

Radicals of the semiring of abelian groups

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i. The isomorphism classes of abelian groups under the operations direct sum, and tensor product, form a semiring which is a class. In section *iii*, several radicals of this semiring will be defined. Concerning computed semirings in general, which are employed in *iii*, will be introduced in section *ii*.

A group will always be meant to be an abelian group.

ii. Let S be a semiring with $0 \in S$ satisfying $a+0=a$, and $a \cdot 0=0 \cdot a=0$ for all $a \in S$.

Definition 2.1. A subset $I \subseteq S$ is said to be an ideal in S if:

- 1) $a, b \in I \Rightarrow a+b \in I$,
- 2) $a \in I, x \in S \Rightarrow ax \in I$, and $xa \in I$, and
- 3) $0 \in I$.

If in addition for all $a \in I$, and $x \in S$, $a+ax \in I$ implies that $x \in I$, then I is said to be a k -ideal.

Undefined terms used here are meant to have the same meaning as their counterparts in ring theory.

Definition 2.2. An ideal I in S is said to be modular if there is an $e \in S$ such that for each $x \in S$, there exists an $i(x) \in I$ with $ex=x+i(x)$.

Clearly, every ideal in a semiring with identity is modular.

Notation 2.3. The sum of all nilpotent ideals in S will be denoted $N_0(S)$. For every ordinal τ , inductively define $N_{\tau+1}(S)$ to be the ideal in S such that $N_{\tau+1}(S)/N_\tau(S)$ is the sum of all the nilpotent ideals in $S/N_\tau(S)$. For τ a limit ordinal, define $N_\tau(S) = \bigcup_{\gamma < \tau} N_\gamma(S)$.

For a definition of S/I , I an ideal in S , see [3, p. 164].

Definition 2.4. The upper nil radical of S , $UN(S)$, is the sum of all the nil ideals in S .

Definition 2.5. The Levitzki radical of S , $L(S)$, is the sum of all the locally nilpotent ideals in S [1, p. 262; 8, p. 91].

Definition 2.6. The prime radical of S , $P(S) = \bigcap \{P \mid P \text{ is a prime ideal in } S\}$.

Definition 2.7. The Jacobson radical of S , $J(S) = \bigcap \{M \mid M \text{ is a modular maximal ideal in } S\}$.

Lemma 2.8. Let S be a semiring with identity, and let K be the set of non-units in S . S has a unique maximal ideal M iff K is an ideal, in which case $M = K$.

PROOF. As in ring theory [2, theorem 3-3].

iii. Let Ab be the class of abelian groups. For every $A \in Ab$, let $[A]$ be the isomorphism class of A . Write $[Ab] = \{[A] \mid A \in Ab\}$. For $A, B \in Ab$, define $[A] + [B] = [A + B]$, and $[A] \cdot [B] = [A \otimes B]$. These two operations clearly make $[Ab]$ a commutative semiring, with identity $[Z]$, Z the group of integers.

Notation 3.1. Let A be a torsion free abelian group. The rank of A will be denoted by $r(A)$. If $r(A) = 1$, then $T(A)$ will signify the type of A [4, vol. 2, p. 109].

Notation 3.2. For all $A \in Ab$, the torsion part of A will be denoted by A_t . For every prime p , A_p will signify the p -component of A_t .

Notation 3.3. $T = \{[A] \mid A \text{ is a torsion group}\}$.

Notation 3.4. $DT = \{[A] \mid A \text{ is a divisible torsion group}\}$.

Notation 3.5. For every prime p , write $D_p = \{[A] \mid A_t \text{ is a } p\text{-divisible group}\}$.

Notation 3.6. $M = \{[A] \mid A \not\cong Z\}$.

Lemma 3.7. Let P be a prime ideal in $[Ab]$, and let A be a torsion group. If $[A] \notin P$, then $[D] \in P$ for every divisible group D .

PROOF. $D \otimes A = 0$ for every divisible group D .

Similarly we have:

Lemma 3.8. Let P be a prime ideal in $[Ab]$, and let D be a divisible group. If $[D] \notin P$, then $[T] \in P$ for every torsion group A .

Corollary 3.9. $DT \subseteq P$ for every prime ideal P in $[Ab]$.

Lemma 3.10. T is a prime k -ideal in $[Ab]$.

PROOF. T is clearly a k -ideal in $[Ab]$.

Let $A, B \in Ab$, and suppose that neither A nor B is a torsion group. Let A' and B' be nonzero, torsion free subgroups of A and B respectively. $A' \otimes B'$ is a nonzero torsion free group, and is isomorphic to a subgroup of $A \otimes B$ [5, theorem 2]. Hence $[A] \cdot [B] \notin T$.

Lemma 3.11. For every prime number p , D_p is a prime k -ideal in $[Ab]$.

PROOF. D_p is clearly a k -ideal in $[Ab]$. Let $A, B \in Ab$, and suppose that $(A \otimes B)_t$ is p -divisible, $A_t \otimes B_t$ is isomorphic to a direct summand of $(A \otimes B)_t$ [4, theorem 61.5], and hence must be p -divisible. $A_p \otimes B_p$ is a direct summand of $A_t \otimes B_t$, and is therefore p -divisible. However, $A_p \otimes B_p$ is a direct sum of cyclic p -groups, [4, theorem 61.3]. Therefore $A_p \otimes B_p = 0$. Let P_A and P_B be p -basic subgroups of A_p

and B_p respectively. $0 = A_p \otimes B_p \cong P_A \otimes P_B$ [4, theorem 61.1]. Since P_A and P_B are both direct sums of cyclic p -groups, $P_A \otimes P_B = 0$ implies that either $P_A = 0$, or $P_B = 0$. This in turn yields that either A_t or B_t is p -divisible.

Theorem 3.12. $P([Ab]) = DT$.

PROOF. $DT \subseteq P([Ab])$ by corollary 3.9. Lemmas 3.10 and 3.11 yield that

$$P([Ab]) \subseteq T \cap \left\{ \bigcap_{p \text{ a prime}} D_p \right\} = DT.$$

Theorem 3.13. For every ordinal τ ,

$$N_\tau([Ab]) = L([Ab]) = UN([Ab]) = DT.$$

PROOF. Clearly $N_0([Ab]) \subseteq L([Ab]) \subseteq UN([Ab])$, and $N_0([Ab]) \subseteq N_\tau([Ab]) \subseteq UN([Ab])$ for every ordinal τ . It therefore suffices to show that $UN([Ab]) = DT$, and that $DT \subseteq N_0([Ab])$.

1) $UN([Ab]) = DT$: Let I be a nil ideal in $[Ab]$, and let $[A] \in [Ab]$. There exists a positive integer n such that $\tilde{A} = A \otimes \dots \otimes A = 0$.

$n\text{-times}$

Let A' be a torsion free subgroup of A . Put $\tilde{A}' = \underbrace{A' \otimes \dots \otimes A'}_{n\text{-times}}$.

The sequence $0 \rightarrow \tilde{A}' \rightarrow \tilde{A}$ is exact. If $A' \neq 0$, then $\tilde{A}' \neq 0$. Therefore A must be a torsion group. If A is not divisible, then A possesses a cyclic, direct summand $B \neq 0$. Write $\tilde{B} = \underbrace{B \otimes \dots \otimes B}_{n\text{-times}}$. Clearly $\tilde{B} \neq 0$, and $\tilde{B}$ is a direct summand of $\tilde{A}$.

A contradiction. Hence A is divisible, and $I \subseteq DT$. This yields that $UN([Ab]) \subseteq DT$.

Since $(DT)^2 = 0$, clearly $DT \subseteq UN([Ab])$.

2) $DT \subseteq N_0([Ab])$: This follows immediately from the fact that $(DT)^2 = 0$.

Theorem 3.14 M is the unique maximal ideal in $[Ab]$.

PROOF. By lemma 2.8 it suffices to show that every $[A] \in M$ is a non-unit in $[Ab]$, or that for $A, B \in Ab$, $A \otimes B \cong Z$ implies that $A \cong B \cong Z$.

Suppose that $A \otimes B \cong Z$.

1) If either A or B is a torsion group, then so is $A \otimes B$.

2) If A and B are both torsion free, then $r(A) \cdot r(B) = r(A \otimes B) = 1$. Therefore $r(A) = r(B) = 1$. However, $(0, \dots, 0, \dots) = T(A \otimes B) \cong T(A) + T(B)$. Therefore $T(A) = T(B) = (0, \dots, 0, \dots)$, and $A \cong B \cong Z$.

3) Suppose that A is a mixed group. $Z \cong A \otimes B \cong (A \otimes B)_t \cong A/A_t \otimes B/B_t$ [4, theorem 61.5]. By 2) we have that $A/A_t \cong B/B_t \cong Z$. Therefore $A_t \cong A_t \otimes B/B_t$. However, $A_t \otimes B/B_t$ is a direct summand of $(A \otimes B)_t$ [4, theorem 61.5]. A contradiction.

Corollary 3.15. $I([Ab]) = M$.

iv. Results concerning the structure of the tensor product of groups, which may be found in [4, Section 61] were employed in section iii. Several of these results were shown to remain true for the tensor product of modules over a Dedekind ring in [5, and 6]. The generalization of a few more of the results in [4, section 61] to modules over a Dedekind ring, would yield that all the results obtained here remain true for the semiring of isomorphism classes of modules over a Dedekind ring.

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