

A remark on universal spaces

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Let us introduce the following well-known

Definition. The real separable $(X, \|\cdot\|)$ Banach space is said to be a universal space, iff for arbitrary real separable $(Y, \|\cdot\|)$ Banach space there exists a subspace of $(X, \|\cdot\|)$, isometrically isomorphic to $(Y, \|\cdot\|)$.

BANACH and MAZUR proved that $C[0, 1]$ is universal. This is a special case of the following [1. p. 229.]

Theorem. Let Ω be a compact metric space. Then the assertions A) and B) are equivalent.

A) $C(\Omega)$ is a universal space,

B) Ω is an uncountable set.

In this paper we construct some more universal spaces.

Theorem 1. Let $A \subset C[0, 1]$ be a closed subalgebra and let $\dim A > 1$ (A is considered as a vector space). Then A is universal (with the $C[0, 1]$ norm).

PROOF. Because of $\dim A > 1$, there exists a $h \in A$, $h(x_0) = \|h\|$, such that h is not a constant. Set

$$\bar{h} = \frac{h}{\|h\|} - \frac{h^2}{\|h\|^2} \in A.$$

The function $\bar{h}$ is not a constant, and $\bar{h}(x_0) = 0$.

It is clear that

$$[0, 1] = \bigcup \bar{h}^{-1}(c), \quad c \in [-\|\bar{h}\|, \|\bar{h}\|]$$

and this is a disjoint union. Let us denote $[y] = \bar{h}^{-1}(\bar{h}(y))$, where $y \in [0, 1]$ arbitrary.

Let us introduce the metric

$$d([x], [y]) = |\bar{h}(x) - \bar{h}(y)|$$

on the set

$$K = \{[x]; x \in [0, 1]\}.$$

It is obvious that (K, d) is a compact metric space. Set

$$K' = \left\{ [x]; [x] \in K, \bar{h}(x) \equiv \frac{\|\bar{h}\|}{2} \right\}.$$

Clearly K' is uncountable, so by the above mentioned theorem, $C(K', d)$ is universal.

It suffices to prove that there exists a subspace in A , isometrical to $C(K', d)$.
Let $\varphi: K \rightarrow [0, 1]$ be continuous, such that

$$(1) \quad \varphi([x_0]) = 0, \quad \varphi([x]) = 1, \quad \text{if } [x] \in K'$$

Let be $D: C(K', d) \rightarrow C(K, d)$ a linear isometrical extension (such a D exists, see [2] p. 365 Theorem 21.1.4).

Let be $T: C(K', d) \rightarrow C(K, d)$,

$$(Tg)([x]) = \varphi([x]) \cdot (Dg)([x]).$$

Clearly T is a linear isometrical operator.

From (1) we have $(Tg)([x_0]) = 0$, thus the Weierstrass-Stone theorem implies that for arbitrary $\varepsilon > 0$ a polynomial P_ε of $\bar{h}$ (with constant coefficient equals to 0) exists such that

$$\max_{[x] \in K} |P_\varepsilon(\bar{h}(x)) - (Tg)([x])| < \varepsilon.$$

Since A is closed, $(Tg) \circ \bar{h} \in A$.

It is evident that the following $U: C(K, d) \rightarrow C[0, 1]$

$$(Uf)(t) = (f \circ \bar{h})(t)$$

operator is a linear isometric one, so

$$U \circ T: C[K', d] \rightarrow A$$

is a linear isometric one too. Qu.e.d.

Theorem 2. Let $(X, \|\cdot\|)$ be a universal Banach space and let $(M, \|\cdot\|)$ be a finite codimensional closed subspace of it. Then $(M, \|\cdot\|)$ is also universal.

PROOF. A simple argument shows that it is enough to prove the following assertion.

Every one codimensional closed subspace of $C[0, 1]$ is universal.

If M is a one-codimensional closed subspace of $C[0, 1]$ then there exists a mapping $m: [0, 1] \rightarrow \mathbb{R}$, such that $0 < \text{tot var } m < +\infty$, and

$$(2) \quad g \in M \Leftrightarrow M(g) \equiv \int_0^1 g(x) dm(x) = 0.$$

Let be $0 \leq a < a' < b' < b \leq 1$, and let be m continuous at the points a and b . Further let be a, b satisfy

$$(3) \quad \text{tot var } m \Big|_{[a, b]} \leq \frac{1}{3} \text{tot var } m \Big|_{[0, 1]} = \frac{1}{3} \|M\|.$$

Hence there exists $s \in C[0, 1]$ such that $\|s\| \leq 1$, $s(x) = 0$ for all $x \in [a, b]$ and

$$M(s) \geq \frac{1}{3} \text{tot var } m.$$

Let $g \in C[a', b']$ be arbitrary, and let us define $\bar{g}: [0, 1] \rightarrow R$ by

$$\bar{g}(x) = \begin{cases} 0, & \text{if } x \in [0, a) \\ \frac{x-a}{a'-a} g(a'), & \text{if } x \in [a, a') \\ g(x), & \text{if } x \in [a', b'] \\ \frac{b-x}{b-b'} g(b'), & \text{if } x \in (b', b] \\ 0, & \text{if } x \in (b, 1]. \end{cases}$$

Using (3), we have

$$|M(\bar{g})| \leq \frac{\|M\|}{3} \|\bar{g}\|.$$

Let $\hat{g} \in C[0, 1]$ be defined as

$$\hat{g}(x) = \bar{g}(x) - \frac{M(\bar{g})}{M(s)} s(x).$$

From (2) we obtain that $\hat{g} \in M$.

At the same time, the mapping $g \rightarrow \hat{g}$ is linear and isometrical, so M contains isometrically $C[a', b']$.

References

- [1] R. B. HOLMES, Geometric functional analysis and its applications. Graduate texts in mathematics, vol. 24. New York—Heidelberg—Berlin 1975.
- [2] Z. SEMADENI, Banach spaces of continuous functions, vol. I. Monografie Matematyczne, Tom. 55. Warszawa 1971.

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