On entire functions of slow growth

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1. Introduction. If $f(z)$ is an entire function and $M(r, f) := \operatorname*{Max}_{|z|=r} |f(z)|$, then $\varrho_{cl}(f) := \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$ is called the *(classical) order* of f. If $0 < \varrho_{cl}(f) < \infty$ then one introduces further $\sigma_{cl}(f) := \limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\varrho_{cl}(f)}}$ and denotes $\sigma_{cl}(f)$ by the *(classical) type* of f. In the theory of entire functions it is well known how $\varrho_{cl}(f)$ and $\sigma_{cl}(f)$ can be expressed by the coefficients a_n of the Taylor series $\sum a_n z^n$ of f.

In their interesting note [3] P. K. JAIN and V. D. CHUGH introduced a *logarithmic order* $\varrho(f)$ for entire functions f by

$$(1) \qquad \varrho(f) := \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log\log r}$$

and proved the analogues of some classical results on entire functions. It is clear that $\varrho(f)=0$, if f is constant, and $\varrho(f) \geqq 1$ otherwise. Especially if f is a nonconstant polynomial, then we have $\varrho(f)=1$; but there are also entire transcendental functions f with $\varrho(f)=1$. For nonconstant entire functions f with $\varrho(f)<\infty$ we introduce further the notion of *logarithmic type* $\sigma(f)$ by [1])

$$(2) \qquad \sigma(f) := \limsup_{r\to\infty} \frac{\log M(r,f)}{(\log r)^{\varrho(f)}};$$

one can ask how ϱ and σ are expressed by the a_n's. This was recently answered by Miss E. JOSKO [4]; concerning ϱ there is already a result of S. M. SHAH and M. ISHAQ [5].

Here we treat these questions more generally giving a connection between ϱ, σ and the coefficients of certain interpolation series for $f(z)$. Such formulas have applications in other mathematical topics, e.g. in Diophantine Approximations (see [1], [2]).

2. Interpolation series. Let $\{z_j\}_{j=1,2,\dots}$ be an infinite sequence of complex numbers and $f(z)$ an entire function. If we define polynomials $P_k(z)$ by

$$(3) \qquad P_0(z) := 1, \quad P_k(z) := \prod_{j=1}^{k} (z - z_j) \quad (k \geqq 1)$$

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[1]) We write only $M(r), \varrho, \sigma$ instead of $M(r, f), \varrho(f), \sigma(f)$ if there is no risk of confusion.

and $A_0, A_1, \ldots; R_n(z)$ by

(4a) $$A_k := \frac{1}{2\pi i} \int_{C_{k+1}} \frac{f(\zeta)\,d\zeta}{P_{k+1}(\zeta)};$$ (4b) $$R_n(z) := P_n(z) \frac{1}{2\pi i} \int_{C_{n,z}} \frac{f(\zeta)\,d\zeta}{(\zeta - z) P_n(\zeta)}$$

then we have

$$f(z) = \sum_{k=0}^{n-1} A_k P_k(z) + R_n(z) \quad (n \geqq 0).$$

C_k resp. $C_{n,z}$ in (4a) resp. (4b) can be chosen as cercles around $\zeta = 0$ containing $z_1, \ldots, z_k$ resp. $z_1, \ldots, z_n$; z.

For transcendental f the following Theorem 1 gives sufficient conditions on the sequence $\{z_j\}$ which guarantee $R_n(z) \to 0$ with $n \to \infty$ for every complex z. If these conditions are satisfied, $f(z)$ can be represented by the series $\sum_{k=0}^{\infty} A_k P_k(z)$ in the whole complex plane. If one has *any* representation $\sum_{k=0}^{\infty} B_k P_k(z)$ for $f(z)$, valid in the whole complex plane, then it follows that $B_k = A_k$ for all $k \geqq 0$. The series $\sum A_k P_k(z)$ is called the *Newton interpolation series* for $f(z)$ with respect to the points $z_1, z_2, \ldots$; in the special case $z_1 = z_2 = \ldots = z_0$ with fixed complex z_0 one gets the power series of $f(z)$ at $z = z_0$ of course. It is also clear, that no conditions on $\{z_j\}$ are needed if f is a polynomial; here $R_n(z)$ is identically zero for all $n >$ degree (f).

Theorem 1. *Let f be transcendental. Then $R_n(z) \to 0$ with $n \to \infty$ for every fixed complex z, if the sequence $\{z_j\}$ satisfies*

(5) $$|z_j| \leqq \exp(cj^{\varkappa}) \quad (j \geqq 1)$$

with one of the following additional conditions

(i) $\varkappa = 0,\ 0 \leqq c$;

(ii) *if* $\varrho = 1$: $0 < \varkappa,\ 0 < c$;

(iii) *if* $1 < \varrho < \infty$: $0 < \varkappa < (\varrho - 1)^{-1},\ 0 < c$;

(iv) *if* $\sigma = 0$: $0 < \varkappa \leqq (\varrho - 1)^{-1},\ 0 < c$;

(v) *if* $0 < \sigma < \infty$: $0 < \varkappa \leqq (\varrho - 1)^{-1},\ 0 < c < (\varrho\sigma)^{-1/(\varrho - 1)}$.

Remark 1. A transcendental f with $\varrho < \infty, \sigma < \infty$ has $\varrho > 1$.

Remark 2. If the sequence $\{z_j\}$ is bounded (see (i)), then each entire function $f(z)$ is represented by its interpolation series $\sum_{k=0}^{\infty} A_k P_k(z)$, independent of the growth of f. The cases (ii) up to (v) are concerned with unbounded $\{z_j\}$: Depending on the growth of f conditions on the growth of $|z_j|$ with j ensuring the convergence of $\sum A_k P_k(z)$ to $f(z)$ are given.

Remark 3. The entire function (treated from the arithmetical point of view in [1]) $f_0(z) := \prod_{k=1}^{\infty} (1 - ze^{-k})$ shows that the result of Theorem 1 is best possible if $\{z_j\}$ is unbounded: First we have $\varrho(f_0) = 2$. If we take $z_j = e^j$ $(j \geqq 1)$, then (5) is satisfied with $\varkappa=c=1$; so $\varkappa<(\varrho-1)^{-1}$ is not valid, but $\sigma(f_0)=1/2$ and $\varkappa=(\varrho-1)^{-1}$. In $c\leqq(\varrho\sigma)^{-1/(\varrho-1)}$ we have not the strong inequality (which would imply $R_n(z)\to 0$ with $n\to\infty$), but we have equality. Here we have indeed $R_n(0)=1$ for all $n\geqq 0$.

PROOF OF THEOREM 1. We have

$$(6)\qquad |P_n(z)|\leqq(|z|+1)^n\exp\big(c\sum_{j=1}^{n}j^{\varkappa}\big)\leqq\exp\left(\frac{c}{\varkappa+1}n^{\varkappa+1}+cn^{\varkappa}+c_1(z)n\right)$$

and from (1) and (2) with arbitrary $\varepsilon>0$

$$(7)\qquad \log M(r)\leqq\begin{cases}(\log r)^{\varrho+\varepsilon} & (\text{if } \varrho<\infty)\\ (\sigma+\varepsilon)(\log r)^{\varrho} & (\text{if furthermore } \sigma<\infty)\end{cases}$$

for all $r\geqq r_0(\varepsilon)$. If the inequality

$$(8)\qquad r\geqq 2\,\mathrm{Max}\big(|z|,\ \exp(cn^{\varkappa})\big)$$

is also satisfied and if we choose $|\zeta|=r$ as $C_{n,z}$ in (4b), then we get

$$(9)\qquad \left|\frac{1}{2\pi i}\int_{C_{n,z}}\frac{f(\zeta)\,d\zeta}{(\zeta-z)P_n(\zeta)}\right|\leqq 2^{n+1}M(r)r^{-n}\leqq\begin{cases}\exp\big(-n\log r+(\log r)^{\varrho+\varepsilon}+n+1\big)\\ \qquad\qquad (\text{if } \varrho<\infty)\\ \exp\big(-n\log r+(\sigma+\varepsilon)(\log r)^{\varrho}+n+1\big)\\ \qquad\qquad (\text{if furthermore } \sigma<\infty).\end{cases}$$

Ad (i): If $\varkappa=0$, then $|P_n(z)|\leqq\exp\big(c_2(z)n\big)$ by (6) and therefore $|R_n(z)|\leqq\leqq 2M(r)(2e^{c_2(z)}r^{-1})^n$ from which we get the assertion by *fixing* r such that $r>\mathrm{Max}\,(r_0,\,2|z|,\,2e^{c},\,2e^{c_2})$.
Ad (ii) up to (v): Here we have $\varrho<\infty$ and furthermore (in (iv), (v)) $\sigma<\infty$. If (5) is satisfied with $\varkappa>0, c>0$, then we can assume w.l.o.g. $\varepsilon>0$ so small, that

$$(10\text{a})\qquad \varkappa<(\varrho+\varepsilon-1)^{-1}\quad (\text{for (ii), (iii)}) \text{ resp.}$$

$$(10\text{b})\qquad c<\big(\varrho(\sigma+\varepsilon)\big)^{-1/(\varrho-1)}\quad (\text{for (iv), (v), if } \varkappa=(\varrho-1)^{-1}).$$

If we choose r by

$$(11\text{a})\qquad \log r=\big(n/(\varrho+\varepsilon)\big)^{1/(\varrho+\varepsilon-1)} \text{ resp. } (11\text{b})\ \log r=\big(n/\varrho(\sigma+\varepsilon)\big)^{1/(\varrho-1)}$$

then $r\geqq r_0(\varepsilon)$ and (8) are satisfied for all $n\geqq n_0(\varepsilon,z)$ by (10a) and (10b). If we use (6), (9) and (11a) resp. (11b) to estimate $R_n(z)$ from (4b), we get in the cases (ii), (iii)

$$\log|R_n(z)|\leqq-(\varrho+\varepsilon-1)\big(n/(\varrho+\varepsilon)\big)^{1+1/(\varrho+\varepsilon-1)}+c(\varkappa+1)^{-1}n^{\varkappa+1}+cn^{\varkappa}+c_3(z)n.$$

From this we find the asserted result by $\varrho+\varepsilon>1$ and (10a). In the cases (iv), (v) we get analogously

$$(12)\qquad \log|R_n(z)|\leqq-\left(1-\frac{1}{\varrho}\right)\big(\varrho(\sigma+\varepsilon)\big)^{-1/(\varrho-1)}n^{1+1/(\varrho-1)}+\frac{c}{\varkappa+1}n^{\varkappa+1}+cn^{\varkappa}+c_4(z)n.$$

$\varkappa<(\varrho-1)^{-1}$ is already settled in (iii) and so we can assume $\varkappa=(\varrho-1)^{-1}$ and the right hand side of (12) becomes

$$-(1-\varrho^{-1})\big((\varrho(\sigma+\varepsilon))^{-1/(\varrho-1)}-c\big)n^{\varrho/(\varrho-1)}+cn^{1/(\varrho-1)}+c_5(z)n$$

from which we get (by (10b)) once more the assertion.

3. Logarithmic order and interpolation coefficients. Now let f be an entire function and $\{z_j\}$ an infinite sequence of complex numbers such that either f is a polynomial or f is transcendental and f, $\{z_j\}$ satisfy one of the conditions (i) up to (v) of Theorem 1. Then

$$f(z) = \sum_{k=0}^{\infty} A_k P_k(z) \tag{13}$$

is valid in the whole complex plane and we look for a connection between $\varrho(f)$ and the interpolation coefficients A_k. To this purpose we define[2])

$$\mu(f) := \limsup_{n\to\infty} \frac{\log n}{\log(-\log|A_n|)}. \tag{14}$$

If f is a polynomial, then we have obviously $\mu(f)=0$ since $A_n=0$ for all $n>\text{degree}(f)$; especially if f is a nonconstant polynomial then we have $\varrho=(1-\mu)^{-1}$. We prove this identity for transcendental f too, if f, $\{z_j\}$ satisfy one of the conditions (i) up to (v) of Theorem 1:

Theorem 2. *Let f be transcendental and let f, $\{z_j\}$ satisfy one of the conditions* (i) *up to* (v) *of Theorem* 1. *Then the coefficients A_n of the series* (13) *for $f(z)$ have the property, that $\mu(f)$ satisfies*

$$0 \leqq \mu(f) \leqq 1 \quad \textit{and} \quad \varrho(f) = \big(1-\mu(f)\big)^{-1}.$$

Remark 4. The coefficients A_n depend on the choice of $\{z_j\}$ but not $\mu(f)$.

Corollary 1. [4] *To every $\lambda\in[1,\infty]$ there are entire transcendental functions f_λ with $\varrho(f_\lambda)=\lambda$.*

Proof. Define $f_\lambda(z) := \sum_{n=0}^{\infty} a_n(\lambda) z^n$ with $a_n(\lambda) := \exp(-n^{\lambda/(\lambda-1)})$, if $\lambda\in(1,\infty)$ and $a_n(1) := \exp(-n^n)$. f_λ is entire transcendental and choosing all $z_j=0$ condition (i) of Theorem 1 is satisfied and from $A_n=a_n(\lambda)$ we see $\mu(f_\lambda)=1-1/\lambda$, if $\lambda\in(1,\infty)$ and $\mu(f_1)=0$. Therefore we have by Theorem 2: $\varrho(f_\lambda)=\lambda$ for $\lambda\in[1,\infty)$. Of course $\varrho(e^z)=\infty$.

Proof of Theorem 2. If we choose $|\zeta|=r$ as C_{n+1} in (4a) with

$$r \geqq 2\exp\big(c(n+1)^{\varkappa}\big) \tag{15}$$

we get immediately from (4a)

$$|A_n| \leqq 2^{n+1} M(r) r^{-n} \quad (n \geqq 0). \tag{16}$$

If $\varkappa=0$ (case (i)), then we *fixe* $r \geqq \text{Max}(2e^c, 4e)$ and obtain $|A_n| \leqq 2^{1-n} M(r) e^{-n} < e^{-n}$ for all large n and therefore $\log(-\log|A_n|) > \log n$ so that $0\leqq\mu\leqq 1$ by (14).

If $\varrho<\infty$, it follows from (16) and the first part of (7), that

$$|A_n| \leqq \exp\big(n-(\varrho+\varepsilon-1)(n/(\varrho+\varepsilon))^{1+1/(\varrho+\varepsilon-1)}\big) \quad (n \geqq n_0), \tag{17}$$

[2]) If $A_n=0$, then $\dfrac{\log n}{\log(-\log|A_n|)}$ is defined to be zero.

choosing r as in (11a); remark that (15) is satisfied if we suppose (10a) for ε (which is obviously no condition in the case $\varkappa=0$). From (17) follows $\log(-\log|A_n|)\geqq \geqq \frac{\varrho+\varepsilon}{\varrho+\varepsilon-1}\log n+O(1)$ which gives $0\leqq\mu\leqq 1-(\varrho+\varepsilon)^{-1}$ for every small $\varepsilon>0$. So we get in the cases (i) with finite ϱ, (ii), (iii) $0\leqq\mu<1$ and

$$\mu \leqq 1-\varrho^{-1}. \tag{17}$$

Since $\mu\leqq 1$, inequality (17) is also correct for $\varrho=\infty$.

If $\sigma<\infty$ it follows from (16) and the second part of (7)

$$|A_n| \leqq \exp\left(n-\left(1-\frac{1}{\varrho}\right)\left(\varrho(\sigma+\varepsilon)\right)^{-1/(\varrho-1)} n^{1+1/(\varrho-1)}\right) \tag{18}$$

choosing r as in (11b); then (15) is satisfied supposing (10b) for ε. Therefore we have $\log(-\log|A_n|)\geqq\frac{\varrho}{\varrho-1}\log n+O(1)$ for all large n so that we get $0\leqq\mu<1$ and (17) also in the cases (iv), (v).

Theorem 2 is shown, if we can further prove

$$\varrho \leqq (1-\mu)^{-1} \tag{19}$$

and in case $\mu=1$ this is trivially true. Thus we can suppose $\mu<1$ and $\varepsilon>0$ so small that $\mu+\varepsilon<1$ is satisfied too. From the definition (14) we have

$$|A_n| \leqq \exp\left(-n^{1/(\mu+\varepsilon)}\right) \quad (n \geqq n_0(\varepsilon)) \tag{20}$$

and therefore from (13), if *we treat first the case* $\varkappa=0$ and if we use

$$|P_n(z)| \leqq (2r)^n \quad \text{on} \quad |z|=r \quad \text{for all} \quad r \geqq e^c, \tag{21}$$

we get

$$M(r) \leqq \sum_{n<n_0} |A_n|(2r)^n + \sum_{n\geqq n_0} \exp\left(n\log 2r - n^{1/(\mu+\varepsilon)}\right). \tag{22}$$

Defining the integer $N_0(r)$ by

$$N_0(r) := \left[(\log 4r)^{(\mu+\varepsilon)/(1-\mu-\varepsilon)}\right] \tag{23}$$

and splitting up the second sum on the right hand side of (22) as $\sum\limits_{n_0\leqq n\leqq N_0(r)} + \sum\limits_{n>N_0(r)}$, we find by (23): $\sum\limits_{n>N_0(r)} \ldots < \sum\limits_{n>N_0(r)} 2^{-n}<1$, whereas

$$\sum_{n_0\leqq n\leqq N_0(r)} \ldots \leqq N_0(r)\exp\left\{(1-\mu-\varepsilon)(\mu+\varepsilon)^{(\mu+\varepsilon)/(1-\mu-\varepsilon)}(\log 2r)^{1/(1-\mu-\varepsilon)}\right\}. \tag{24}$$

Using these estimations in (22), we find from the definition (1) that $\varrho\leqq(1-\mu-\varepsilon)^{-1}$ and so we have (19) since ε was arbitrary.

The (uniform) *treatment of the case* $\varkappa>0$ is a bit more delicate. Since (19) is certainly correct for $\varrho=1$, inequality (19) has only to be shown in the cases (iii), (iv), (v) of Theorem 1 and here we have always $\varkappa\leqq(\varrho-1)^{-1}$. If $(\mu+\varepsilon)^{-1}\leqq\varkappa+1$, from the last inequality we get $\varrho\leqq(1-\mu-\varepsilon)^{-1}$ and therefore (19). Thus we can assume

$$1+\varkappa < (\mu+\varepsilon)^{-1}. \tag{25}$$

With the $c>0$ from Theorem 1 we define

$$N(r) := [c^{-1/\varkappa}(\log r)^{1/\varkappa}]; \tag{26}$$

on the cercle $|z|=r$ we have

$$|P_n(z)| \leqq \begin{cases} (2r)^n & \text{if} \quad n \leqq N(r), \\ 2^n r^{N(r)} \exp\big(c \sum\limits_{j=1+N(r)}^{n} j^{\varkappa}\big) & \text{if} \quad n > N(r). \end{cases} \tag{27}$$

Choosing r so that $N(r)>n_0(\varepsilon)$ we have from (13), (20), (27)

$$\begin{aligned} M(r) \leqq \sum_{n<n_0} |A_n|(2r)^n + \sum_{n_0 \leqq n \leqq N(r)} \exp\big(n \log 2r - n^{1/(\mu+\varepsilon)}\big) + \\ + \sum_{n>N(r)} \exp\Big(n - n^{1/(\mu+\varepsilon)} + N(r)\log r + \frac{c}{\varkappa+1} n^{\varkappa+1} + cn^{\varkappa} - \frac{c}{\varkappa+1} N(r)^{\varkappa+1}\Big). \end{aligned} \tag{28}$$

For the second sum on the right hand side we have once more (24), but now with $N(r)$ from (26) instead of $N_0(r)$. Choosing r so that we have $\frac{1}{2} n^{1/(\mu+\varepsilon)} \geqq \frac{c}{\varkappa+1} n^{\varkappa+1} + cn^{\varkappa}+n$ for all $n>N(r)$ (which is possible by (25)) we get from (26)

$$\begin{aligned} \sum_{n>N(r)} \ldots < \exp\big(c^{-1/\varkappa}(\log r)^{(\varkappa+1)/\varkappa}\big) \sum_{n>N(r)} \exp\Big(-\frac{1}{2} n^{1/(\mu+\varepsilon)}\Big) = \\ \exp\big(c^{-1/\varkappa}(\log r)^{(\varkappa+1)/\varkappa} - \\ -\frac{1}{2}(N(r)+1)^{1/(\mu+\varepsilon)}\big) \sum_{j=0}^{\infty} \exp\Big\{-\frac{1}{2}\big((N(r)+1+j)^{1(\mu+\varepsilon)} - (N(r)+1)^{1/(\mu+\varepsilon)}\big)\Big\}. \end{aligned}$$

Using once more (26) and Bernoulli's inequality we find

$$\begin{aligned} \sum_{n>N(r)} \ldots < \exp\Big(c^{-1/\varkappa}(\log r)^{(\varkappa+1)/\varkappa} - \frac{1}{2} c^{-1/\varkappa(\mu+\varepsilon)} (\log r)^{1/\varkappa(\mu+\varepsilon)}\Big) \times \\ \times \sum_{j=0}^{\infty} \exp\Big(-\frac{j}{2(\mu+\varepsilon)}\big(N(r)+1\big)^{(1-\mu-\varepsilon)/(\mu+\varepsilon)}\Big) \end{aligned} \tag{29}$$

and $\sum\limits_{j} \ldots$ is bounded by an absolute constant. The first factor on the right hand side of (29) is also bounded as $r \to \infty$ in virtue of (25) and we get $\varrho \leqq (1-\mu-\varepsilon)^{-1}$ and so (19).

4. Logarithmic type and interpolation coefficients. Here we investigate the connection between $\sigma(f)$ and the A_n's for entire functions f with $1 \leqq \varrho(f) < \infty$. To this purpose we define [3])

$$v(f) := \limsup_{n\to\infty} \frac{n^{\varrho}}{(-\log|A_n|)^{\varrho-1}}. \tag{30}$$

If f is a nonconstant polynomial, then $\nu(f)=0$ and $\sigma(f)=\text{degree}(f)$. This shows that the supposition of the transcendence of f cannot be canceled in the next theorem.

Theorem 3. *Let f be transcendental with $\varrho(f)<\infty$ and let $f, \{z_j\}$ satisfy one of the conditions* (i) *up to* (v) *of Theorem* 1. *Then the coefficients A_n of the series* (13) *for $f(z)$ have the property that we have for $\nu(f)$*

$$\sigma = \nu(\varrho-1)^{\varrho-1}\varrho^{-\varrho} \quad (0^0 := 1).$$

Remark 5. By Remark 1 $\varrho=1$ implies here $\sigma=\infty$ and by (30) we have also $\nu=\infty$ since $A_n \neq 0$ infinitely often. So we can confine us to $1<\varrho<\infty$ for the proof.

Remark 6. Lemma 1, ii) in [1] is a very special case of Theorem 3.

Corollary 2. *Let λ, ω be given with $1<\lambda<\infty$, $0\leqq\omega\leqq\infty$. Then there are entire functions $f_{\lambda,\omega}(z)$ with $\varrho(f_{\lambda,\omega})=\lambda$, $\sigma(f_{\lambda,\omega})=\omega$.*

Proof. Take $\sum z^n \exp\left(-(n^\lambda \log n)^{1/(\lambda-1)}\right)$ if $\omega=0$, $\sum z^n \exp\left(-(n^\lambda \log^{-1} n)^{1/(\lambda-1)}\right)$ if $\omega=\infty$ and $\sum z^n \exp\left(-(n^\lambda \nu^{-1})^{1/(\lambda-1)}\right)$ if $0<\omega<\infty$, where $\nu:=\omega\lambda^\lambda(\lambda-1)^{1-\lambda}$.

Proof of Theorem 3. If $\sigma<\infty$ we have immediately from (18) that $\nu\leqq\sigma\varrho^\varrho(\varrho-1)^{1-\varrho}$ which is also correct for $\sigma=\infty$. To prove Theorem 3 we show

$$\sigma \leqq \nu\varrho^{-\varrho}(\varrho-1)^{\varrho-1} \tag{31}$$

which is true for $\nu=\infty$. So we can suppose $\nu<\infty$ and we have from the definition (30)

$$|A_n| \leqq \exp\left(-(\nu+\varepsilon)^{-1/(\varrho-1)} n^{\varrho/(\varrho-1)}\right) \quad \left(n \geqq \bar{n}_0(\varepsilon)\right) \tag{32}$$

instead of (20). To treat the case $\varkappa=0$ we argue exactly as in the proof of Theorem 2 replacing (20) by (32) and $N_0(r)$ in (23) by $\overline{N}_0(r):=[(\nu+\varepsilon)(\log 4r)^{\varrho-1}]$. We find $\sum\limits_{n>\overline{N}_0(r)} \ldots <1$ and instead of (24)

$$\sum_{\bar{n}_0\leqq n\leqq \overline{N}_0(r)} \ldots \leqq \overline{N}_0(r) \exp\left((\nu+\varepsilon)(\varrho-1)^{\varrho-1}\varrho^{-\varrho}(\log 2r)^\varrho\right) \tag{33}$$

from which (31) follows.

To treat the case $\varkappa>0$ we start from (28) with $n_0, n^{1/(\mu+\varepsilon)}$ replaced by $\bar{n}_0$, $(\nu+\varepsilon)^{-1/(\varrho-1)} n^{\varrho/(\varrho-1)}$ but with the same $N(r)$ as in (26). Then the sum $\sum\limits_{\bar{n}_0\leqq n\leqq N(r)}$ has the same bound as in (33) with $N(r)$ instead of $\overline{N}_0(r)$. The sum $\sum\limits_{n>N(r)}$ is estimated by

(34)
$$\exp\left(\frac{\varkappa}{1+\varkappa} c^{-1/\varkappa}(\log r)^{(\varkappa+1)/\varkappa}\right) \sum_{n>N(r)} \exp\left(-(\nu+\varepsilon)^{-1/(\varrho-1)} n^{\varrho/(\varrho-1)} + \frac{c}{\varkappa+1} n^{\varkappa+1} + cn^\varkappa + n\right)$$

[3]) If $A_n=0$, then $n^\varrho(-\log|A_n|)^{1-\varrho}$ is defined to be zero.

where we used (for later purposes) the term $-c(\varkappa+1)^{-1}N(r)^{\varkappa+1}$ in (28) too. If $\varkappa<(\varrho-1)^{-1}$ we choose r so large that the sum $\sum\limits_{n>N(r)}$ in (34) is bounded by

$$\sum_{n>N(r)} \exp\left(-\frac{1}{2}(\nu+\varepsilon)^{-1/(\varrho-1)} n^{\varrho/(\varrho-1)}\right) < \tag{35}$$

$$< \exp\left(-\frac{1}{2}(\nu+\varepsilon)^{-1/(\varrho-1)} c^{-\varrho/\varkappa(\varrho-1)}(\log r)^{\varrho/\varkappa(\varrho-1)}\right)\cdot \sum_{j=0}^{\infty}\cdots$$

where $\sum\limits_{j}$ is a sum of the same type as in (29) being bounded by an absolute constant. Inserting (35) in (34) and using $\varkappa+1<\varrho/(\varrho-1)$ we find that (34) is bounded by an absolute constant so that (31) is proved for $\varkappa<(\varrho-1)^{-1}$.

To prove it finally in the case $\varkappa=(\varrho-1)^{-1}$ we can assume w.l.o.g. that

$$c < \varrho(\varrho-1)^{-1}(\nu+\varepsilon)^{-1/(\varrho-1)} \tag{36}$$

since otherwise we have from a part of condition (v) of Theorem 1 $\varrho(\varrho-1)^{-1}(\nu+\varepsilon)^{-1/(\varrho-1)}\leqq c<(\varrho\sigma)^{-1/(\varrho-1)}$ from which (31) follows. $\sum\limits_{n>N(r)}$ in (34) is by $\varkappa=(\varrho-1)^{-1}$

$$\sum_{n>N(r)} \exp\left(-\left((\nu+\varepsilon)^{-1/(\varrho-1)}-\frac{c(\varrho-1)}{\varrho}\right) n^{\varrho/(\varrho-1)}+cn^{1/(\varrho-1)}+n\right)$$

with a positive factor of $n^{\varrho/(\varrho-1)}$ by (36). Choosing r large enough we conclude as in the proof of (29), that (34) is bounded by $\exp\left((c-(\nu+\varepsilon)^{-1/(\varrho-1)}+\varepsilon)c^{-\varrho}(\log r)^{\varrho}\right)\leqq$ $\leqq\exp\left(((\nu+\varepsilon)^{-1/(\varrho-1)}-\varepsilon)^{1-\varrho}\frac{(\varrho-1)^{\varrho-1}}{\varrho^{\varrho}}(\log r)^{\varrho}\right)$; here we supposed ε so small that $\varepsilon<(\nu+\varepsilon)^{-1/(\varrho-1)}$ which is possible for each $\nu\in[0,\infty)$. Collecting all estimations we find

$$\sigma\leqq\left((\nu+\varepsilon)^{-1/(\varrho-1)}-\varepsilon\right)^{1-\varrho}(\varrho-1)^{\varrho-1}\varrho^{-\varrho}$$

and therefore (31) in the case $\varkappa=(\varrho-1)^{-1}$ too.

Theorem 3 tells us that $\nu(f)$ is independent of the choice of $\{z_j\}$.

References

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