

Powers of the augmentation ideal in the Witt ring of a field

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Let F be a formally real field. Denote the Witt ring of F , its augmentation ideal, and its nil radical by $W(F)$, $I(F)$, and $W_t(F)$, respectively. Pfister's local-global principle says that $W_t(F)$ is the kernel of the total signature map on $W(F)$ [7, Satz 22].

Theorem. *Let s be a positive integer. Suppose F has only finitely many distinct places into the real numbers. Then $I(F)^s + W_t(F)$ is exactly the set of elements of $W(F)$ whose signatures at every ordering of F are divisible by 2^s .*

This theorem gives an affirmative answer to Lam's "Open Problem B" [5, p. 49] in the case of fields admitting only a finite number of real-valued places.

I believe that in a narrow sense the proof given here of the theorem is the first. However, M. MARSHALL has recently observed (see [6, footnote on p. 611]) that the theorem follows fairly directly from work he had done a few years ago [6, Theorem 7]. Even more recently, TOM CRAVEN discovered a third proof [3]. The proof presented here, which is quite different from that of Marshall or Craven, is an application of the arithmetic structure theory in [2, § 6] for reduced Witt rings of fields with only finitely many real-valued places. We hope interest in such a proof is increased by the recent result of BECKER and BRÖCKER generalizing this arithmetic structure theory to arbitrary fields [1, Theorem 6.5]. The methods of [2] will be used to reduce the proof of the above theorem to that of the following lemma on subdirect products of integral group rings.

Lemma. *Suppose $n \geq 2$. Let $\Lambda_1, \dots, \Lambda_n$ and $\Delta_2, \dots, \Delta_n$ be groups of exponent two and let $u_{i-1}: \Lambda_{i-1} \rightarrow \Delta_i$ and $v_i: \Lambda_i \rightarrow \Delta_i$ be surjective group homomorphisms, for $i=2, 3, \dots, n$. For each i , $1 \leq i \leq n$, let I_i be the kernel of the ring homomorphism from the group ring $Z(\Lambda_i)$ to $Z_2 = Z/2Z$ which maps each element of Λ_i to $1+2Z$. Define $\Theta = \Theta_n: \prod_{i=1}^n Z(\Lambda_i) \rightarrow \prod_{i=2}^n Z_2(\Delta_i)$ by the formula*

$$\Theta((\lambda_i)_{1 \leq i \leq n}) = (\bar{u}_{i-1}(\lambda_{i-1}) + \bar{v}_i(\lambda_i))_{2 \leq i \leq n}.$$

(For notation, see below.) For each $s \geq 0$ let $J_s = \text{Ker } \Theta \cap \prod_{i=1}^n I_i^s$. Then for each $s \geq 0$, $\text{Ker } \Theta$ is a commutative unitary ring, J_s is an ideal of $\text{Ker } \Theta$, and $J_s = J_1^s$.

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In the above lemma $\bar{u}_{i-1}$ and $\bar{v}_i$ are the compositions of the homomorphisms $u_{i-1}^*: Z(\Delta_{i-1}) \rightarrow Z(\Delta_i)$ and $v_i^*: Z(\Delta_i) \rightarrow Z(\Delta_i)$ induced by u_{i-1} and v_i with the canonical map $Z(\Delta_i) \rightarrow Z_2(\Delta_i)$. We will write u_{i-1} for u_{i-1}^* and v_i for v_i^* .

We now prove the lemma. It is easy to check that $\text{Ker } \Theta$ is a commutative unitary ring with ideal J_s and that $J_s \supseteq J_1^s$. It remains to prove the reverse inclusion. Let $a = (a_1, \dots, a_n)$ be in J_s . We proceed by induction on n . Pick homomorphisms $f: \Delta_n \rightarrow \Delta_n$ and $g: \Delta_2 \rightarrow \Delta_1$ which are right inverses to the surjections v_n and u_1 , respectively. First suppose that $n > 2$. By induction on n one can assume that $(a_1, \dots, a_{n-1})$ is in $(\text{Ker } \Theta_{n-1} \cap \prod_{i=1}^{n-1} I_i)^s$. Θ_{n-1} denotes here the obvious analogue of $\Theta = \Theta_n$. One checks that $a' = (a_1, \dots, a_{n-1}, fu_{n-1}(a_{n-1}))$ is in J_1^s . (The map $(b_1, \dots, b_{n-1}) \rightarrow (b_1, \dots, b_{n-1}, fu_{n-1}(b_{n-1}))$ is a ring homomorphism.) Similarly, we have $a'' = (gv_2(a_2), a_2, \dots, a_n)$ in J_1^s . We may suppose without loss of generality that $a_i = 0$ for all $i < n$ (replace a by $a - a'$). But then $a = a''$ (since $a_1 = a_2 = 0$), which is in J_1^s . Now suppose that $n = 2$. Let $b = a_2 - fu_1(a_1)$ and note $\bar{v}_2(b) = 0$. Hence $v_2(b) = 2c$ for some c in $Z(\Delta_2)$. Note that since b is in I_2^s , we have $2g(c)$ in I_1^s , and so $g(c)$ is in I_1^{s-1} . (For the last assertion apply [4, Theorem 5.13, (7) and (8)]: $Z(\Delta_1)$ is isomorphic to the Witt ring of some superpythagorean field.) Write

$$a = (a_1, fu_1(a_1)) + (gv_2(b), b) - (2, 0)(g(c), f(c)).$$

Clearly now, a is in J_1^s ; the lemma is proved.

PROOF OF THEOREM. Let $s \geq 0$. Let $\sigma_1, \dots, \sigma_n$ denote the distinct real-valued places on F , indexed so that

$$\sigma_i^{-1}(R) \cdot \sigma_{i-1}^{-1}(R) \subseteq \sigma_{i-1}^{-1}(R) \cdot \sigma_j^{-1}(R)$$

whenever $i \leq j \leq n$ (R denotes the real numbers). For each $i = 1, \dots, n$, let Δ_i denote the square factor group of the value group of σ_i , and let F_i denote the ultracompletion of F at σ_i (i.e., the maximal extension of F admitting a real-valued place extending σ_i and having the same value group as σ_i [2, Lemma 2.1]). Further, for $i = 2, 3, \dots, n$, let Δ_i denote the square factor group of the value group of the valuation ring $\sigma_{i-1}^{-1}(R) \cdot \sigma_i^{-1}(R)$; we have canonical homomorphisms $u_{i-1}: \Delta_{i-1} \rightarrow \Delta_i$ and $v_i: \Delta_i \rightarrow \Delta_i$ [2, 3.2]. Let $W_{\text{red}}(F) = W(F)/W_t(F)$ and let t denote the total signature map from $W_{\text{red}}(F)$ to $\prod_X Z$ (the set of orderings of F is denoted by X). Let $I_{\text{red}}(F)$ denote the image of $I(F)$ in $W_{\text{red}}(F)$. We have a commutative diagram

$$\begin{array}{ccccc} W_{\text{red}}(F) & \xrightarrow{t_0} & \prod_{i=1}^n W(F_i) & \xrightarrow{t_1} & \prod_X Z \\ \varphi \downarrow & & \Phi \downarrow & & \\ \text{Ker } \Theta & \xrightarrow{j} & \prod_{i=1}^n Z(\Delta_i) & \xrightarrow{\Theta} & \prod_{i=2}^n Z_2(\Delta_i). \end{array}$$

Here, t_0 is induced by the inclusions $F \rightarrow F_i$ and t_1 by the total signature maps on the $W(F_i)$ (so, $t = t_1 t_0$ [2, Lemma 3.1]). j denotes the inclusion map and Θ is from the Lemma. The map Φ is the product of isomorphisms $W(F_i) \rightarrow Z(\Delta_i)$ obtained as follows. For each i pick an ordering P_i of F_i . Then there is a unique

isomorphism from $W(F_i)$ to $Z(\Lambda_i)$ which takes each one-dimensional form $\langle a \rangle$ to the image of a in Λ_i if a is in P_i and to the negative of this image otherwise [2, Theorem 2.5]. Φ induces an isomorphism $\varphi: W_{\text{red}}(F) \rightarrow \text{Ker } \Theta$, and in fact

$$I_{\text{red}}(F)^s = \varphi^{-1} \varphi(I_{\text{red}}(F)^s) = \varphi^{-1}(J_1^s)$$

[2, Corollary 6.5]. The Lemma tells us that this equals

$$\varphi^{-1}(J_s) = \varphi^{-1} j^{-1} \left(\prod_{i=1}^n I_i^s \right)$$

which by the commutativity of our diagram equals

$$t_0^{-1} \Phi^{-1} \left(\prod_{i=1}^n I_i^s \right) = t_0^{-1} \left(\prod_{i=1}^n I(F_i)^s \right).$$

Since the F_i are superpythagorean we may apply [4, Theorem 5.13 (7)] to conclude that

$$t_0^{-1} \left(\prod_{i=1}^n I(F_i)^s \right) = t_0^{-1} t_1^{-1} (2^s \prod_X Z) = t^{-1} (2^s \prod_X Z).$$

Thus, $I_{\text{red}}(F)^s = t^{-1} (2^s \prod_X Z)$, which was to be proved.

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