

Concrete radicals in general modules

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§ 1. Introduction

Let A be an associative ring, (without any assumptions on commutativity or the existence of a unity element), and denote by ${}_A\mathcal{M}$ the category of all left A -modules. For this category we take over the concept of a radical as it is employed in *torsion theory* for unitary modules over a ring with unity element; (see e.g. [6]). We define: A preradical of ${}_A\mathcal{M}$ is a functor $r: {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$ such that for all $M, N \in {}_A\mathcal{M}$,

- (i) $M \mapsto r(M)$, a submodule of M ;
- (ii) $(f \in \text{Hom}_A(M, N)) \mapsto r(f) = f|_{r(M)} \in \text{Hom}_A(r(M), r(N))$.

A preradical r is said to be *idempotent* if $r(r(M)) = r(M)$ for all $M \in {}_A\mathcal{M}$, and r is called a *radical* if $r(M/r(M)) = 0$ for all $M \in {}_A\mathcal{M}$. If r is a radical the submodule $r(M)$ is called the *r -radical* of the module M , and M is said to be *r -radical* (*r -semisimple*), if $r(M) = M$, ($r(M) = 0$).

Our purpose is to construct a class of radicals of this type and to produce a few concrete examples of such radicals.

§ 2. Radicals of ${}_A\mathcal{M}$ defined by intersections

The basic tool in our construction is the well known one used in the theory of radicals in rings, namely *taking intersections*. Some terminology will be necessary: We consider a property Σ of submodules which defines within each module $M \in {}_A\mathcal{M}$ a subsystem Σ_M of submodules, called the Σ -submodules of M . For a given Σ the system Σ_M may be empty, e.g. when the Σ -submodules are defined to be the maximal submodules. A defining property as described above will be called an *isolator*.

2.1 Definition. An isolator Σ is said to be *stable* on ${}_A\mathcal{M}$ if for each A -epimorphism $f: M \rightarrow N$ the assignment $P \mapsto f(P)$ defines a bijection between $\{P \in \Sigma_M | \text{Ker } f \subset P\}$ and Σ_N .

A direct consequence of this definition is that $S \in \Sigma_M$ if and only if $0 \in \Sigma_{M/S}$.

2.2 Definition. A *transferring* isolator Σ is one which satisfies the following condition: If $M \in {}_A\mathcal{M}$, T a submodule of M and $S \in \Sigma_M$, then $T \cap S \neq T$ implies that $T \cap S \in \Sigma_T$.

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With each stable transferring isolator Σ we now associate the assignment $r: {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}; M \mapsto r(M)$ and $(f: M \rightarrow N) \mapsto r(f) = f|_{r(M)}: r(M) \rightarrow N$; where we define $r(M) = \bigcap_{S \in \Sigma_M} S$ if $\Sigma_M \neq \emptyset$ and $r(M) = M$ if $\Sigma_M = \emptyset$. We show that this assignment defines a radical of ${}_A\mathcal{M}$.

2.3 Lemma. *If $M \in {}_A\mathcal{M}$ and T is a submodule of M , then $r(T) \subset T \cap r(M)$.*

PROOF. If either $\Sigma_M = \emptyset$ or $T \subset S$ for all $S \in \Sigma_M$, there is nothing to prove. If $T \cap S \neq T$ for at least one $S \in \Sigma_M$, then $T \cap S \in \Sigma_T$ and hence $r(T) = \bigcap_{K \in \Sigma_T} K \subset \bigcap_{S \in \Sigma_M} (T \cap S) = T \cap \left(\bigcap_{S \in \Sigma_M} S \right) = T \cap r(M)$.

2.4 Theorem. *r is a radical of ${}_A\mathcal{M}$, i.e. every A -homomorphism $f: M \rightarrow N$ induces $r(M) \rightarrow r(N)$ by restriction and $r(M/r(M)) = 0$ for all $M \in {}_A\mathcal{M}$.*

PROOF. To prove the first part of the theorem we consider an arbitrary A -homomorphism $f: M \rightarrow N$. We need obviously only consider the case where $r(N) \neq N$. If $r(M) = M$, then the stableness of Σ shows that $r(\text{Im} f) = \text{Im} f$. Since now $\text{Im} f \subset r(N)$, by 2.3, we have that $r(M) \rightarrow r(N)$. If $r(M) \neq M$ we consider the set $\{S_\alpha \in \Sigma_M \mid \text{Ker } f \subset S_\alpha\}$, which is non-empty since $r(N) \neq N$. Now

$$f(r(M)) \subset f\left(\bigcap_{\alpha} S_\alpha\right) \subset \bigcap_{\alpha} f(S_\alpha) = r(\text{Im} f) \subset \text{Im} f \cap r(N) \subset r(N).$$

Hence $r(M) \rightarrow r(N)$ in this case also.

Concerning the second part of the theorem we may clearly confine ourselves to modules $M \in {}_A\mathcal{M}$ with $0 \neq r(M) \neq M$. Now considering the canonical projection $M \rightarrow M/r(M)$ we have the bijection $\Sigma_M \leftrightarrow \Sigma_{M/r(M)}$, $K_\beta \leftrightarrow K_\beta/r(M)$. So if $x + r(M) \in r(M/r(M))$ then $x + r(M) \in K_\beta/r(M)$ for all β so that $x \in K_\beta$ for all β . Hence $x \in r(M)$, or equivalently, $x + r(M) = 0$. This completes the proof.

As an immediate consequence of the construction of $r(M)$ we also have the following criterion.

2.5 Corollary. *A non-zero module M in ${}_A\mathcal{M}$ is isomorphic to a subdirect product of all modules M/S , ($S \in \Sigma_M$), if and only if $r(M) = 0$.*

§ 3. Concrete radicals in ${}_A\mathcal{M}$

In view of theorem 2.4 we may now substitute different stable transferring isolators for Σ to obtain concrete examples of radicals of ${}_A\mathcal{M}$. Our basic isolator (which we shall denote by Σ^1) is defined as follows:

3.1 Definition. A submodule S of a module $M \in {}_A\mathcal{M}$ shall be called a Σ^1 -submodule of M if $x \in M$ and $Ax \subset S$ imply that $x \in S$.

3.2 Lemma. Σ^1 is a stable transferring isolator.

PROOF. Let $f: M \rightarrow N$ be an A -epimorphism and let $P \in \Sigma_M^1$ such that $\text{Ker } f \subset P$. Let $An \subset f(P)$ and m a fixed element of M with $f(m) = n$. Then $Af(m) \subset f(P)$ implies that $f(am) \in f(P)$ for all $a \in A$. Hence for each $a \in A$ there is a $p_a \in P$ such that $f(am - p_a) = 0$ so that $am - p_a \in \text{Ker } f \subset P$. This shows that $Am \subset P$. Since

$P \in \Sigma_M^1$ we must have that $m \in P$ and consequently $n \in f(P)$. Hence $f(P) \in \Sigma_N^1$. On the other hand, if $Q \in \Sigma_N^1$, then $Am_1 \subset f^{-1}(Q)$ implies that $Af(m_1) \subset Q$ so that $f(m_1) \in Q$ and hence $m_1 \in f^{-1}(Q)$. Thus we have that $f^{-1}(Q) \in \Sigma_M^1$. From the identity $f(f^{-1}(Q)) = Q$ for all $Q \in \Sigma_N^1$ we obtain the surjectivity of the mapping $\{P \in \Sigma_M^1 \mid \text{Ker } f \subset P\} \rightarrow \Sigma_N^1, P \mapsto f(P)$. The injectivity follows from equally well known elementary arguments. Thus the stableness of Σ^1 is established. That this isolator also has the transferring property is a direct consequence of its definition.

The following properties of Σ^1 relative to an arbitrary $M \in \mathcal{A}\mathcal{M}$ are stated for reference; the verifications are straightforward.

3.3 Lemma. (i) Σ_M^1 is closed under intersections. (ii) If P, Q and R are submodules of M such that $P \in \Sigma_Q^1$ and $Q \in \Sigma_R^1$, then $P \in \Sigma_R^1$. (iii) If S is a submodule of M and $\bar{S} = \{x \in M \mid Ax \subset S\}$, then $S \in \Sigma_M^1$ if and only if $S = \bar{S}$.

We now consider the radical r_1 associated with Σ^1 . Let $M \in \mathcal{A}\mathcal{M}$. Then 3.3 (i) shows that $r_1(M) = 0$ if and only if $0 \in \Sigma_M^1$. Consequently, the subdirect decomposition of an r_1 -semisimple module according to 2.5 is a trivial representation. Next we observe that r_1 is idempotent. For if $r_1(r_1(M)) \subseteq r_1(M)$ for some $M \in \mathcal{A}\mathcal{M}$, then there is a $Q \subseteq r_1(M)$ in $\Sigma_{r_1(M)}^1$. Since $r_1(M) \in \Sigma_M^1$ by 3.3 (i), the property 3.3 (ii) shows that $Q \in \Sigma_M^1$. This contradicts the definition of $r_1(M)$.

Finally in connection with r_1 we characterize $r_1(M)$ under various conditions, mainly on the ground ring A . First we note that the submodules

$$M_n = \{x \in M \mid A^n x = 0\},$$

$n \in \mathbb{N}$, are all contained in $r_1(M)$, for $x \in M_n$ implies that $A^n x = 0$ so that $A^n x \subset r_1(M)$. This shows that $A(a'x) \subset r_1(M)$ for all $a' \in A^{n-1}$ so that $A^{n-1}x \subset r_1(M)$, because $r_1(M) \in \Sigma_M^1$. Continuing this process, we eventually obtain $Ax \subset r_1(M)$ and then $x \in r_1(M)$. Since clearly $M_n \subset M_{n+1}$ for all $n \in \mathbb{N}$ we have an ascending chain

$$(1) \quad M_1 \subset M_2 \subset \dots \subset M_n \subset \dots \subset r_1(M),$$

in which $M_1 = \{x \in M \mid Ax = 0\}$ is the maximal trivial submodule of M .

3.4 Corollary. If there is an $n \in \mathbb{N}$ with $A^n = A^{n+1} = \dots$, then $r_1(M) = M_n$.

PROOF. If $Ax \subset M_n$ then $A^n(ax) = 0$ for all $a \in A$. Hence $A^{n+1}x = 0$ so that $A^n x = 0$, or equivalently $x \in M_n$. Thus we have that $M_n \in \Sigma_M^1$ which together with $M_n \subset r_1(M)$ show that $r_1(M) = M_n$. In particular, if A is left or right artinian there exists an $n \in \mathbb{N}$ such that $r_1(M) = M_n$, and if A is idempotent then $r_1(M) = M_1$. Apart from conditions on A we note that the chain (1) may also be employed in the same direct manner as in 3.4 to derive the following characterization.

3.5 Corollary. If M is noetherian, then $r_1(M) = M_n$ for some $n \in \mathbb{N}$.

Returning to our arbitrary $M \in \mathcal{A}\mathcal{M}$ we assume for the moment that A has a unity element e . Then 3.4 shows that $r_1(M) = M_1$. Furthermore, if $\lambda: A \rightarrow \text{End}^l(M)$ is the ring homomorphism which supplies the module structure of M and if we identify $\lambda(e)$ with e , we note that the submodule $E = \{ex - x \mid x \in M\}$ is contained in $M_1 = r_1(M)$, because $a(ex - x) = 0$ for all $a \in A$. However, $E \in \Sigma_M^1$, for $Ay \subset E$ implies that $ey \in E$ so that $y \in E$. Hence we have that $r_1(M) = M_1 = E$. Using the

characterization $r_1(M) = M_1$ we see that r_1 -radicality is equivalent with triviality, while the characterization $r_1(M) = E$ establishes the equivalence between r_1 -simplicity and unitariness.

Once again we let A be an arbitrary ring and we now consider the Σ^1 -maximal submodules of an arbitrary module $M \in \mathcal{A}\mathcal{M}$; these submodules will be termed the Σ^2 -submodules of M . It is obvious that $\Sigma^2 \subset \Sigma^1$, and the restriction of $P \mapsto f(P)$ in the proof of 3.2 to $\{P \in \Sigma_M^2 \mid \text{Ker } f \subset P\}$ immediately yields the stableness of Σ^2 . Moreover, if T is a submodule of M and $L \in \Sigma_M^2$, a direct application of the definition of Σ^1 shows that $T \cap L \neq T$ implies that $T \cap L \in \Sigma_T^1$. The relation $T \cap L \in \Sigma_T^2$ now follows from $T/(T \cap L) \cong (T+L)/L = M/L$, which is a simple module. Hence we have shown that Σ^2 is a stable transferring isolator, and we may consider the radical r_2 associated with it. First we observe that for every unitary Z -module (abelian group) M , $r_2(M) = \Phi(M)$, the Frattini submodule of M , and hence that r_2 is not idempotent. The latter observation immediately yields the expected fact that $r_1 \neq r_2$. We shall employ the radical r_2 to fit into our scheme a well known module radical:

3.6 Theorem. *The radical r_2 coincides with the Kertész radical.*

Remark. The Kertész radical, which we shall denote by k , is discussed in [2; 3]. For an arbitrary $M \in \mathcal{A}\mathcal{M}$ it is defined by $k(M) = \{x \in M \mid Ax \subset \Phi(M)\}$. Using the easily verifiable fact that $L \in \Sigma_M^2$ if and only if M/L is irreducible, we note that our theorem is already partially covered in [2; 3], where modules which are not k -radical are being characterized. We give here a complete proof within the framework of our approach.

PROOF. If M has no maximal submodules, then also $\Sigma_M^2 = \emptyset$, so that $r_2(M) = k(M) = M$. In the other alternative we consider the set $\{L_\gamma \mid \gamma \in C\}$ of all maximal submodules of M and the corresponding set $\{\bar{L}_\gamma \mid \gamma \in C\}$. (See 3.3.) For each $\gamma \in C$ we have that $\bar{L}_\gamma$ is a submodule of M , that $L_\gamma \subset \bar{L}_\gamma \subset M$ and hence (by the maximality of L_γ) that $L_\gamma = \bar{L}_\gamma$ or $\bar{L}_\gamma = M$. Since $x \in k(M)$ if and only if $Ax \subset L_\gamma$ for all γ , and since the latter condition is equivalent with $x \in \bar{L}_\gamma$ for all γ , we have that

$$(2) \quad k(M) = \bigcap_{\gamma} \bar{L}_\gamma.$$

Furthermore, 3.3 (iii) shows that $L_\gamma \in \Sigma_M^2$ if and only if $L_\gamma = \bar{L}_\gamma$. Hence if $\bar{L}_\gamma = M$ for all γ , then $\Sigma_M^2 = \emptyset$ and, applying (2), we obtain $r_2(M) = M = \bigcap_{\gamma} \bar{L}_\gamma = k(M)$.

On the other hand, if at least one $\bar{L}_\gamma \neq M$, i.e. $\bar{L}_\gamma = L_\gamma$, then the redundancy of the (possible) M 's in $\bigcap_{\gamma} \bar{L}_\gamma$ in (2) shows that $k(M) = \bigcap_{L \in \Sigma_M^2} L = r_2(M)$. This completes the proof.

Our final application of theorem 2.4 is the construction of one more radical based on a concept of 'modularity' of maximal submodules in the following sense.

3.7 Definition. A maximal submodule L of a module $M \in \mathcal{A}\mathcal{M}$ is called a Σ^3 -submodule of M if there exists an $a \in A$ such that $ax - x \in L$ for all $x \in M$.

It easily follows that $\Sigma_M^3 \subset \Sigma_M^2$, and if A has a unity element e , the reverse inclusion also holds; for in this case, if $L \in \Sigma_M^2$ we have that $ex - x \in r_1(M) \subset L$ for all $x \in M$. Thus we have:

3.8 Lemma. For each $M \in {}_A\mathcal{M}$: (i) $\Sigma_M^3 \subset \Sigma_M^2 \subset \Sigma_M^1$; (ii) if A has a unity element then $\Sigma_M^2 = \Sigma_M^3$.

The validity of the following auxilliary result may also be checked directly.

3.9 Lemma. Σ^3 is a stable transferring isolator on ${}_A\mathcal{M}$.

Denoting the radical associated with Σ^3 by r_3 we may state the following immediate consequence of lemma 3.8.

3.10 Corollary. For each $M \in {}_A\mathcal{M}$: (i) $r_1(M) \subset r_2(M) \subset r_3(M)$; (ii) if A has a unity element then $r_2(M) = r_3(M)$.

Regarding the second part of this corollary we note that in the case of a unitary left A -module M the radicals r_2 and r_3 coincide with the radical usually employed in this case, namely $\Phi(M)$. This also shows that r_3 is not idempotent.

For an arbitrary $M \in {}_A\mathcal{M}$, (A arbitrary), the stableness of Σ^3 has the implication that $L \in \Sigma_M^3$ if and only if $0 \in \Sigma_{M/L}^3$. This means that for $L \in \Sigma_M^3$ the factor module M/L has a 'quasi-unitary' property in the sense that there is an $a \in A$ such that $a(x+L) = x+L$ for all $x+L \in M/L$. Since L is maximal we have the additional property that M/L is irreducible. Calling a module $N \in {}_A\mathcal{M}$ *quasi-unitary* if there exists an element $a \in A$ with $ax = x$ for all $x \in N$, we have the following result in view of 2.5.

3.11 Corollary. A non-zero module M in ${}_A\mathcal{M}$ is isomorphic to a subdirect product of quasi-unitary irreducible modules in ${}_A\mathcal{M}$ if and only if $r_3(M) = 0$.

We observe that if A has a unity element, the r_i -semisimple modules, ($i=1, 2, 3$), are in fact unitary, for $r_i(M) = 0$ implies that $M = M/r_i(M) = M/r_1(M) \cong AM$. Finally in this section we must mention that the exact relationship between r_2 and r_3 has not yet been settled. The probability seems to weigh in the direction of a difference.

§ 4. The module radicals r_i of ${}_A\mathcal{M}$ confined to the ground ring A

We conclude our discussion by comparing the module radicals $r_i(A)$, ($i=1, 2, 3$), with ring radicals of A . The Baer lower radical β , the Jacobson radical J and the Brown—McCoy radical B were the best known candidates for this purpose.

4.1 Theorem. For every associative ring A : (i) $r_1(A) \subset \beta(A)$, (ii) $(r_2A) \subset J(A)$, (iii) $r_3(A) \subset B(A)$.

PROOF. We need only prove (i) and (iii) since (ii) has already been established in [3, 7]. (i) Let P be a prime ideal of A and let $x \in A$ with $Ax \subset P$. Each element of the ideal product $A(x)$ may be written as a finite sum of elements of the form $ax + \sum a_i x a'_i$, where $a, a_i, a'_i \in A$. Since $Ax \subset P$ we therefore have that $A(x) \subset P$ and since P is a prime ideal we obtain $x \in P$. This shows that $P \in \Sigma_A^1$, and we may conclude that $r_1(A) \subset \beta(A)$. (iii) Clearly, every modular maximal ideal of A , (cf. [5]), belongs to Σ_A^3 . Hence $r_3(A) \subset B(A)$. This completes the proof.

In the case where A has a unity element it was already noted that $r_2(M)=r_3(M)$ for all $M \in {}_A\mathcal{M}$. Considering the module $A \in {}_A\mathcal{M}$ in this case, we observe that Σ_A^2 and Σ_A^3 coincide with the set of modular maximal left ideals of A , so that $r_2(A)=r_3(A)=J(A)$. If A is a commutative ring (with or without unity element), we know that $J(A)=B(A)$. Moreover, the modular maximal ideals of A are exactly the Σ^3 -ideals of A . Finally in this case, if $L \in \Sigma_A^2$, then $A^2 \not\subseteq L$ and hence a result in [4] ensures the modularity of L . Thus if A is a commutative ring we have that $r_2(A)=r_3(A)=J(A)=B(A)$.

Concerning r_1 no positive results are ensured by either of the above mentioned 'natural' conditions on A . If A has a unity element then $r_1(A)=0$, and in the case of commutative rings the Zassenhaus example A mentioned in [1], p. 20, supplies a counter example. Here $\beta(A)=A$, while $A^2=A$ shows that $r_1(A)=A_1 \neq A$.

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