

The geometry of surfaces in parasymplectic spaces

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I. Parasymplectic space

1. Definition of the parasymplectic space

Let R^m denotes m -dimensional cartesian space and let A be an automorphism of R^m . We shall denote by this same symbol its matrix with respect to the canonical basis. Denote by $S_A(m)$ the subgroup of the group $GL(m)$ defined by:

$$(1) \quad S_A(m) = \{G \in GL(m) | G^T A G = A\}.$$

Consider a Klein space for which the basic space is R^m and its fundamental automorphisms group is the affine group reduced to the transformations $x \rightarrow Ax + a$ for which $A \in S_A(m)$. This space will be denoted by (R^m, A) .

We shall deal with some special space, namely such one, for which A is anti-symmetric and singular.

Definition 1. If A is antisymmetric and singular then we call the space (R^m, A) a parasymplectic one.

We have to find a fundamental differential invariant of second order and we investigate a differential geometry of surfaces in our space.

Theorem 1. Let $A^T \neq A$ and $A^T \neq -A$. We put $\Phi = A - A^T$. Then $S_A(m)$ is a subgroup of $S_\Phi(m)$.

PROOF. Let $G \in S_A(m)$, so we have

$$G^T A G = A$$

and then

$$G^T A^T G = A^T.$$

Hence

$$G^T \Phi G = \Phi.$$

The main consequence of this Theorem 1 is that each space (R^m, A) where $A^T \neq A$ and $A^T \neq -A$ is either a subspace of the symplectic space or of the parasymplectic space $(R^m, A - A^T)$.

2. Length of a curve in parasymplectic space

Let $t \rightarrow z(t)$ be a curve in the parasymplectic space (R^m, A) .

Definition 2. The length of the curve $t \rightarrow z(t)$ contained between points $z(t_1)$, $z(t_2)$ is the number

$$(2) \quad \int_{t_1}^{t_2} (\dot{z}(t)^T A \ddot{z}(t))^{\frac{1}{3}} dt.$$

This kind of definition of length was first proposed by R. KRASNOŹBSKI [2]. It is easy to see that if we take any new parametrization $t \rightarrow s(t)$ then we have

$$(z \circ s)' = (\dot{z} \circ s) \dot{s}$$

$$(z \circ s)'' = (\ddot{z} \circ s) \dot{s}^2 + (\dot{z} \circ s) \ddot{s}$$

$$(z \circ s)'(t)^T A (z \circ s)''(t) = \dot{s}(t)^3 \dot{z}(s(t))^T A \ddot{z}(s(t))$$

and this implies that the quantity (2) does not depend on parametrization.

Theorem 2. The length of the curve defined by (2) is an invariant with respect to automorphisms of the parasymplectic space.

PROOF. Let $t \rightarrow z(t)$ be a curve in (R^m, A) . By any affine transformation it is mapped onto the curve

$$t \rightarrow (Az + a)(t) := Az(t) + a$$

where $A \in S_A(m)$.

Simple calculations show that

$$(Az + a)'(t)^T A (Az + a)''(t) = \dot{z}(t)^T A \ddot{z}(t)$$

what ends the proof.

Now we are going to construct the geometry of the surfaces in parasymplectic space basing on the notion of introduced above length.

Contrary to the euclidean geometry this one is to be the geometry of second order due to the second order jets appearing in the definition of the length.

II. Geometry of the surface in parasymplectic space

1. Interior geometry

Let M be a k -dimensional surface in (R^m, A) . Suppose

$$u = (u^1, \dots, u^k) \rightarrow x(u) = \begin{bmatrix} x^1(u^1, \dots, u^k) \\ \vdots \\ x^m(u^1, \dots, u^k) \end{bmatrix}$$

be a local parametrization of a subdomine of M . Put

$$p_\alpha(u) = \begin{bmatrix} x_{|\alpha}^1(u) \\ \vdots \\ x_{|\alpha}^m(u) \end{bmatrix} \quad p_{\alpha\beta}(u) = \begin{bmatrix} x_{|\alpha\beta}^1(u) \\ \vdots \\ x_{|\alpha\beta}^m(u) \end{bmatrix} \quad \alpha, \beta = 1, \dots, k$$

where $x_{|\alpha}^i$ denotes the partial derivative of x^i with respect to α -coordinate. Obviously $p_\alpha(u), p_{\alpha\beta}(u)$ are the coordinates of the jet $j_u^2 x$.

We introduce the invariants

$$(3) \quad P_{\alpha\beta}(u) = p_\beta(u)^T \Lambda p_\alpha(u)$$

$$(4) \quad Q_{\alpha\beta\gamma}(u) = p_\gamma(u)^T \Lambda p_{\alpha\beta}(u) \quad \alpha, \beta, \gamma = 1, \dots, k.$$

If we change the parametrisation

$$(u^{1'}, \dots, u^{k'}) \rightarrow (u^1(u^{1'}, \dots, u^{k'}), \dots, u^k(u^{1'}, \dots, u^{k'}))$$

the functions $P_{\alpha\beta}$ and $Q_{\alpha\beta\gamma}$ transforms as follow

$$(3') \quad P_{\alpha' \beta'} = A_{\alpha'}^\alpha A_{\beta'}^\beta P_{\alpha\beta}$$

$$(4') \quad Q_{\alpha' \beta' \gamma'} = A_{\alpha'}^\alpha A_{\beta'}^\beta A_{\gamma'}^\gamma Q_{\alpha\beta\gamma} + A_{\alpha'}^\sigma A_{\beta'}^\rho A_{\gamma'}^\epsilon P_{\sigma\rho\epsilon}$$

where

$$A_{\alpha'}^\alpha = u_{|\alpha'}^\alpha, \quad A_{\alpha' \beta'}^\alpha = u_{|\alpha' \beta'}^\alpha.$$

Thus the functions $P_{\alpha\beta}$ are coordinates of some tensor field P on M .

Theorem 3. *The length of the curve $t \rightarrow x(u(t))$ on the surface M induced by (2) is given by the formula*

$$(5) \quad \int_{t_1}^{t_2} [P_{\beta\gamma}(u(t)) \dot{u}^\beta(t) \dot{u}^\gamma(t) + Q_{\alpha\beta\gamma}(u(t)) \dot{u}^\alpha(t) \dot{u}^\beta(t) \dot{u}^\gamma(t)]^{\frac{1}{3}} dt.$$

PROOF. We have

$$(6) \quad \begin{aligned} (x \circ u)^\cdot(t) &= p_\alpha(u(t)) \dot{u}^\alpha(t) \\ (x \circ u)^\cdot(t) &= p_\alpha(u(t)) \ddot{u}^\alpha(t) + p_{\alpha\beta}(u(t)) \dot{u}^\alpha(t) \dot{u}^\beta(t) \end{aligned}$$

and using (3), (4), (6) we get

$$(x \circ u)^\cdot(t)^T \Lambda (x \circ u)^\cdot(t) = P_{\beta\gamma}(u(t)) \dot{u}^\beta(t) \dot{u}^\gamma(t) + Q_{\alpha\beta\gamma}(u(t)) \dot{u}^\alpha(t) \dot{u}^\beta(t) \dot{u}^\gamma(t).$$

Comparing the last equality with (2) we obtain (5).

Definition 3. The tensor field P is said to be nonsingular iff

$$(7) \quad \det [P_{\alpha\beta}(u)] \neq 0.$$

Suppose now that M is a surface for which the tensor field P is nonsingular. The condition (7) in view of obvious

$$(8) \quad P_{\alpha\beta}(u) = -P_{\beta\alpha}(u)$$

implies that k must be an even integer.

Put

$$(9) \quad G_{\alpha\beta}^{\gamma}(u) = P^{\gamma\sigma}(u) Q_{\alpha\beta\sigma}(u)$$

where $[P^{\alpha\beta}]$ is uniquely determined from the equation

$$P_{\alpha\sigma}(u) P^{\beta\sigma}(u) = \delta_{\alpha}^{\beta}.$$

The condition (4') implies that the functions $G_{\alpha\beta}^{\gamma}$ behave under transformations analogously as Christoffel's coefficients in the Riemann space. Moreover in view of (4)

$$(9') \quad G_{\alpha\beta}^{\gamma}(u) = G_{\beta\alpha}^{\gamma}(u).$$

Definition 4. The pair $(P_{\alpha\beta}, Q_{\alpha\beta\gamma})$ will be called the metrical object of the surface M .

Denote ∇ the operator of covariant differentiation with respect $G_{\alpha\beta}^{\gamma}$.

Theorem 4.

$$(10) \quad \nabla_{\gamma} P_{\alpha\beta} = 0.$$

PROOF. Starting from (3), (4) we get

$$(11) \quad P_{\alpha\beta|\gamma}(u) = Q_{\alpha\gamma\beta}(u) - Q_{\beta\gamma\alpha}(u)$$

and then using (8), (9), (11)

$$\begin{aligned} \nabla_{\gamma} P_{\alpha\beta} &= P_{\alpha\beta|\gamma} - P_{\alpha\sigma} G_{\beta\gamma}^{\sigma} - P_{\sigma\beta} G_{\alpha\gamma}^{\sigma} = \\ &= Q_{\alpha\gamma\beta} - Q_{\beta\gamma\alpha} + Q_{\beta\gamma\alpha} - Q_{\alpha\gamma\beta} = 0. \end{aligned}$$

This is a counterpart of the Ricci Lemma in riemannian geometry.

Theorem 5.

$$(12) \quad P_{\alpha\beta|\gamma} + P_{\beta\gamma|\alpha} + P_{\gamma\alpha|\beta} = 0,$$

$$(13) \quad P^{\sigma\varrho} P_{\sigma\varrho|\alpha} = 2 \sum_{\sigma} G_{\alpha\sigma}^{\sigma}.$$

The PROOF is a consequence of (11).

Definition 5. The curve in (R^m, A) is said to be isotropic iff the length of any arc contained in it is equal zero.

Definition 6. The geodesic curve $t \rightarrow x(u(t))$ on M is any curve satisfying.

$$(14) \quad \ddot{u}^{\gamma}(t) + G_{\alpha\beta}^{\gamma}(u(t)) \dot{u}^{\alpha}(t) \dot{u}^{\beta}(t) = \lambda(t) \dot{u}^{\gamma}(t).$$

Theorem 6. Each geodesic curve is isotropic.

PROOF. Looking on (8), (9), (14) we get

$$P_{\beta\gamma} \ddot{u}^{\beta} \dot{u}^{\gamma} + Q_{\alpha\beta\gamma} \dot{u}^{\alpha} \dot{u}^{\beta} \dot{u}^{\gamma} = P_{\sigma\gamma} (\ddot{u}^{\sigma} + G_{\alpha\beta}^{\sigma} \dot{u}^{\alpha} \dot{u}^{\beta}) \dot{u}^{\gamma} = 0$$

what together with (5) ends the proof.

2. Exterior geometry of hypersurfaces

Consider the parasymplectic space (R^m, Λ) of odd dimension m with $\text{rank } \Lambda = m-1$.

In view of singularity of the matrix Λ there exists a non-zero vector $n \in R^m$ such that

$$(15) \quad \Lambda n = 0.$$

Consider the constant vector field $x \rightarrow N(x)$ on M obtained from the vector n by translation to the point $x \in M$.

Theorem 7. *If $\text{rank } \Lambda = m-1$ then the conditions*

1° the tensor field P is nonsingular,

2° the vectors $p_1(u), \dots, p_{m-1}(u), N(u)$ are linearly independent,
are equivalent.

PROOF. Consider the matrix $[p_1(u), \dots, p_{m-1}(u), N(u)]$ consisting of the column vectors $p_1(u), \dots, p_{m-1}(u), N(u)$. Notice that

$$\begin{aligned} & [p_1(u), \dots, p_{m-1}(u), N(u)]^T [-\Lambda p_1(u), \dots, -\Lambda p_{m-1}(u), N(u)] = \\ & = \begin{bmatrix} P_{\alpha\beta}(u) & * \\ 0 & \sum_{i=1}^m (n^i)^2 \end{bmatrix}. \end{aligned}$$

The nonsingularity of P implies 2°.

If the vectors $p_1(u), \dots, p_{m-1}(u), N(u)$ are linearly independent then the condition $\text{rank } \Lambda = m-1$ implies the linear independence of vectors $-\Lambda p_1(u), \dots, -\Lambda p_{m-1}(u), N(u)$. Thus $2^\circ \Rightarrow 1^\circ$.

Restrict the vector field p_α to the field of β 's parametric line. We can differentiate along this line in R^m obtaining the vector field $p_{\alpha\beta}$.

Hence by using Theorem 7 we get the following "Gauss decomposition".

$$(16) \quad p_{\alpha\beta}(u) = \Gamma_{\alpha\beta}^\sigma(u) p_\sigma(u) + B_{\alpha\beta}(u) N(u)$$

so for the vector field N "the Weingarten's formulas" are trivial

$$(17) \quad N|_\alpha(u) = 0.$$

Theorem 8.

$$\Gamma_{\alpha\beta}^\gamma(u) = G_{\alpha\beta}^\gamma(u).$$

PROOF. Composing (16) and (4) we get

$$Q_{\alpha\beta\gamma}(u) = p_\gamma(u)^T (\Gamma_{\alpha\beta}^\sigma(u) p_\sigma(u) + B_{\alpha\beta}(u) N(u)) = \Gamma_{\alpha\beta}^\sigma(u) P_{\sigma\gamma}(u),$$

what implies (15) completing the proof.

Theorem 9.

$$(19) \quad B_{\alpha\beta}(u) = \frac{\det [p_1(u) \dots p_{m-1}(u) p_{\alpha\beta}(u)]}{\det [p_1(u) \dots p_{m-1}(u) N(u)]}.$$

PROOF. From the decomposition (16) we have

$$\begin{aligned}\det [p_1(u) \dots p_{m-1}(u) p_{\alpha\beta}(u)] &= \det [p_1(u) \dots p_{m-1}(u) B_{\alpha\beta}(u) N] = \\ &= B_{\alpha\beta}(u) \det [p_1(u) \dots p_{m-1}(u) N(u)].\end{aligned}$$

Hence we obtain (19).

It is easy to see that the functions $B_{\alpha\beta}$ are coordinates of some tensor field B on M and

$$(20) \quad B_{\alpha\beta}(u) = B_{\alpha\beta}(u).$$

Definition 7. The tensor field B we will call the imbedding tensor of the surface M .

Let us look now for some counterparts of Codazzi's formulas and count down the coordinates of the curvature tensor.

Theorem 10. *The following formulas hold*

$$(21) \quad \nabla_\gamma B_{\alpha\beta} = \nabla_\beta B_{\alpha\gamma}$$

$$(22) \quad R_{\alpha\beta\gamma}^\delta = 0.$$

PROOF. From Gauss formula (16) we derive

$$p_{\alpha\beta|\gamma} = (G_{\alpha\beta|\gamma}^\sigma + G_{\alpha\beta}^\delta G_{\delta\gamma}^\sigma) p_\sigma + (B_{\alpha\beta|\gamma} - G_{\alpha\beta}^\delta B_{\delta\gamma}) N.$$

Hence

$$0 = p_{\alpha\beta|\gamma} - p_{\alpha\gamma|\beta} = R_{\gamma\alpha\beta}^\sigma p_\sigma + (\nabla_\gamma B_{\alpha\beta} - \nabla_\beta B_{\alpha\gamma}) N$$

and the proof easily follows.

Theorem 11. *Let $D \subset R^{2m}$ be an open and connected domain of three matrices fields $[P_{\alpha\beta}]$, $[Q_{\alpha\beta\gamma}]$, $[B_{\alpha\beta}]$, $\alpha, \beta, \gamma = 1, \dots, 2m$ such that*

$$P_{\alpha\beta} = -P_{\beta\alpha}, \quad \det [P_{\alpha\beta}] \neq 0$$

$$Q_{\alpha\beta\gamma} = Q_{\beta\alpha\gamma}, \quad B_{\alpha\beta} = B_{\beta\alpha}$$

$$(P) \quad P_{\alpha\beta|\gamma} + Q_{\beta\gamma\alpha} - Q_{\alpha\gamma\beta} = 0$$

$$(B) \quad B_{\alpha\beta|\gamma} - B_{\alpha\gamma|\beta} + G_{\alpha\gamma}^\sigma B_{\sigma\beta} - G_{\alpha\beta}^\sigma B_{\sigma\gamma} = 0$$

$$(R) \quad G_{\alpha\beta|\gamma}^\delta - G_{\alpha\gamma|\beta}^\delta + G_{\alpha\beta}^\sigma G_{\sigma\gamma}^\delta - G_{\alpha\gamma}^\sigma G_{\sigma\beta}^\delta = 0$$

where $G_{\alpha\beta}^\gamma = P^{\gamma\sigma} Q_{\alpha\beta\sigma}$.

Then for any point $u \in D$ there exists a neighbourhood U of u and a diffeomorphism x of U into a some parasymplectic space (R^{2m+1}, Λ) with the following properties:

The pair $(P_{\alpha\beta}, Q_{\alpha\beta\gamma})$ is the metrical object of the surface M parametrized by x ; moreover the function $B_{\alpha\beta}$ are the coordinates of the imbedding tensor of M .

PROOF. Consider the Pfaff's system

$$(a) \quad \begin{cases} dx^i = x_{|\alpha}^i du^\alpha \\ dx_{|\alpha}^i = G_{\alpha\beta}^\gamma x_{|\gamma}^i du^\beta + B_{\alpha\beta} N^i du^\beta \\ dN^i = 0. \end{cases}$$

The conditions $B_{\alpha\beta} = B_{\beta\alpha}$, $G_{\alpha\beta}^\gamma = G_{\beta\alpha}^\gamma$, (B) i (R) implies the integrability of our system.

Let $u_0 \in D$ and let the vectors $a_1, \dots, a_{2m}$ in R^{2m+1} be linearly independent. We define the vectors $a_{\alpha\beta}$ by the formula

$$(1^\circ) \quad A_\alpha = AX^{-1}Y_\alpha$$

where

$$A = [a_1 \dots a_{2m}] \quad (\text{rectangle matrix})$$

$$X = [P_{\alpha\beta}(u_0)] \quad (\text{nonsingular matrix})$$

$$Y_\alpha = [Q_{\alpha\beta\gamma}(u_0)]_{\beta,\gamma} \quad (\text{a sequence of square matrices})$$

$$A_\alpha = [a_{\alpha 1} \dots a_{\alpha 2m}] \quad (\text{a sequence of rectangle matrices}).$$

We find some antisymmetric matrix A of degree $2m+1$ such that $\text{rank } A = 2m$, $P_{\alpha\beta}(u_0) = a_{\beta}^T A a_\alpha$, $Q_{\alpha\beta\gamma}(u_0) = a_\gamma^T A a_{\alpha\beta}$, i.e.

$$(2^\circ) \quad \text{rank } A = 2m$$

$$(3^\circ) \quad X = A^T A A$$

$$(4^\circ) \quad Y_\alpha = A^T A A_\alpha.$$

We note that (1°) and (3°) implies (4°) . Since $\text{rank } A = \text{rank } X = 2m$ so from (3°) we obtain (2°) . Thus (3°) defined some matrix A . Let $\hat{N}$ be defined by the condition $A\hat{N} = 0$.

Let x be a solution of the system (a) with the initial condition

$$x(u_0) = a, \quad x_{|\alpha}(u_0) = a_\alpha, \quad N(u_0) = \hat{N}.$$

Put

$$\hat{P}_{\alpha\beta}(u) = x_{|\beta}(u)^T A x_{|\alpha}(u)$$

$$\hat{Q}_{\alpha\beta\gamma}(u) = x_{|\gamma}(u)^T A x_{|\alpha\beta}(u).$$

Of course, we have

$$(b) \quad \begin{cases} \hat{P}_{\alpha\beta}(u_0) = P_{\alpha\beta}(u_0) \\ \hat{Q}_{\alpha\beta\gamma}(u_0) = Q_{\alpha\beta\gamma}(u_0). \end{cases}$$

Now we put

$$\hat{G}_{\alpha\beta}^\gamma(u) = \hat{P}^{\gamma\sigma}(u) Q_{\alpha\beta\sigma}(u),$$

the functions $\hat{G}_{\alpha\beta}^\gamma$ are defined in some neighbourhood of the point u_0 .

We find the differential equations for the functions $\hat{P}_{\alpha\beta}$ and $\hat{Q}_{\alpha\beta\gamma}$.

$$\hat{P}_{\alpha\beta|\gamma} = x_{|\beta\gamma}^T A x_{|\alpha} + x_{|\beta}^T A x_{|\alpha\gamma} = \hat{G}_{\beta\gamma}^\sigma \hat{P}_{\alpha\sigma} + \hat{G}_{\alpha\gamma}^\sigma \hat{P}_{\sigma\beta} = \hat{Q}_{\alpha\gamma\beta} - \hat{Q}_{\beta\gamma\alpha}$$

$$\hat{Q}_{\alpha\beta\gamma|\delta} = x_{|\gamma\delta}^T A x_{|\alpha\beta} + x_{|\gamma}^T A x_{|\alpha\beta\delta} = \hat{G}_{\gamma\delta}^\sigma \hat{G}_{\alpha\beta}^\rho \hat{P}_{\rho\sigma} + (\hat{G}_{\alpha\beta|\delta}^\sigma + \hat{G}_{\alpha\delta}^\rho \hat{G}_{\rho\beta}^\sigma) \hat{P}_{\sigma\gamma}.$$

Thus we obtain the following system of differential equations

$$(c) \quad \begin{cases} \hat{P}_{\alpha\beta|\gamma} = \hat{Q}_{\alpha\gamma\beta} - \hat{Q}_{\beta\gamma\alpha} \\ \hat{Q}_{\alpha\beta\gamma|\delta} = \hat{G}_{\gamma\delta}^\sigma \hat{G}_{\alpha\beta}^\rho \hat{P}_{\rho\sigma} + (\hat{G}_{\alpha\beta|\delta}^\sigma + \hat{G}_{\alpha\delta}^\rho \hat{G}_{\rho\beta}^\sigma) \hat{P}_{\sigma\gamma}. \end{cases}$$

We show that

$$(Q) \quad Q_{\alpha\beta\gamma|\lambda} = P_{\sigma\gamma}(G_{\alpha\beta|\lambda}^{\sigma} + G_{\sigma\gamma}^{\sigma} G_{\alpha\beta}) + G_{\gamma\lambda}^{\sigma} Q_{\alpha\beta\sigma}.$$

The condition

$$P_{\alpha\varphi} P^{\beta\varphi} = \delta_{\alpha}^{\beta}$$

and (P) implies that

$$P^{\beta\lambda}_{|\gamma} = -P^{\sigma\lambda} P_{\sigma\varphi|\gamma} P^{\beta\varphi} = -P^{\sigma\lambda} (Q_{\sigma\gamma\varphi} - Q_{\varphi\gamma\sigma}) P^{\beta\varphi} = -P^{\sigma\lambda} G_{\sigma\gamma}^{\beta} - P^{\beta\varphi} G_{\varphi\gamma}^{\lambda}.$$

Hence

$$\begin{aligned} G_{\alpha\beta|\gamma}^{\delta} &= P^{\delta\varphi}_{|\lambda} Q_{\alpha\beta\varphi} + P^{\delta\varphi} Q_{\alpha\beta\varphi|\lambda} = (-G_{\sigma\lambda}^{\delta} P^{\sigma\varphi} - G_{\sigma\lambda}^{\varphi} P^{\delta\sigma}) Q_{\alpha\beta\varphi} + P^{\delta\varphi} Q_{\alpha\beta\varphi|\lambda} = \\ &= -G_{\sigma\lambda}^{\delta} G_{\alpha\beta}^{\sigma} - G_{\sigma\lambda}^{\varphi} P^{\delta\sigma} Q_{\alpha\beta\varphi} + P^{\delta\varphi} Q_{\alpha\beta\varphi|\lambda}. \end{aligned}$$

Multiplying by $P_{\delta\gamma}$ we obtain (Q). The functions $P_{\alpha\beta}$ and $Q_{\alpha\beta\gamma}$ with respect to (P) and (Q) satisfying the system (c). The initial conditions (b) implies

$$\hat{P}_{\alpha\beta} \equiv P_{\alpha\beta} \quad \text{and} \quad \hat{Q}_{\alpha\beta\gamma} \equiv Q_{\alpha\beta\gamma}.$$

Moreover B are coordinates of the imbedding tensor of M .

3. Exterior geometry of surfaces in parasymplectic space (R^3, Λ)

S. WATANABE [4, 5] investigated the geometry of surfaces in the affine space K^3 . Our aim in this chapter is to prove the following

Theorem 12.

$$K^3 = \left(R^3, \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \right).$$

PROOF. Before starting the proof let us recall the definition of K^3 . To do so it is enough to show the centroaffine group which for K^3 is

$$(24) \quad \{A \in GL(3) \mid \sum_{i=1}^3 a_i^j = \det A \text{ for } j=1, 2, 3\}.$$

It is easy to see that if

$$(25) \quad a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

then this group can be defined also as

$$(24') \quad \{A \in GL(3) \mid Aa = \det A \cdot a\}$$

as it was done in [1].

Suppose

$$(26) \quad \Lambda = \begin{bmatrix} 0 & \lambda & \mu \\ -\lambda & 0 & \kappa \\ -\mu & -\kappa & 0 \end{bmatrix} \quad n = \begin{bmatrix} \kappa \\ -\mu \\ \lambda \end{bmatrix}.$$

Theorem 13. For any $G \in GL(3)$ the conditions

- (a) $G^T \Lambda G = \Lambda$
 (b) $Gn = \det G \cdot n$

are equivalent.

PROOF. Suppose

$$G = \begin{bmatrix} a & b & c \\ u & v & w \\ p & q & r \end{bmatrix} \in S_A(3).$$

then the conditions $G^T \Lambda G = \Lambda$ leads to the equations

$$\begin{aligned} (-\lambda u - \mu p)b + (\lambda a - \kappa p)v + (\mu a + \kappa u)q &= \lambda \\ (-\lambda v - \mu q)c + (\lambda a - \kappa p)w + (\mu a + \kappa u)r &= \mu \\ (-\lambda v - \mu q)c + (\lambda b - \kappa q)w + (\mu b + \kappa v)r &= \kappa \end{aligned}$$

or in an equivalent way to

$$\begin{aligned} \kappa(uq - pv) &= \lambda + \lambda ub + \mu bp - \lambda av - \mu aq \\ \kappa(ur - pw) &= \mu + \lambda uc + \mu bc - \lambda aw - \mu ar \\ \kappa(vr - qw) &= \kappa + \lambda vc + \mu qc - \lambda bw - \mu br \end{aligned}$$

and counting down $\kappa \det G$, $-\mu \det G$ and $\lambda \det G$ we get

$$\begin{aligned} \kappa \det G &= a\kappa(vr - wq) - b\kappa(ur - wp) + c\kappa(uq - vp) = a\kappa - b\mu + c\lambda, \\ -\mu \det G &= u\kappa - v\mu + w\lambda, \\ \lambda \det G &= p\kappa - q\mu + r\lambda. \end{aligned}$$

On the other hand

$$Gn = \begin{bmatrix} a & b & c \\ u & v & w \\ p & q & r \end{bmatrix} \begin{bmatrix} \kappa \\ -\mu \\ \lambda \end{bmatrix} = \begin{bmatrix} a\kappa - b\mu + c\lambda \\ u\kappa - v\mu + w\lambda \\ p\kappa - q\mu + r\lambda \end{bmatrix}$$

so $Gn = \det G \cdot n$ implying (a) $\Rightarrow$ (b).

New the condition (b) may be written as

$$n = \det G \cdot G^{-1}n,$$

hence

$$\varkappa = (vr - qw)\varkappa + (br - cq)\mu + (bw - vc)\lambda,$$

$$\mu = (ur - wp)\varkappa + (ar - cp)\mu + (aw - cu)\lambda,$$

$$\lambda = (uq - vp)\varkappa + (aq - bp)\mu + (av - ub)\lambda$$

which is exactly what we need to show that (b) $\Rightarrow$ (a).

Let pass now to the proof of Theorem 12.

PROOF. Putting in Theorem 13 $n=a$ and using the definition (24') we get the conclusion of Theorem 12 immediately.

4. Exterior geometry of k -dimensional surfaces in $(R^m, \cdot)$

Our aim now is to investigate k -dimensional surfaces M which are subject to

$$(*) \quad \det [P_{\alpha\beta}(u)] \neq 0$$

$$\left(\begin{array}{c} * \\ * \end{array} \right) \quad k = \text{rank } A.$$

These two conditions imply that $\dim \text{Ker } A = m - k$ where k is an even integer.

Let the vectors $n_1, \dots, n_{m-k}$ constitute the bases in $\text{Ker } A$. Let us define a constant vector field N_i on M translating vectors n_i to the point $x(u) \in M$.

Theorem 14. *If $\text{rank } A = k$ then the conditions:*

(a) *the tensor field P is nonsingular,*

(b) *the vectors $p_1(u), \dots, p_k(u), N_1(u), \dots, N_{m-k}(u)$ are linearly independent are equivalent.*

PROOF. We have

$$\begin{aligned} & [p_1(u) \dots p_k(u) N_1(u) \dots N_{m-k}(u)]^T [-Ap_1(u) \dots -Ap_k(u) N_1(u) \dots N_{m-k}(u)] = \\ & = \left[\begin{array}{c|c} [P_{\alpha\beta}(u)]_{\alpha, \beta=1, \dots, k} & * \\ \hline 0 & \left[\sum_{r=1}^m n_i^r n_j^r \right]_{i, j=1, \dots, m-k} \end{array} \right]. \end{aligned}$$

In view of the independence of $N_1, \dots, N_{m-k}$ and the generalized Lagrange's theorem on determinants, the matrix

$$\left[\sum_{r=1}^m n_i^r n_j^r \right]_{i, j=1, \dots, m-k}$$

is nonsingular so (a) $\Rightarrow$ (b).

If the vectors $p_1(u), \dots, p_k(u), N_1(u), \dots, N_{m-k}(u)$ are linearly independent then the condition $\text{rank } A = k$ imply that the vectors $-Ap_1(u), \dots, -Ap_k(u), N_1(u), \dots, N_{m-k}(u)$ are also linearly independent. Thus (b) $\Rightarrow$ (a).

The consequence of this is the following formula

$$p_{\alpha\beta}(u) = \tilde{F}_{\alpha\beta}^{\sigma}(u) p_{\sigma}(u) + B_{\alpha\beta}^i(u) N_i(u).$$

Analogously as in chapter 2 we can show

Theorem 15.

$$(27) \quad \tilde{F}_{\alpha\beta}^{\gamma} = G_{\alpha\beta}^{\gamma}$$

$$(28) \quad R_{\alpha\beta\gamma}^{\delta} = 0$$

$$(29) \quad \nabla_{\gamma} B_{\alpha\beta}^i = \nabla_{\beta} B_{\alpha\gamma}^i$$

$$(30) \quad B_{\alpha\beta}^i = \frac{\det[p_1, \dots, p_k, N_1 \dots N_{i-1}, p_{\alpha\beta}, N_{i+1} \dots N_{m-k}]}{\det[p_1 \dots p_k, N_1 \dots N_{m-k}]}.$$

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