On certain graphs composed of algebraic integers of a number field and their applications I.

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In [8] we solved, for a large class of irreducible polynomials $g(x)$, an old problem of A. BRAUER, R. BRAUER and H. HOPF [4] concerning the reducibility of polynomials of the form $g(f(x))$ where $g, f \in \mathbf{Z}[x]$. In [8], [9], [10], [12], [13] and [21] we considerably generalized and improved this result. To formulate and prove our irreducibility theorems we associated to every pair f, g in question a certain graph with vertex set consisting of the roots of f. As it turned out, the question of reducibility of $g(f(x))$ over $\mathbf{Q}$ is closely connected with the structure of the corresponding graph. Namely, we proved [8] that if this graph has a connected component with s vertices then the number of irreducible factors of $g(f(x))$ is not greater than $\left[\frac{\deg f}{s}\right]$ and this estimate is in general best possible (see [9]). To generalize our earlier irreducibility theorems, in [9] and [10] we studied these graphs in the following more general form. Let K be an algebraic number field with ring of integers $\mathbf{Z}_K$. Let $\mathscr{A} = \{\alpha_1, \ldots, \alpha_m\}$ be a finite subset of $\mathbf{Z}_K$ and, for given $N \geqq 1$, define the graph $\mathscr{G}$ with vertex set $\mathscr{A}$ so that $[\alpha_i, \alpha_j]$ is an edge of $\mathscr{G}$ if and only if

$$|N_{K/Q}(\alpha_i - \alpha_j)| > N.$$

By solving some diophantine problems and using certain deep effective results on diophantine equations we proved in [9] (see also [10]) that if m is sufficiently large, then $\mathscr{G}$ has a connected component with at least $\left[\frac{m+1}{2}\right]$ vertices. Further, we conjectured ([9], p. 311) that here the lower bound $\left[\frac{m+1}{2}\right]$ can be further improved (i.e. that for g considered in [9] $g(f(x))$ is always irreducible over $\mathbf{Q}$ if f has distinct real roots and $\deg f$ is sufficiently large).

Some of our recent investigations showed that some further properties of the graphs defined above (e.g. the completeness of $\mathscr{G}$ or $\overline{\mathscr{G}}$, the existence of large complete subgraphs in $\overline{\mathscr{G}}$, the connectivity and the maximum degree of $\overline{\mathscr{G}}^T$) play an important role in the resolution of several other number-theoretic problems as well. We discovered many specific properties of these graphs (and their p-adic analogues) and with the aid of these properties (but without using graph terminol-

ogy [1]) we obtained general and effective results among others on polynomials and algebraic integers with given discriminant [11], [12], [13], [14], [15], [16], on algebraic integers with given index [13], [15], [16], on the arithmetical structure of discriminants and of indices of algebraic integers [16], on power bases and on the number of generators of the ring of integers of a number field [13], [15], on diophantine equations of discriminant type [13], [14], on discriminant form, index form and norm form equations in an arbitrary number of unknowns and on their generalizations [13], [14], [29], [18], [20] and on the prime factors of decomposable forms [29], [18]. We mention that these results enabled us to solve, in effective and more general forms, certain problems of DELONE and FADDEEV [7], NAGELL [36] and NARKIEWICZ [38], respectively. Further, our results generalized a number of earlier theorems of the above mentioned domains, among others some well-known theorems of NAGELL [34], [35], [36], [37], MAHLER [32], BAKER [1], COATES [5], [6], SPRINDŽUK [44], [45], KOTOV [30], SPRINDŽUK and KOTOV [46] and SHOREY, VAN DER POORTEN, TIJDEMAN and SCHINZEL [43], respectively.

In the present paper we generalize the above graphs $\mathcal{G}$ to the $\mathfrak{p}$-adic case. As a considerable generalization and improvement of our earlier results on the graphs $\mathcal{G}$ (obtained with or without the use of graph terminology) we give such a description of these graphs which proves very useful from the point of view of applications. Our main results (Theorems 1 and 2) have many various number-theoretic applications. From our Theorems 1 and 2 it follows e.g. an affirmative answer to our conjecture mentioned above and the resolution of the Brauer—Hopf problem in a more general form (see [21]). Moreover, all our results quoted above can be obtained (in more general and improved forms) as consequences of our main theorems (see e.g. [21], [22], [23]). Further new applications are given in our papers [25], [26] and [27].

In order to prove our main results in the present form we had to resolve some difficult diophantine problems (see e.g. [19] or Lemma 5 in the present paper). To solve these problems we used, among other things, two recent explicit estimates [41], [42] for linear forms in the logarithms of algebraic numbers. We remark that the earlier weaker estimates concerning the linear forms in logarithms do not enable one to prove our Theorem 2.

2. Notations, definitions and preliminary remarks

Let again K be an algebraic number field of degree $k \geqq 1$ with ring of integers $\mathbf{Z}_K$. Denote by R_K, h_K and D_K the regulator, the class number and the discriminant of K. Let $R_K^* = \max(R_K, e)$ and let r be the number of fundamental units in K. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ denote $s(\geqq 0)$ distinct prime ideals of K lying above rational primes not exceeding $P(\geqq 2)$. Let S be a finite set of normalized valuations $|\ldots|_v$ of K containing the set S_∞ of the archimedean valuations and suppose that the valuations of $S \setminus S_\infty$ belong to the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. We denote by U_s the group of S-units in K. If $S = S_\infty$ (i.e. if $s=0$) U_s obviously coincides with the group U_K of units of K. Denote by $\mathcal{N}$ the set of those elements $\beta \in \mathbf{Z}_K$ for which $0 < |N_{K/Q}(\beta)| \leqq N$ where

[1]) In our paper ([12], p. 129) we remarked that in the proofs of Theorems 2 and 3 of [12] we used our graph method.

$N \geqq 1$. We suppose throughout this paper that K, S and $\mathcal{N}$ are fixed (except possibly in the examples given after our theorems).

If $\mathcal{A}=\{\alpha_1, \ldots, \alpha_m\}$ is a set of algebraic integers in K, let $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ denote the *graph* whose vertex set is $\mathcal{A}$ and whose edges are the pairs $[\alpha_i, \alpha_j]$ [2]) satisfying

$$\alpha_i - \alpha_j \notin \mathcal{N}(U_s \cap \mathbf{Z}_K).$$

As we mentioned in the introduction, in the special case $s=0$ the above graphs were introduced in our papers [8] and [9] (see also [10], [12], [13]).

The graphs $\mathcal{G}$ defined above have many specific properties. It is easy to see that if $m=|\mathcal{G}|$ (i.e. the order of $\mathcal{G}$) is large then the number e of edges of $\mathcal{G}$ is also large. To formulate this statement in a quantitative form we denote by L the smallest integer for which $L=N(\mathfrak{A})>N$ with an integral ideal $\mathfrak{A}$ in K relatively prime to $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$.

Proposition. *With the above notations we have*

$$e \geqq \frac{m(m-L)}{2L}. \tag{1}$$

To prove this, let $\alpha(\mathcal{G})$ denote the maximum of the orders of the complete subgraphs of the complementary graph $\overline{\mathcal{G}}$ of $\mathcal{G}$. [3]) Since $\alpha_i - \alpha_j \in \mathfrak{A}$ implies $L=N(\mathfrak{A}) \mid N_{K/Q}(\alpha_i - \alpha_j)$, hence $\alpha(\mathcal{G}) \leqq L$. On the other hand, by a theorem of TURÁN [48] (see also [49]) $\frac{m^2}{2e+m} \leqq \alpha(\mathcal{G})$ and so (1) is proved.

For $s=0$, $L \leqq (N^{1/k}+1)^k$. Thus, in the special case $s=e=0$, $N=1$, (1) gives $m \leqq L \leqq 2^k$ (where L denotes now the smallest positive integer which is norm of a prime ideal of K). This latter estimate was proved (without using graphs and graph terminology) by LENSTRA [31] in connection with his results on euclidean number fields.

In some number-theoretic investigations (see e.g. WASÉN [50] and LENSTRA [31]) it is important to have good bounds (in our graph terminology) for $\alpha(\mathcal{G})$. As we showed above, L is a universal upper bound for $\alpha(\mathcal{G})$. On the other hand, there exist algebraic number fields K of arbitrarily large degree k and graphs $\mathcal{G}$ defined in K with the property $\alpha(\mathcal{G}) > \frac{\log k}{\log \log k}$. Indeed, if $\varepsilon_1=0$, ε_i is a root of the polynomial $f_i(x)=(x-\varepsilon_1)\ldots(x-\varepsilon_{i-1})-1$ for $i=2, \ldots, m$, $\mathcal{E}=\{\varepsilon_1, \ldots, \varepsilon_m\}$ and m is large then $\mathcal{G}=\mathcal{G}(\mathcal{E}, S, \mathcal{N})$ possesses the required property in $K=\mathbf{Q}(\varepsilon_3, \ldots, \varepsilon_m)$ for every S and $\mathcal{N}$. This example has a consequence [27] in connection with a problem of Browkin and Schinzel.

3. The main results

Keeping the notations of Section 2, we describe now the structure of the graphs $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ of given order. To state our Theorem 1 we need some further definitions.

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[2]) We use the notations and terminology of [3].

[3]) In other words $\alpha(\mathcal{G})$ denotes the stability number of $\mathcal{G}$ (see e.g. [3]).

Let $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ be a graph as above with at least one edge. Consider the hypergraph [4]) whose vertices are the edges of $\mathcal{G}$, and whose edges are the triples of edges of $\mathcal{G}$ that form a triangle. This hypergraph is called the *triangle hypergraph* of $\mathcal{G}$ and is denoted by $\mathcal{G}^T$ (see e.g. [3], p. 440). In $\mathcal{G}$ a connected subgraph $\mathcal{H}$ of order $\geqq 2$ will be called *triangular connected component* of $\mathcal{G}$ if $\mathcal{H}^T$ is a connected component of $\mathcal{G}^T$. Evidently any subgraph of $\mathcal{G}$ which has no isolated vertices can be covered [5]) with edge disjoint triangular connected components of $\mathcal{G}$ (i.e. with connected components of $\mathcal{G}^T$).

Let $m\geqq 2$ be a fixed integer, and let $\mathcal{F}=\mathcal{F}(\mathcal{B}, S, \mathcal{N})$ be a graph with vertex set $\mathcal{B}=\{\beta_1, \ldots, \beta_n\}$ from $\mathbf{Z}_K$, $2\leqq n\leqq m$. We say that $\mathcal{F}$ and $\overline{\mathcal{F}}$ have the *property* (P_m) if there exist $\varepsilon\in U_S\cap\mathbf{Z}_K$ and $\beta_{ij}\in\mathbf{Z}_K$ such that $\beta_i-\beta_j=\varepsilon\beta_{ij}$ for all distinct i, j and

$$\max_{i,j}|\overline{\beta_{ij}}| < \exp\{10k^3m^2c_1^2(s+1)^2P^{2k}(\log 2P)^4\cdot \tag{2}$$

$$\cdot[s(R_K+h_K\log P)\log(1+sR_Kh_K)+1]^2R_K^2((s+1)R_K+sh_K\log P)^2(R_K+h_K\log P)^{2s}\cdot$$

$$\cdot[\log R_K^*+s\log(1+R_Kh_K\log P)]^4[R_K+sh_K\log P+\log N]\} = C_1$$

with $c_1=(25(r+s+3)k)^{20r+13s+2rs+36}$.

With the above notations and definitions we have the following *)

Theorem 1. *Let $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ be a graph with $m\geqq 3$ vertices. Then at least one of the following cases holds:*

(i) *$\mathcal{G}$ is connected and, if $\mathcal{G}$ is not complete, all triangular connected components of $\overline{\mathcal{G}}$ have the property (P_m),*

(ii) *$\mathcal{G}$ consists of two connected components $\mathcal{G}_1, \mathcal{G}_2$, $\overline{\mathcal{G}}_1$ is not connected, $|\mathcal{G}_2|=1$ and $\mathcal{H}\cup\mathcal{G}_2$ has the property (P_m) for each connected component $\mathcal{H}$ of $\overline{\mathcal{G}}_1$,*

(iii) *$\mathcal{G}$ consists of two connected components $\mathcal{G}_1, \mathcal{G}_2$, both are complete and $|\mathcal{G}_1|, |\mathcal{G}_2|\geqq 2$,*

(iv) *$\mathcal{G}$ has the property (P_m).*

As will be apparent from the proof, each triangular connected component of $\overline{\mathcal{G}}$ has the property (P_m) with mC_4 in place of C_1 (where mC_4 is less than C_1; see Section 4).

It is easy to see that up to the obvious multiplications by elements of $U_S\cap\mathbf{Z}_K$ and the translations by algebraic integers of K the number of graphs $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ having the property (P_m) is finite and all these graphs can be effectively determined. Except for these graphs, by our Theorem 1 all graphs $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ of order $m\geqq 3$ possess the property (i), (ii) or (iii).

Theorems 1 and 2 together with (14) show that if in Theorem 1 (iii) holds then $\max(|\mathcal{G}_1|, |\mathcal{G}_2|)\leqq\psi^2(N)(r+4s+1)$ with the $\psi(N)$ defined in Section 4 or $\mathcal{G}$ has the property $(P_{[2C_2]})$ with N replaced by N^{**} of (13) in C_1.

It follows from our Proposition that in the case (iv) the number of connected components of $\mathcal{G}$ is at most L. Thus, if in particular $L=2$, by Theorem 1 every graph $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ has at most two connected components.

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[4]) If $X=\{x_1, \ldots, x_n\}$ is a finite set and $\mathcal{E}=(E_i|i\in I)$ is a family of subsets of X such that $E_i\neq\emptyset$ $(i\in I)$, then $H=(X, \mathcal{E})$ is called a *hypergraph*. x_j are the vertices and E_i are the edges of H.

[5]) In other words every edge of the subgraph in question is in at least one triangular connected component of $\mathcal{G}$.

*) *Added in proof.* In my paper "On certain graphs composed of elements of an integral domain and their applications" (to appear), I extended some results of the present paper to integral domains.

It is easy to construct graphs with the property (P_m). We shall now show that in Theorem 1 each of the cases (i), (ii) and (iii) really occurs. Further, in the following examples we can choose the vertex sets so that the constructed graphs $\mathcal{G}$ have not the property (P_m).

Examples for the case (i). 1) Clearly there are infinitely many complete graphs $\mathcal{G}$ of order m.

2) Let now $\mathcal{A}=\{\alpha_1, \ldots, \alpha_m\}$ where $\alpha_1=0$ and $\alpha_2, \ldots, \alpha_k \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$ for a given $k<m$. We can choose $\mathcal{A}$ so that $\alpha_i-\alpha_j \notin \mathcal{N}(U_S \cap \mathbf{Z}_K)$ for any i, j with $1 \leqq i \leqq k$ and $k<j \leqq m$. In this case for $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ (i) holds and the degree of α_1 is $m-k$. Here $m-k$ can take any value between 1 and $m-2$. Further, $\alpha_2, \ldots, \alpha_m$ can be chosen so that $\overline{\mathcal{G}}$ contains no triangle.

3) Let $m \geqq 4$, $\alpha_1=0, \alpha_2=1$ and $\alpha_i=1+\varepsilon_3+\ldots+\varepsilon_i$ with units $\varepsilon_3, \ldots, \varepsilon_i$ $(i=3, \ldots, m)$ so that $\alpha_i \notin \mathcal{N}(U_S \cap \mathbf{Z}_K)$ for $i \geqq 3$ and $\alpha_4-\alpha_2 \notin \mathcal{N}(U_S \cap \mathbf{Z}_K)$. Then $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ with $\mathcal{A}=\{\alpha_1, \ldots, \alpha_m\}$ is connected and $\overline{\mathcal{G}}$ has a hamiltonian path. We can construct in a similar manner graphs $\mathcal{G}$ such that $\overline{\mathcal{G}}$ contains several kinds of trees.

4) Suppose that K has an exceptional unit ε (i.e. ε and $1-\varepsilon \in U_K$) [6]), and let $m_1, \ldots, m_t \geqq 2$ be integers with $m_1+\ldots+m_t+2=m$. Let $\mathcal{H}_i=\mathcal{H}_i(\mathcal{A}_i, S, \mathcal{N})$ with vertex set $\mathcal{A}_i=\{0, \varepsilon_i, \varepsilon_i\varepsilon, \ldots, \varepsilon_i\varepsilon^{m_i-1}\}$ for $i=1, \ldots, t$, where the units ε_i are chosen so that $\varepsilon_1=1$ and for all distinct i, j $\varepsilon_i\varepsilon^k-\varepsilon_j\varepsilon^l \notin \mathcal{N}(U_S \cap \mathbf{Z}_K)$ if $0 \leqq k<m_i$, $0 \leqq l<m_j$. Let α be an algebraic integer in K with $\alpha, \varepsilon_i\varepsilon^k-\alpha \notin \mathcal{N}(U_S \cap \mathbf{Z}_K)$ for all i and $k<m_i$. Consider the graph $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ with vertex set $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t \cup \{\alpha\}$. Then $\mathcal{G}$ is connected and $\overline{\mathcal{H}_1}, \ldots, \overline{\mathcal{H}_t}$ are the triangular connected components of $\overline{\mathcal{G}}$.

Examples for the case (ii). 1) Let $\mathcal{A}=\{0, \alpha_2, \ldots, \alpha_m\}$ and $\mathcal{A}'=\mathcal{A} \setminus \{0\}$ where $\alpha_2, \ldots, \alpha_m$ are units such that $\mathcal{G}_1=\mathcal{G}_1(\mathcal{A}', S, \mathcal{N})$ is complete. Then for $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ (ii) holds.

2) Consider again the above example 4) with $t \geqq 2$. Let now $\mathcal{A}_i'=\{\varepsilon_i, \varepsilon_i\varepsilon, \ldots, \varepsilon_i\varepsilon^{m_i-1}\}$, $\mathcal{H}_i'=\mathcal{H}_i'(\mathcal{A}_i', S, \mathcal{N})$ for $i=1, \ldots, t$, $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ with vertex set $\mathcal{A}=\mathcal{A}_1' \cup \ldots \cup \mathcal{A}_t' \cup \{0\}$ and consider the subgraphs $\mathcal{G}_1, \mathcal{G}_2$ with vertex sets $\mathcal{A}_1' \cup \ldots \cup \mathcal{A}_t'$ and $\{0\}$, respectively. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are the connected components of $\mathcal{G}$, $\overline{\mathcal{G}}_1$ is not complete, $\overline{\mathcal{H}_1'}, \ldots, \overline{\mathcal{H}_t'}$ are the connected components of $\overline{\mathcal{G}}_1$ and $\overline{\mathcal{H}_i' \cup \mathcal{G}_2}$ are the triangular connected components of $\overline{\mathcal{G}}$. Thus, by our Lemma 3 all these components have the property (P_m).

Examples for the case (iii). 1) If K contains a real quadratic subfield, then our Proposition 5 in [9] shows that there are infinitely many graphs $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ with $|\mathcal{G}|=4$ for which (iii) holds.

2) For every even $m=2n>2$ there exist algebraic number fields K and graphs $\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ of order m in K with the property (iii). To verify it we consider distinct rational integers $a_1=0, a_2, \ldots, a_n$ such that $\min_{i,j} |a_i-a_j|$ is large relative to N and n. As is known (see e.g. [39] or [40]) the polynomial $f(x)=x(x-a_2)\ldots$

[6]) For any $k \geqq 3$ there exist number fields K of degree k which have exceptional units. E.g. for large and distinct positive integers $a_3, \ldots, a_k$ each root ε of the irreducible polynomial $x(x-1)(x-a_3)\ldots(x-a_k)-1$ is an exceptional unit in $K=\mathbf{Q}(\varepsilon)$.

$\dots(x-a_n)-1$ is irreducible over $\mathbf{Q}$ and, by a theorem of Weisner [51], $|\varepsilon^{(i)}-\varepsilon^{(j)}|$ are also large for any distinct roots $\varepsilon^{(i)}, \varepsilon^{(j)}$ of f. Let $K=\mathbf{Q}(\varepsilon^{(1)}, \dots, \varepsilon^{(n)})$. It is easily seen that if $\mathscr{A}=\{a_1, \dots, a_n, \varepsilon^{(1)}, \dots, \varepsilon^{(n)}\}$ then, for any $S \supseteq S_\infty$ for which the prime ideals belonging to $S \setminus S_\infty$ do not divide $D(f) \cdot \prod_{1 \leqq i < j \leqq n} (a_j - a_i)$, the graph $\mathscr{G}=\mathscr{G}(\mathscr{A}, S, \mathscr{N})$ has only two connected components $\mathscr{G}_1, \mathscr{G}_2$ with vertex sets $\{a_1, \dots, a_n\}$ and $\{\varepsilon^{(1)}, \dots, \varepsilon^{(n)}\}$ respectively and $\mathscr{G}_1, \mathscr{G}_2$ are complete.

As remarked in the introduction, the main results of the present paper have a number of applications. By using our Theorem 1 (or its certain particular cases [7]) we obtained general and effective results among others on irreducible polynomials [8], [9], [10], [21], on polynomials and algebraic integers with given discriminant [11], [12], [13], [14], [15], [16], [22], on power bases and on the number of generators of the ring of integers of a number field [13], [15], [22], on discriminant form equations, index form equations, norm form equations and their generalizations [13], [14], [29], [18], [20], [23], on prime factors of decomposable forms [29], [18], [23] and on prime divisors of certain sequences of algebraic integers [25].

We describe now the structure of those graphs $\mathscr{G}=\mathscr{G}(\mathscr{A}, S, \mathscr{N})$ whose number of vertices is large. As we shall see, in this case there exist no graphs $\mathscr{G}$ with the property (iii) occurring in Theorem 1. This fact plays a role of crucial importance in some applications (see e.g. [9], [10], [21] and [26]).

Theorem 2. *Under the above notations let $\mathscr{G}=\mathscr{G}(\mathscr{A}, S, \mathscr{N})$ be a graph with $|\mathscr{G}|>2C_2$ where*

$$C_2 = \max\left[N^5, |D_K|^{5k^2}(\log|2D_K|)^{2(r+s)} \exp\left\{c_2 P^k R_K \cdot \right.\right.$$

$$\left.\left. \cdot (R_K+h_K \log P)^s (R_K+sh_K \log P)[s(R_K+h_K \log P)+1] \log\left(R_K^*(1+sh_K P)\right)\right\}\right]$$

and $c_2=10(25(r+s+3)k)^{20(r+2)+13s}$. Then either

(i) *$\mathscr{G}$ is connected and has at most one vertex with degree $<(|\mathscr{G}|-C_2)/2$,*

or

(ii) *$\mathscr{G}$ consists of two connected components $\mathscr{G}_1, \mathscr{G}_2$, $|\mathscr{G}_2|=1$ and all vertices of $\mathscr{G}_1$ are of degree $\geqq|\mathscr{G}|-C_2$.*

Under the additional hypothesis $m=|\mathscr{G}|>2C_2$ the examples given for the cases (i) and (ii) of Theorem 1 are at the same time examples for the cases (i) and (ii) of Theorem 2 as well.

In our Theorems 1 and 2 the dependence on N of C_1 and C_2 is rather good. If in Theorem 2 N is sufficiently large, we may take $C_2=N^5$. In consequence of the application of Baker's method and because of the generality of our Theorems C_1 and C_2 are very large in terms of the other parameters. It is very likely that C_1 and C_2 can be considerably improved in these parameters. However, the structure of our graphs $\mathscr{G}$ does not depend on the possible improvements of C_1 or C_2.

In proving our Theorems we use some recent results of ours on diophantine equations [19]. In [9] we reduced our conjecture mentioned in the introduction to the Tarry-Escott diophantine problem, but this problem has not been resolved so far. Our Theorem 2 proves our conjecture by showing that every graph

[7]) In some of our papers quoted here (e.g. in [11], [12], [13], [15] and [16]) we applied some weaker versions of Theorem 1 without using graph terminology.

$\mathcal{G}=\mathcal{G}(\mathcal{A}, S, \mathcal{N})$ with $m>2C_2$ vertices has a connected component with at least $m-1$ vertices and this bound is already in general best possible.

Some further properties and applications of the graphs $\mathcal{G}$ discussed above are given in Part II [24].

4. The proofs of the main results

To prove Theorems 1 and 2 we shall need some lemmas. We keep the notations of Section 2.

Lemma 1. *If x_1, x_2 and x_3 are non-zero algebraic integers in K satisfying*

$$x_1+x_2+x_3=0 \quad \text{and} \quad x_1, x_2, x_3 \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$$

then we have $x_i=\sigma\varrho_i$ where $\sigma \in U_S \cap \mathbf{Z}_K$, $\varrho_i \in \mathbf{Z}_K$, $i=1, 2, 3$, and

$$\max_{1\leqq i\leqq 3} |\overline{\varrho_i}| < \exp\{c_1 P^k (\log P)[s(R_K+h_K \log P)\log(1+sR_K h_K)+1]\cdot$$

$$\cdot R_K((s+1)R_K+sh_K \log P)(R_K+h_K \log P)^s [\log R_K^* + s\log(1+R_K h_K \log P)]^2 \cdot$$

$$\cdot [R_K+sh_K \log P+\log N]\} = C_3$$

with the c_1 defined in (2).

This is a special case of Lemma 6 of [19]. In the case $s=0$ we obtained in [17] a slightly better estimate.

We remark that in the proofs of Lemmas 1 and 5 we used, among other things, Baker's method.

Let $\mathcal{B}=\{\beta_1, \ldots, \beta_m\}$ be a system of elements in $\mathbf{Z}_K$ with $m\geqq 3$. We shall say that this system $\mathcal{B}$ is *connected* if for any distinct i, j with $1\leqq i, j\leqq m$ there is a sequence $\beta_i=\beta_{i_1}, \ldots, \beta_{i_v}=\beta_j$ in $\mathcal{B}$ such that for each u with $1\leqq u\leqq v-1$

$$\beta_{i_u}+\beta_{i_{u+1}}+\beta_{i_u, u+1}=0$$

with some $\beta_{i_u, u+1} \in \mathcal{B}$.

Lemma 2. *Let $\mathcal{B}$ be as above and suppose that $\mathcal{B}$ is connected. If $\beta_i \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$, $i=1, \ldots, m$, then $\beta_i=\sigma\varrho_i$ for each i with some $\sigma \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$ and $\varrho_i \in \mathbf{Z}_K$ satisfying*

$$\max_{1\leqq i\leqq m} |\overline{\varrho_i}| < \exp\{3km(s+1)\log P \log C_3\} = C_4. \tag{4}$$

Proof of Lemma 2. We use an idea applied several times in our earlier papers (cf. [12], [13], [15], [16], [28], [18]). Suppose, for convenience, that for $\beta_1, \beta_2, \beta_3$

$$\beta_1+\beta_2+\beta_3=0$$

holds. Then, by Lemma 1 we have $\beta_i=\sigma'\delta_i$ with some $\sigma' \in U_S \cap \mathbf{Z}_K$ and $\delta_i \in \mathbf{Z}_K$ such that $\max_{1\leqq i\leqq 3} |\overline{\delta_i}| < C_3$, where C_3 denotes the same expression as in (3). Consider now any β_j with $3\leqq j\leqq m$. By the assumption there is a sequence $\beta_2=\beta_{j_1}, \ldots, \beta_{j_v}=\beta$ in $\mathcal{B}$ such that for each u with $1\leqq u\leqq v-1$

$$\beta_{j_u}+\beta_{j_{u+1}}+\beta_{j_u, u+1}=0.$$

Further, we may assume $v \leqq m$. By Lemma 1 we have

$$\beta_1 = \sigma' \delta_1, \quad \beta_2 = \sigma' \delta_2 \tag{5}$$

and

$$\beta_{j_u} = \sigma_u \delta'_{j_u}, \beta_{j_{u+1}} = \sigma_u \delta''_{j_{u+1}} \tag{6}$$

for $u=1, \ldots, v-1$ where $\delta'_{j_u}, \delta''_{j_{u+1}} \in \mathbf{Z}_K$ with

$$\max_{1 \leqq u \leqq v-1} (\overline{|\delta'_{j_u}|}, \overline{|\delta''_{j_{u+1}}|}) < C_3$$

and $\sigma_u \in U_S \cap \mathbf{Z}_K$. It follows from (5) and (6) that

$$\beta_j = \beta_{j_v} = \sigma' \varphi_j / \psi_j, \quad 3 \leqq j \leqq m,$$

with

$$\varphi_j = \delta_2 \prod_{u=1}^{v-1} \delta''_{j_{u+1}} \quad \text{and} \quad \psi_j = \prod_{u=1}^{v-1} \delta'_{j_u}.$$

Write $\psi_1 = \psi_2 = 1$ and $\varphi_j = \delta_j$ for $j=1, 2$. It is clear that

$$\max (\overline{|\varphi_j|}, \overline{|\psi_j|}) < C_3^m, \quad j = 1, \ldots, m.$$

We recall that $(\sigma') = \mathfrak{p}_1^{a_1} \ldots \mathfrak{p}_s^{a_s}$. Denote by $\mathfrak{p}_i^{b_i}$ the highest power of $\mathfrak{p}_i$ with $b_i \leqq a_i$ that divides at least one of the $\psi_1, \ldots, \psi_m$. By taking norms we see that

$$b_i \leqq km \log C_3, \quad i = 1, \ldots, s.$$

The ideal $\mathfrak{p}_i^{h_K}$ is principal for all i. Write $b_i^* = \min (a_i, b_i + r_i)$ where $0 \leqq r_i < h_K$ such that $a_i \equiv b_i + r_i \pmod{h_K}$. Putting $d_i = a_i - b_i^*$, $(\xi) = \mathfrak{p}_1^{b_1^*} \ldots \mathfrak{p}_s^{b_s^*}$, $(\sigma) = \mathfrak{p}_1^{d_1} \ldots \mathfrak{p}_s^{d_s}$ with $\xi \sigma = \sigma'$ and $\varrho_j = \xi \varphi_j / \psi_j$ we get

$$\beta_j = \sigma \varrho_j, \quad j = 1, \ldots, m,$$

where $\sigma \in U_S \cap \mathbf{Z}_K$ and ϱ_j are algebraic integers in K. Since by Lemma 3 of [17] ξ can be chosen so that

$$\overline{|\xi|} < \exp \{2km(s+1) \log P \log C_3\},$$

hence for ϱ_j (4) holds.

Consider again the graphs defined in Section 2.

Lemma 3. *Let $\mathscr{H} = \mathscr{H}(\mathscr{A}', S, \mathscr{N})$ be a graph with $m \geqq 2$ vertices. Suppose that both $\overline{\mathscr{H}}$ and $\overline{\mathscr{H}}^T$ are connected. Then there exists $\sigma \in U_S \cap \mathbf{Z}_K$ such that for all distinct $\alpha_i, \alpha_j \in \mathscr{A}'$*

$$\alpha_i - \alpha_j = \sigma \alpha_{ij}$$

with some $\alpha_{ij} \in \mathbf{Z}_K$ satisfying

$$\max_{i,j} \overline{|\alpha_{ij}|} < mC_4.$$

Proof of Lemma 3. By virtue of Lemma 3 of [17], for $m=2$ our Lemma 3 is obvious. Suppose now $m \geqq 3$. By definition any triple $[\alpha_i, \alpha_j]$, $[\alpha_j, \alpha_l]$, $[\alpha_l, \alpha_i]$ of

edges of $\overline{\mathscr{H}}$ forms an edge in $\overline{\mathscr{H}}^T$. It is clear that

$$(\alpha_i-\alpha_j)+(\alpha_j-\alpha_l)+(\alpha_l-\alpha_i)=0.$$

So we may apply Lemma 2 and we get

$$\alpha_i-\alpha_j=\sigma\varrho_{ij}$$

for each edge $[\alpha_i, \alpha_j]$ of $\overline{\mathscr{H}}$ where $\sigma\in U_S\cap\mathbf{Z}_K$ and $\varrho_{ij}\in\mathbf{Z}_K$ satisfies

$$\overline{|\varrho_{ij}|}<C_4.$$

By hypothesis $\overline{\mathscr{H}}$ is connected. Thus, for any distinct $\alpha_i, \alpha_j\in\mathscr{A}'$ there exists a path $\alpha_j=\alpha_{i_1}, \ldots, \alpha_{i_v}=\alpha_i$ of length at most m in $\overline{\mathscr{H}}$ and so

$$\alpha_i-\alpha_j=(\alpha_{i_2}-\alpha_{i_1})+\ldots+(\alpha_{i_v}-\alpha_{i_{v-1}})=$$
$$=\sigma(\varrho_{i_2i_1}+\ldots+\varrho_{i_vi_{v-1}})=\sigma\varrho_{ij},$$

whence $\varrho_{ij}\in\mathbf{Z}_K$ and

$$\overline{|\varrho_{ij}|}<mC_4.$$

PROOF OF THEOREM 1. Let $\mathscr{A}=\{\alpha_1, \ldots, \alpha_m\}$ be the vertex set of $\mathscr{G}$. First suppose that $\mathscr{G}$ is connected. If $\mathscr{G}$ is not complete, then $\overline{\mathscr{G}}$ has at least one edge and by Lemma 3 (i) holds.

Suppose now that $\mathscr{G}$ is not connected and let $\mathscr{G}_1, \ldots, \mathscr{G}_l$, $l\geqq 2$, be the connected components of $\mathscr{G}$. In case $l\geqq 3$ $\overline{\mathscr{G}}$ and $\overline{\mathscr{G}}^T$ are connected. Consequently, by Lemma 3 and $mC_4<C_1$, $\mathscr{G}$ has the property (P_m).

Consider the case $l=2$. First assume that at least one of $\mathscr{G}_1$ and $\mathscr{G}_2$, say $\mathscr{G}_2$, is not complete and that $|\mathscr{G}_1|, |\mathscr{G}_2|\geqq 2$. Let $[\alpha_u, \alpha_v]$ be an edge of $\overline{\mathscr{G}}_2$ and denote by $\mathscr{F}=\mathscr{F}(\mathscr{A}', S, \mathscr{N})$ that subgraph of $\mathscr{G}$ whose vertex set $\mathscr{A}'$ consists of α_u, α_v and of the vertices of $\mathscr{G}_1$. Since $\mathscr{F}$ satisfies the conditions of Lemma 3, we have

$$\alpha_i-\alpha_j=\sigma\delta_{ij},\quad \sigma\in U_S\cap\mathbf{Z}_K,\quad 0\neq\delta_{ij}\in\mathbf{Z}_K$$

and

$$\max_{i,j}\overline{|\delta_{ij}|}<mC_4$$

for any two vertices α_i, α_j of $\mathscr{G}_1$. Here

$$|N_{K/Q}(\delta_{ij})|<(mC_4)^k=N^*.$$

Let $\mathscr{N}^*$ denote the set of $\beta\in\mathbf{Z}_K$ with $0<|N_{K/Q}(\beta)|\leqq N^*$ and consider the subgraph $\mathscr{G}^*=\mathscr{G}^*(\mathscr{A}, S, \mathscr{N}^*)$ of $\mathscr{G}$. It is easily seen that $\mathscr{G}^*$ also satisfies the conditions of Lemma 3 with $\mathscr{N}$ replaced by $\mathscr{N}^*$. So, by Lemma 3 $\mathscr{G}$ has the property (P_m).

If both $\mathscr{G}_1$ and $\mathscr{G}_2$ are complete and $|\mathscr{G}_1|, |\mathscr{G}_2|\geqq 2$, then (iii) holds.

Assume now that $|\mathscr{G}_2|=1$. If $\overline{\mathscr{G}}_1$ is connected, we may apply Lemma 3 to $\mathscr{G}$ and so $\mathscr{G}$ has the property (P_m). It remained the case when $\overline{\mathscr{G}}_1$ is not connected. If $\mathscr{H}$ is any connected component of $\overline{\mathscr{G}}_1$, by applying Lemma 3 we see that $\mathscr{H}\cup\mathscr{G}_2$ has the property (P_m).

To prove Theorem 2 we need some further lemmas.

Lemma 4. *Let $\gamma_1, \gamma_2, \gamma_3$ be non-zero algebraic integers in K with $\max_{1\leq i\leq 3} |\overline{\gamma_i}| \leqq \Gamma$ and let $\beta \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$. Then the number of solutions of the equation*

$$\gamma_1 x_1 + \gamma_2 x_2 = \gamma_3 \beta \tag{7}$$

in $x_1, x_2 \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$ is not greater than $C_7 = 4^{k+r+s}(C_6^k C_5^{r+s})^2$ with the C_5, C_6 defined below.

Proof of Lemma 4. We follow the proof of Lemma 1 (i.e. that of Lemma 6 in [19]) and only a minimal amount of the discussion of that proof will be repeated here.

Using the notation of [19], let $\beta = -x_3$, $\mathfrak{p}_l^{h_K} = (\pi_l)$ with $\pi_l \in \mathbf{Z}_K$ and, if $r > 0$, let $\eta_1, \ldots, \eta_r$ be units with the same properties as in [19]. Then

$$x_i = \sigma \varrho_i, \quad i = 1, 2, 3,$$

where $\sigma = \varepsilon_3 \pi_1^{a_1} \ldots \pi_s^{a_s}$ with $\varepsilon_3 \in U_K$ and $a_l \geqq 0$,

$$\varrho_i = \gamma_i' \eta_1^{w_{1i}} \ldots \eta_r^{w_{ri}} \pi_1^{v_{1i}} \ldots \pi_s^{v_{si}}, \quad \min_i v_{li} = 0 \quad \text{for each } l, \quad w_{13} = \ldots = w_{r3} = 0$$

and $\gamma_i' \in \mathbf{Z}_K$ with

$$\max_{1\leq i\leq 3} |\overline{\gamma_i'}| \leqq N^{1/k} P^{sh_K} \exp\{c_3 r R_K\} = C_8, \tag{8}$$

where $c_3 = (6rk^2)^r$. It is clear that the number of solutions of (7) does not exceed the number of pairs ϱ_1, ϱ_2. But it was proved in [19] that [8])

$$\max_{i,j,l} (|w_{ji}|, |v_{li}|) < c_4 P^k (\log P)[s(R_K + h_K \log P) \log(1 + sh_K R_K) + 1] \cdot$$

$$\cdot R_K (R_K + h_K \log P)^s [\log R_K^* + s \log(1 + R_K h_K \log P)]^2 [R_K + sh_K \log P + \log(\Gamma N)] = C_5$$

where $c_4 = (25(r+s+3)k)^{19r+13s+2rs+34,5}$. Further, by a result of Mahler [33] and Bartz [2] there is an integral basis $\omega_1, \ldots, \omega_k$ in K such that

$$\max_{1\leq j\leq k} |\overline{\omega_j}| \leqq k^k |D_K|^{1/2}.$$

If

$$\gamma_i' = a_{i_1}\omega_1 + \ldots + a_{ik}\omega_k, \quad i = 1, 2, 3,$$

with suitable $a_{ij} \in \mathbf{Z}$, by taking the conjugates of γ_i' we get

$$\max_{i,j} |a_{ij}| < k^{k^2 - \frac{k}{2}} |D_K|^{\frac{k-2}{2}} C_8 = C_6.$$

So, the number of pairs ϱ_1, ϱ_2 is at most $[(2C_6)^k (2C_5)^{r+s}]^2$ which completes the proof.

For $\alpha \in K$ put $\|\alpha\|_v = |\alpha|_v^{n_v}$ where $n_v = [K_v : \mathbf{Q}_v]$. Let $\alpha_1, \alpha_2, \beta$ be non-zero algebraic integers in K and consider the equation

$$\alpha_1 x_1 + \alpha_2 x_2 = \beta \tag{9}$$

in S-units x_1, x_2 of K. We may suppose without loss of generality that $m \geqq \max(m_1, m_2)$ where $m = \prod_{v\in S} \|\beta\|_v$ and $m_i = \prod_{v\in S} \|\alpha_i\|_v$.

[8]) When $s = 0$, we can use the estimate (22) of [17].

Lemma 5. *Suppose that*

$$\log m > \varepsilon^{-1} \log\left(\frac{2}{\varepsilon}\right)(25(r+s+3)k)^{20(r+2)+13s} P^k R_K (R_K + h_K \log P)^s \cdot$$

$$\cdot (R_K + sh_K \log P)[s(R_K + h_K \log P)+1] \log\left(R_K^*(1+sh_K P)\right) = C_9$$

and that $\min(m_1, m_2) \leqq m^{1-\varepsilon}$ *for some real number* ε *with* $0 < \varepsilon \leqq 1$. *Then the number of solutions of* (9) *in S-units* x_1, x_2 *of* K *is not greater than* $r+4s+1$.

PROOF OF LEMMA 5. This is the main result in [19].

Under the conditions of Lemma 5 the number of solutions of (9) does not depend on m_1, m_2 and m. This fact is of basic importance in the proof of Theorem 2.

Denote by $\psi(N)$ the number of nonassociate numbers β in $\mathbf{Z}_K$ with $|N_{K/Q}(\beta)| \leqq N$. It follows from certain explicit estimates of Sunley [47] that

$$\psi(N) < e^{20k^2} |D_K|^{\frac{1}{k+1}} (\log |2D_K|)^k N. \tag{10}$$

Write in (9) $\beta = \sigma\beta^*$ where $\sigma \in U_S \cap \mathbf{Z}_K$, $\beta^* \in \mathbf{Z}_K$ and $|N_{K/Q}(\beta^*)|$ is minimal. Since $\mathfrak{p}_i^{u_i} \mid \beta^*$ implies $u_i < h_K$, hence

$$|N_{K/Q}(\beta^*)| \leqq m P^{skh_K}.$$

With the above notation we have

Lemma 6. *Suppose that*

$$\log |N_{K/Q}(\beta^*)/P^{skh_K}| > C_9$$

and that

$$N \max(m_1, m_2) \leqq |N_{K/Q}(\beta^*)/P^{skh_K}|^{1-\varepsilon}$$

for some ε *with* $0 < \varepsilon \leqq 1$. *Then the number of solutions of* (9) *in* $x_1, x_2 \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$ *is not greater than* $\psi^2(N)(r+4s+1)$. [9])

PROOF OF LEMMA 6. Let $x_1, x_2 \in \mathcal{N}(U_S \cap \mathbf{Z}_K)$ be an arbitrary solution of (9). Then

$$x_1 = \delta_1 y_1, \quad x_2 = \delta_2 y_2$$

where $|N_{K/Q}(\delta_i)| \leqq N$ and $y_1, y_2 \in U_S \cap \mathbf{Z}_K$. From (9) we get at most $\psi^2(N)$ equations

$$\alpha_1 \delta_1 y_1 + \alpha_2 \delta_2 y_2 = \beta \tag{11}$$

in $y_1, y_2 \in U_S \cap \mathbf{Z}_K$. Since

$$\prod_{v \in S} \|\alpha_i \delta_i\|_v \leqq m_i N$$

and

$$|N_{K/Q}(\beta^*)|/P^{skh_K} \leqq m,$$

[9]) It is easy to prove that under certain stronger conditions concerning β the number of solutions of (9) (and thereby $\max(|\mathcal{G}_1|, |\mathcal{G}_2|)$ in (iii) of Theorem 1) is $\leqq r+1+4(\pi_K(N)+s)$. But, using this form of Lemma 6, C_2 would become much larger in terms of N.

we may apply Lemma 5 to each of the equations (11) and the assertion follows.

Denote by C_{10} the expression obtained from C_7 by taking $\Gamma=1$.

Lemma 7. *Let* $\mathscr{H}=\mathscr{H}(\mathscr{A}', S, \mathscr{N})$ *be a graph with* $|\mathscr{H}|\geqq 3$. *Suppose that* $\overline{\mathscr{H}^T}$ *has a vertex of degree* $|\mathscr{H}|-2$.[10]) *Then*

$$|\mathscr{H}| \leqq C_{10}+2. \tag{12}$$

PROOF OF LEMMA 7. Suppose that $[\alpha_i, \alpha_j]$ is a vertex of $\overline{\mathscr{H}^T}$ with the required property. We have

$$(\alpha_i-\alpha_u)+(\alpha_u-\alpha_j) = (\alpha_i-\alpha_j)$$

for each $\alpha_u\in\mathscr{A}'$ which is different from α_i and α_j. Since

$$\alpha_i-\alpha_u,\ \alpha_u-\alpha_j\in\mathscr{N}(U_S\cap \mathbf{Z}_K),$$

(12) follows from Lemma 4.

PROOF OF THEOREM 2. First suppose that $\mathscr{G}$ is not connected and let $\mathscr{G}_1, \ldots, \mathscr{G}_l$ $(l\geqq 2)$ be the connected components of $\mathscr{G}$. Put $|\mathscr{G}|=m$ and assume $|\mathscr{G}_l|\leqq \leqq|\mathscr{G}_{l-1}|\leqq\ldots\leqq|\mathscr{G}_1|$.

If $l\geqq 3$, consider that subgraph $\mathscr{F}=\mathscr{F}(\mathscr{A}', S, \mathscr{N})$ of $\mathscr{G}$ whose vertex set $\mathscr{A}'$ consists of those of $\mathscr{G}_{l-2}, \ldots, \mathscr{G}_1$ and of α_i, α_j where α_i and α_j are fixed vertices of $\mathscr{G}_l$ and $\mathscr{G}_{l-1}$, respectively. It is clear that $|\mathscr{F}|\geqq\frac{m}{3}+2$. We may apply Lemma 7 to $\mathscr{F}$ and we obtain

$$\frac{m}{3} \leqq C_{10} < C_2/3.$$

But this yields a contradiction.

Assume now that $l=2$ and $|\mathscr{G}_2|\geqq 2$. Let

$$N^{**} = P^{skh}\kappa \max(N^2, \exp\{C_{11}\}) \tag{13}$$

where C_{11} is obtained from C_9 by the choice $\varepsilon=1/2$. Denote by $\mathscr{N}^{**}$ the set of algebraic integers β in K satisfying $0<|N_{K/Q}(\beta)|\leqq N^{**}$ and consider the subgraph $\mathscr{G}^{**}=\mathscr{G}^{**}(\mathscr{A}, S, \mathscr{N}^{**})$ of $\mathscr{G}$. First suppose that $\mathscr{G}_2$ and $\mathscr{G}^{**}$ have a common edge, say $[\alpha_i, \alpha_j]$. Let α_u be an arbitrary vertex of $\mathscr{G}_1$. Then we can apply Lemma 6 to the number of solutions of

$$(\alpha_i-\alpha_u)+(\alpha_u-\alpha_j) = (\alpha_i-\alpha_j)$$

in $(\alpha_i-\alpha_u)$, $(\alpha_u-\alpha_j)\in\mathscr{N}(U_S\cap\mathbf{Z}_K)$ and we get

$$|\mathscr{G}_1| \leqq \psi^2(N)(r+4s+1). \tag{14}$$

We obtained (14) without assuming $m>2C_2$. If $m>2C_2$, then, by (10), (14) contradicts the assumption made on m.

Suppose now that all edges of $\mathscr{G}_2$ belong to $\overline{\mathscr{G}^{**}}$ and let $[\alpha_i, \alpha_j]$ be one of these edges. Consider that subgraph $\mathscr{H}=\mathscr{H}(\mathscr{A}'', S, \mathscr{N}^{**})$ of $\mathscr{G}^{**}$ whose vertex set $\mathscr{A}''$ consists of α_i, α_j and the vertex set of $\mathscr{G}_1$. Denote by C_{12} the expression obtained

[10]) That is, in $\overline{\mathscr{H}}$ there exist $|\mathscr{H}|-2$ triangles with a common edge (cf. [3], p. 429)

from C_{10} by taking N^{**} in place of $N.\mathscr{H}$ satisfies the conditions of Lemma 7 and so

$$2+\frac{m}{2} \leqq 2+C_{12} < 2+C_2/2.$$

This gives again a contradiction.

Consider now the case when $l=2$ and $|\mathscr{G}_2|=1$. If $\mathscr{G}_1$ contains a vertex, say α_u, of degree $<m-C_2$, then removing the vertices of $\mathscr{G}_1$ which are adjacent to α_u we get a graph with more than C_2 vertices and with at least three connected components. But, as we showed above, it is impossible. Thus, all vertices of $\mathscr{G}_1$ are of degree $\geqq m-C_2$.

Finally, consider the case when $\mathscr{G}$ is connected. Suppose that $\mathscr{G}$ has two vertices, say α_i and α_j, with degrees $<(m-C_2)/2$. Remove those vertices of $\mathscr{G}$ which are different from α_i, α_j and are adjacent to at least one of α_i, α_j. Denote by $\mathscr{K}$ the subgraph of $\mathscr{G}$ obtained in this way. Then $|\mathscr{K}|>C_2$ and $\mathscr{K}$ has either at least three connected components or has two connected components with vertices $\geqq 2$. But we showed above that this gives a contradiction. So, in this case $\mathscr{G}$ has at most one vertex of degree $<(m-C_2)/2$.

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