \nabla_s v_i)) = 2g \left(K \sum_{j=1}^i c_j w_{j+1}, \sum_{j=1}^i c_j w_{j+1} \right) = 0$$

since K is antisymmetric. But we have by (5)

$$g(\nabla_s v_i, \nabla_s v_i) = \kappa_{i-1}^2 + \kappa_i^2,$$

thus $\kappa_{i-1} = \text{constant}$ implies that $\kappa_i = \text{constant}$ as well.

Theorem 3. Let $(x(s), e_1(s), \dots, e_n(s))$ be a geodesic on $O(M)$. Then the i^{th} vector of curvature of the curve $x(s)$ can be expressed in the form

$$\kappa_i v_{i+1} = (\kappa_1 \cdot \dots \cdot \kappa_{i-1})^{-1} (\mu_0^{(i)} w_{i+1} + \mu_1^{(i)} w_{i-1} + \dots + \mu_m^{(i)} w_{i+1-2m}),$$

($i+1-2m \geq 1$), where $\mu_j^{(k)} = p_{k,j}(\kappa_1, \dots, \kappa_{k-1})$ and the polynomials $p_{k,j}(x_1, \dots, x_{k-1})$ ($k \geq 0$) are defined by the recurrence

$$p_{k,0} = 1,$$

$$p_{k,j} = p_{k-1,j} + \kappa_{k-1}^2 p_{k-2,j-1}, \quad \text{if } k+1 > 2j,$$

$$\text{and} \quad p_{k,j} = 0 \quad \text{if } k+1 \leq 2j, \quad j \neq 0.$$

The proof is the same as of Theorem 3 in [3].

6§. On the orthonormal frame bundle of the n -sphere

We denote by S_r^n the n -sphere of radius r . It is well known that there is a natural diffeomorphism φ of $O(S_r^n)$ on the manifold V^{n+1} of orthonormal frames in R^{n+1} , which maps an n -frame $(x, e_1, \dots, e_n) \in O(S_r^n)$ on the $n+1$ -frame $(\frac{1}{r}x, e_1, \dots, e_n) \in V^{n+1}$.

Theorem 4. The map $\varphi: O(S_r^n) \rightarrow V^{n+1}$ is a homothety with respect to the Riemannian metrics (2) and (1), respectively, if and only if $r^2 = g$. In this case the ratio of homothety is r^2 .

PROOF. Let $u = (x, e_1, \dots, e_n) \in O(S_r^n)$ and $\varphi(u) = (e_0, e_1, \dots, e_n) \in V^{n+1}$, where $e_0 = \frac{1}{r}x$. We can write

$$dx = \omega_i e_i, \quad de_i = -\omega_i x + \omega_{ik} e_k,$$

and

$$de_0 = \omega_{0i} e_i, \quad de_i = \omega_{i0} e_0 + \omega_{ik} e_k,$$

respectively, where $i, k = 1, \dots, n$. From these follows $\omega_i = r\omega_{oi}$, since $x = re_0$. The Riemannian metrics on $O(S_r^n)$ and on V^{n+1} are

$$d\tilde{s}^2 = \sum_i (\omega_i)^2 + \varrho \sum_{i>k} (\omega_{ik})^2 = r^2 \sum_i (\omega_{i0})^2 + \varrho \sum_{i>k} (\omega_{ik})^2$$

and

$$d\tilde{s}^2 = \sum_i (\omega_{i0})^2 + \sum_{i>k} (\omega_{ik})^2.$$

respectively, from which the Theorem follows.

References

- [1] W. KLINGENBERG and S. SASAKI, On the tangent sphere bundle of a 2-sphere, *Tohoku Math. J.* **27** (1975), 49—56.
- [2] P. T. NAGY, On the tangent sphere bundle of a Riemannian 2-manifold, *Tohoku Math. J.* **29** (1977), 203—208.
- [3] P. T. NAGY, Geodesics on the tangent sphere bundle of a Riemannian manifold, *Geometriae Dedicata* **7** (1978), 233—243.
- [4] S. SASAKI, Differential geometry of tangent bundles I, *Tohoku Math. J.* **10** (1958), 338—354; II, *ibid.* **14** (1962), 146—155.

BOLYAI INSTITUTE OF UNIVERSITY
H—6720 SZEGED, HUNGARY

(Received December 14, 1976.)