On the representation of integers by decomposable forms in several variables

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1. Introduction

The purpose of the present paper is to generalize some results of [10], [11] and [12] concerning diophantine equations and to emphasize the role of our graph method in the investigation of certain diophantine equations.

Let $\mathbf{L}$ be an algebraic number field with ring of integers $\mathbf{Z}_L$. Let $\beta, \pi_1, \ldots, \pi_t$ be distinct non-zero algebraic integers in $\mathbf{L}$ and assume that $\pi_1, \ldots, \pi_t$ are not units. Let $F(\mathbf{x}) = F(x_1, \ldots, x_m) \in \mathbf{Z}_L[x_1, \ldots, x_m]$ be a form of degree $n \geqq 3$ in $m \geqq 2$ variables and suppose that $F(\mathbf{x})$ is *decomposable* (i.e. that it is a product of linear forms with algebraic coefficients). Let $d \geqq 1$. In [10], [11], [12] we proved that under certain assumptions made on the linear factors of F (cf. Section 2) the diophantine equation

$$F(\mathbf{x}) = \beta \pi_1^{z_1} \ldots \pi_t^{z_t}, \quad \text{Norm}\,((x_1, \ldots, x_m)) \leqq d \tag{1}$$

has only finitely many solutions in $\mathbf{x} \in \mathbf{Z}_L^m$, $z_1, \ldots, z_t \in \mathbf{Z}$ with $z_1, \ldots, z_t \geqq 0$ and these solutions can be effectively determined. Further, in [11] and [12] explicit upper bounds have been established for the size of all the solutions of (1).

In this paper we give a common generalization of some results of [10], [11] and [12]. We get as a special case of our main result that if $\mathbf{K}$ is an arbitrary but fixed number field containing $\mathbf{L}$, $\beta, \pi_1, \ldots, \pi_t \in \mathbf{Z}_K$ and F satisfies the same conditions as in [10], [11] or in Corollary 1 of [12] then the number of solutions of (1) in $\mathbf{x} \in \mathbf{Z}_K^m$, $z_1, \ldots, z_t \geqq 0$ is finite[1]). More precisely, we give effectively computable upper bounds for the size of all solutions $x_1, \ldots, x_m \in \mathbf{Z}_K$, $z_1, \ldots, z_t \geqq 0$ of (1). Apart from the form of the bounds, our theorems generalize several earlier effective results on the Thue—Mahler equation and on norm form, discriminant form and index form equations. Further, we give a generalization of some effective theorems on the greatest prime factors of decomposable forms at integer points.

In [4], [5] we introduced certain graphs composed of algebraic integers of a given number field. In [4], [5] and in some recent papers of ours it turned out (for references see e.g. [8]) that certain properties of these graphs play an important role in the investigation of several number-theoretic problems. In the proofs of [10] and [11] we had to combine the classical methods used in the case $m = 2$ with our graph method, but we did not employ graph terminology. In [8] we considerably

[1]) In this case Norm $((x_1, \ldots, x_m))$ denotes the absolute norm of the ideal $(x_1, \ldots, x_m)$ of $\mathbf{K}$.

improved and generalized our earlier results obtained on the graphs mentioned above. To prove these improvements and generalizations we used, among other things, Baker's method. The main theorems of [8] have various applications (see e.g. [8]). In this paper we give an application of Theorem 1 of [8] to the equation (1).

2. Results

Let $\mathbf{L}$ and $\mathbf{K}$ be defined as above. Let $F(\mathbf{x})\in\mathbf{Z}_L[x_1, \ldots, x_m]$ be a decomposable form of degree $n\geqq 3$ in $m\geqq 2$ variables. We may suppose without loss of generality that the coefficient of x_1^n in F is not zero (see e.g. [10]). Let $F(\mathbf{x})=a_0 l_1(\mathbf{x}) \ldots l_n(\mathbf{x})$ be the factorization of F where the $l_i(\mathbf{x})=x_1+\alpha_{i2}x_2+\ldots+\alpha_{im}x_m$ denote linear forms with algebraic coefficients and suppose[2] $\overline{|a_0\alpha_{ij}|}\leqq A$ (with $\alpha_{i1}=1$ for $i=1, \ldots, n$). Assume that the equation system

$$l_i(\mathbf{x}) = 0, \quad i = 1, \ldots, n, \tag{2}$$

has no solution $\mathbf{x}\neq\mathbf{0}$ in $\mathbf{L}^m$ and that the system $\mathscr{L}$ of linear forms $l_1, \ldots, l_n$ can be divided into pairwise disjoint subsystems $\mathscr{L}_1, \ldots, \mathscr{L}_k$ such that each $\mathscr{L}_h$ $(1\leqq h\leqq k)$ is connected (i.e. for any distinct i, j with $l_i, l_j\in\mathscr{L}_h$ there exists a sequence $l_i=l_{j_1}, \ldots, l_{j_v}=l_j$ in $\mathscr{L}_h$ such that $\lambda'_{j_u}l_{j_u}+\lambda''_{j_{u+1}}l_{j_{u+1}}\in\mathscr{L}_h$ for each u, $1\leqq u\leqq v-1$, with some $\lambda'_{j_u}, \lambda''_{j_{u+1}}\in\overline{\mathbf{Q}}\setminus\{0\}$; see e.g. [13] or [10])[3]. As we shall see in the proof of Theorem 1, under these hypotheses $m\leqq n$ and (2) has no solution $\mathbf{x}\neq\mathbf{0}$ in $\mathbf{C}^m$.

Throughout this paper $\mathbf{G}$ denotes a number field containing $\mathbf{K}$ and the splitting field of $F(\mathbf{x})$ over $\mathbf{L}$. Let g, R_G, h_G and r be the degree, regulator, class number and number of fundamental units of $\mathbf{G}$ and write $f=[\mathbf{G}:\mathbf{K}]$.

Let $\beta, \pi_1, \ldots, \pi_t$ denote non-zero algebraic integers in $\mathbf{K}$ and suppose that $\pi_1, \ldots, \pi_t$ are not units. Let s denote the number of distinct prime ideals of $\mathbf{K}$ dividing $\pi_1 \ldots \pi_t$ and let P be the greatest rational prime for which $(\pi_1 \ldots \pi_t, P)\neq 1$. Further, suppose $|N_{K/Q}(\beta)|\leqq b$ and $\max_j \overline{|\pi_j|}\leqq\mathscr{P}$.

Our main result is then as follows:*)

Theorem 1. *Suppose that under the above assumptions there is an l $(1\leqq l\leqq m)$ such that if $l_i(\mathbf{x})=0$ with $\mathbf{x}=(x_1, \ldots, x_m)\in\mathbf{G}^m$ for all $l_i\in\mathscr{L}_h$ then $x_l=0$ holds for each fixed h $(1\leqq h\leqq k)$[4]. Then all solutions $\mathbf{x}\in\mathbf{Z}_K^m, z_1, \ldots, z_t\in\mathbf{Z}$ of (1) with $x_l\neq 0$, $z_1, \ldots, z_t\geqq 0$ satisfy*

$$\max_{1\leqq i\leqq m} \overline{|x_i|} < \overline{|\beta|}^{1/n}(d^{1/f}C^m)^{f+t\log\mathscr{P}} \tag{3}$$

and

$$\prod_{j=1}^{t} |N_{K/Q}(\pi_j)|^{z_j} \leqq (dC^m)^n \tag{4}$$

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[2]) As usual, $\overline{|\alpha|}$ denotes the maximum absolute value of the conjugates of an algebraic integer α (in other words the size of α).

[3]) It is easy to verify that if $m=2$ then every system $\mathscr{L}$ containing at least three pairwise non-proportional linear forms satisfies these conditions with $k=1$.

[4]) It is clear that if $k=1$ then the other hypotheses of Theorem 1 imply this condition for each l.

*) See the remark at the end of the paper.

where

$$C = \exp\{(c_1(s+1))^{sf(2r+13)+20r+42} n P^g (\log P)^{sf+7}(1+n\log(Ab))\}$$

with an effectively computable positive number c_1 *depending only on* g, R_G *and* h_G.

Apart from the fact that [11] contains explicit estimates in terms of each parameter, the above quoted results of [10] and [11] correspond to the case $\mathbf{K}=\mathbf{L}$, $k=1$ of our Theorem 1 (when the assumption $x_l \neq 0$ can be omitted). Here a straightforward application of our graph method is emphasized in place of obtaining explicit bounds. By using the bounds of [8] in explicit form we could easily derive an explicit value for c_1.

Keeping the above notations, we present now some consequences (Theorems 2, 3, 4, 5, 6, 7) of our main result. The following classes of forms satisfy the conditions of Theorem 1.

Let $m=2$, and let $F(\mathbf{x})=F(x_1, x_2)\in \mathbf{Z}_L[x_1, x_2]$ be a binary form of degree n with $F(1, 0)\neq 0$ such that $F(x_1, 1)$ has at least three distinct zeros and $2\overline{|F|}\leqq A$ (where $\overline{|F|}$ denotes the maximum absolute value of the conjugates of the coefficients of F). Then (1) is just the Thue—Mahler equation

$$F(x_1, x_2) = \beta\pi_1^{z_1}\dots\pi_t^{z_t}. \tag{5}$$

Theorem 2. *All solutions of* (5) *in* $x_1, x_2\in \mathbf{Z}_K, z_1, \dots, z_t\in\mathbf{Z}$ *with* $\mathrm{Norm}((x_1, x_2))\leqq$ $\leqq d, z_1, \dots, z_t\geqq 0$ *satisfy* (3) *and* (4).

This result was in fact proved in [11] with slightly different estimates. Apart from the form of the bounds, Theorem 2 is a generalization of theorems of COATES [2], [3], SPRINDŽUK [17], [18], KOTOV [15] and KOTOV and SPRINDŽUK [16] on the Thue—Mahler equation.

By a solution $\mathbf{x}$ of (6), (7) and (8) we mean an $\mathbf{x}\in\mathbf{Z}_K^m$ satisfying $l_1(\mathbf{x})\dots l_n(\mathbf{x})=$ $=\beta\pi_1^{z_1}\dots\pi_t^{z_t}$, where the l_i are linear factors of the corresponding decomposable form.

Let $\mathbf{M}$ be an extension of degree $n\geqq 3$ of $\mathbf{L}$, and let $\alpha_1=1, \alpha_2, \dots, \alpha_m\in\mathbf{M}$ $(m\geqq 2)$ with $\mathbf{M}=\mathbf{L}(\alpha_2, \dots, \alpha_m)$. Suppose that α_{i+1} is of degree $\geqq 3$ over $\mathbf{L}(\alpha_1, \dots, \alpha_i)$ for $i=1, \dots, m-1$. Let

$$F(\mathbf{x}) = a_0 N_{M/L}(x_1+\alpha_2 x_2+\dots+\alpha_m x_m)\in\mathbf{Z}_L[x_1, \dots, x_m]$$

and $\max\limits_{1\leqq i\leqq m}\overline{|a_0\alpha_i|}\leqq A$. In this case (1) is a norm form equation

$$a_0 N_{M/L}(x_1+\alpha_2 x_2+\dots+\alpha_m x_m) = \beta\pi_1^{z_1}\dots\pi_t^{z_t}. \tag{6}$$

Theorem 3. *Under the above assumptions all solutions of* (6) *in* $x_1, \dots, x_m\in\mathbf{Z}_K$, $z_1, \dots, z_t\geqq 0$ *with* $\mathrm{Norm}((x_1, \dots, x_m))\leqq d$ *satisfy* (3) *and* (4).*)

If $x_m\neq 0$, the conditions of Theorem 3 can be weakened.

Theorem 4. *Suppose that in* (6) $\mathbf{M}=\mathbf{L}(\alpha_2, \dots, \alpha_m)$, $\alpha_1=1, \alpha_2, \dots, \alpha_{m-1}$ *are linearly independent over* $\mathbf{L}$ *and* α_m *is of degree* $\geqq 3$ *over* $\mathbf{L}(\alpha_1, \dots, \alpha_{m-1})$. *Then all*

*) *Added in proof.* In a recent work of S. V, KOTOV (Inst. Math. Akad. Nauk BSSR, Preprint No. 10 (90), Minsk, 1980), this theorem (in the case $\mathbf{K}=\mathbf{L}$ and under slightly stronger conditions) is proved in a different way.

solutions $\mathbf{x} \in \mathbf{Z}_K^m$, $z_1, \ldots, z_t \geqq 0$ *of* (6) *with* $x_m \neq 0$, Norm $((x_1, \ldots, x_m)) \leqq d$ *satisfy* (3) *and* (4).*)

When $m=2$, (6) becomes the Thue—Mahler equation with an irreducible binary form. Consequently, Theorems 3 and 4 can be regarded as another generalization of the above quoted theorems on the Thue—Mahler equation. Further, Theorems 3 and 4 generalize some results of [11] and [12] concerning norm form equations.

Again let $\mathbf{M}$ denote an extension of degree $n \geqq 3$ of $\mathbf{L}$, and let $1, \alpha_1, \ldots, \alpha_m \in \mathbf{Z}_M$ be linearly independent elements over $\mathbf{L}$ such that $\mathbf{M}=\mathbf{L}(\alpha_1, \ldots, \alpha_m)$ and $\max_i |\overline{\alpha_i}| \leqq A$. If $x_0, x_1, \ldots, x_m \in \mathbf{Z}_L$ are variables, the discriminant $D_{M/L}(\alpha_1 x_1 + \ldots + \alpha_m x_m)$ of $x_0 + \alpha_1 x_1 + \ldots + \alpha_m x_m$ over $\mathbf{L}$ is a decomposable form of degree $n(n-1)$ in $x_1, \ldots, x_m$ with integer coefficients in $\mathbf{L}$. It is called a discriminant form (see e.g. [6] or [14]), and, if $F(\mathbf{x}) = D_{M/L}(\alpha_1 x_1 + \ldots + \alpha_m x_m)$, (1) becomes a discriminant form equation

$$D_{M/L}(\alpha_1 x_1 + \ldots + \alpha_m x_m) = \beta \pi_1^{z_1} \ldots \pi_t^{z_t}, \quad \text{Norm}((x_1, \ldots, x_m)) \leqq d. \tag{7}$$

Theorem 5. *All solutions of* (7) *in* $x_1, \ldots, x_m \in \mathbf{Z}_K$, $z_1, \ldots, z_t \geqq 0$ *satisfy* (3) *and* (4).

In the special case $\mathbf{K}=\mathbf{L}$ Theorem 5 implies (with other estimates) theorems of GYŐRY [6], [7], [11] and GYŐRY and PAPP [14] on discriminant form equations.

If $\mathcal{O}$ is an order of the field extension $\mathbf{M}/\mathbf{L}$ with the above $\mathbf{M}$ such that $\mathcal{O}$ has a relative integral basis of the form $1, \alpha_2, \ldots, \alpha_n$ over $\mathbf{L}$, then

$$D_{M/L}(\alpha_2 x_2 + \ldots + \alpha_n x_n) = [F(x_2, \ldots, x_n)]^2 D_{M/L}(1, \alpha_2, \ldots, \alpha_n)$$

where $F(x_2, \ldots, x_n) \in \mathbf{Z}_L[x_2, \ldots, x_n]$ is a decomposable form of degree $n(n-1)/2$. This form is called the index form of the basis $1, \alpha_2, \ldots, \alpha_n$ of $\mathcal{O}$ over $\mathbf{L}$. Suppose $\max_i |\overline{\alpha_i}| \leqq A$. If $F(\mathbf{x}) = F(x_2, \ldots, x_n)$, then (1) is an index form equation

$$F(x_2, \ldots, x_n) = \beta \pi_1^{z_1} \ldots \pi_t^{z_t}, \quad \text{Norm}((x_2, \ldots, x_n)) \leqq d. \tag{8}$$

Theorem 6. *All solutions* $x_2, \ldots, x_n \in \mathbf{Z}_K$, $z_1, \ldots, z_t \geqq 0$ *of* (8) *satisfy* (3) *and* (4) *with* $D_{M/L}(1, \alpha_2, \ldots, \alpha_n)\beta^2$ *in place of* β.

In the case $\mathbf{K}=\mathbf{L}=\mathbf{Q}$ GYŐRY [6], [7] and TRELINA [19], and in the case of arbitrary $\mathbf{K}=\mathbf{L}$ GYŐRY and PAPP [14] and GYŐRY [11] obtained general effective results on index form equations. Theorem 6 generalizes these results with other estimates.

We signify by $\omega(\alpha)$ the number of distinct prime ideal divisors of a non-zero algebraic integer α in $\mathbf{K}$, and by $P(\alpha)$ the greatest of the norms of these prime ideals (with the convention that $P(\alpha)=1$ if α is a unit).

Theorem 7. *Let* $\mathbf{L}$, $\mathbf{K}$, $F(\mathbf{x})$, d, l *and* $\mathbf{G}$ *be defined as in Theorem* 1. *If* $\mathbf{x} \in \mathbf{Z}_K^m$ *with* $F(\mathbf{x}) \neq 0$, $x_l \neq 0$, Norm $((x_1, \ldots, x_m)) \leqq d$ *and* $N = \max_i |N_{K/Q}(x_i)| \geqq N_0$, *then*

$$s \log(s+1) + \log P > c_2 \log \log N \tag{9}$$

*) *Added in proof.* Very recently KOTOV (private communication) obtained a similar result in the special case $\mathbf{K}=\mathbf{L}$, $t=0$.

and

$$P > c_3 \log\log N \tag{10}$$

where $s=\omega(F(\mathbf{x}))$, $P=P(F(\mathbf{x}))$ *and* $c_2=c_2(g)$, $c_3=c_3(g)$, $N_0=N_0(F, \mathbf{G}, \mathbf{K}, d)$ *are effectively computable positive numbers.*

In the special case $\mathbf{K}=\mathbf{L}, k=1$ (when $x_1 \neq 0$ can be omitted) we obtained [10] (9) and (10) with explicit constants c_2, c_3. From (3) we could easily derive explicit values for c_2 and c_3.

Our Theorem in [10] and Theorem 2 of [12] generalize several earlier effective results on the greatest prime factors of norm forms, discriminant forms and index forms (see [10]). Theorem 7 is a further generalization of these results. For the forms $F(\mathbf{x})$ satisfying the conditions of Theorems 3, 4, 5 or 6 the conclusion of Theorem 7 obviously holds.

We remark that in the first version our main result was proved in a special case.

3. Proofs

We keep the notations of Section 2. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s'}$ denote distinct prime ideals in $\mathbf{G}$ lying above rational primes $\leqq P$. Let S be a finite set of valuations of $\mathbf{G}$ containing all the archimedean valuations and suppose that the non-archimedean valuations of S belong to $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s'}$. We denote the group of S-units of $\mathbf{G}$ by U_S. $\mathcal{N}$ will signify the set of those elements $\varkappa \in \mathbf{Z}_G$ for which $0 < |N_{G/Q}(\varkappa)| < N$ where $N>1$ is a given integer. Let $\mathcal{B}=\{\beta_1, \ldots, \beta_m\}$ be a set of algebraic integers of $\mathbf{G}$ with $m \geqq 3$. Consider the graph $\mathcal{G}=\mathcal{G}(\mathcal{B}, S, \mathcal{N})$ with vertex set $\mathcal{B}$ so that the pair $[\beta_i, \beta_j]$ is an edge of $\mathcal{G}$ if and only if $\beta_i-\beta_j \notin \mathcal{N} \cdot (U_S \cap \mathbf{Z}_G)$. We denote by $\mathcal{G}^T$ the triangle hypergraph of $\mathcal{G}$, i.e. that hypergraph whose vertices are the edges of $\mathcal{G}$, and whose edges are the triples of edges of $\mathcal{G}$ that form a triangle (for this concept see e.g. [1], p. 440). We say that $\mathcal{G}$ is triangular connected if both $\mathcal{G}$ and $\mathcal{G}^T$ are connected.

With the above notations we have the following

Lemma. *If the complementary graph of* $\mathcal{G}(\mathcal{B}, S, \mathcal{N})$ *is triangular connected then there exist* $\sigma \in U_S \cap \mathbf{Z}_G$ *and* $\beta_{ij} \in \mathbf{Z}_G$ *such that* $\beta_i-\beta_j=\sigma\beta_{ij}$ *for all distinct* i, j *and*

$$\max_{i,j} |\beta_{ij}| < \exp\left\{(c_4(s'+1))^{s'(2r+13)+20r+41} m P^g (\log P)^{s'+6} \log N\right\} \tag{11}$$

where c_4 *is an effectively computable positive number depending only on* g, R_G *and* h_G.

This lemma is an immediate consequence of Theorem 1 of [8]. In fact, this theorem provides (11) with a slightly larger bound, but in view of the remark following Theorem 1 of [8] we get (11).

In [8] c_4 is given explicitly in terms of each parameter. Using the explicit form of c_4 we could derive an explicit value for the number c_1 occurring in our Theorem 1.

By applying our above Lemma we can easily reduce the equation (1) to a linear equation system and then (3) and (4) readily follows. Thanks to the application of this lemma, our below proof is much shorter than those of [10] and [11] in the case $\mathbf{K}=\mathbf{L}$, $k=1$.

PROOF OF THEOREM 1. Let $\mathbf{x}\in\mathbf{Z}_K^m$, $z_1, \ldots, z_t\in\mathbf{Z}$ be an arbitrary but fixed solution of (1) with $z_1, \ldots, z_t\geqq 0$. Put $a_0 l_j(\mathbf{x})=\beta_j$, $j=1, \ldots, n$. Evidently $\beta_j\in\mathbf{Z}_G$ for each j. It follows from (1) that the principal ideal (β_j) generated by β_j can be written in the form

$$(\beta_j) = \mathfrak{a}_j \mathfrak{p}_1^{u_{1j}} \ldots \mathfrak{p}_{s'}^{u_{s'j}}, \quad j = 1, \ldots, n,$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s'}$ are distinct prime ideals in $\mathbf{G}$ dividing $(\pi_1 \ldots \pi_t)$ and $\mathfrak{a}_1 \ldots \mathfrak{a}_n | (\beta a_0^{n-1})$. Clearly $s'\leqq sf$. Writing $u_{ij}=v_{ij}h_G+r_{ij}$ with $v_{ij}, r_{ij}\in\mathbf{Z}$, $0\leqq r_{ij}<h_G$, $\mathfrak{a}_j \mathfrak{p}_1^{r_{1j}} \ldots \mathfrak{p}_{s'}^{r_{s'j}} = (\gamma_j)$ and $(\mathfrak{p}_1^{v_{1j}} \ldots \mathfrak{p}_{s'}^{v_{s'j}})^{h_G}$ are principal ideals in $\mathbf{G}$. We have

$$|N_{G/Q}(\gamma_j)| \leqq b^f A^{g(n-1)} P^{s'gh_G}, \quad j = 1, \ldots, n. \tag{12}$$

Further, there is an integer σ_j in $\mathbf{G}$ such that

$$\beta_j = \gamma_j \sigma_j, \quad j = 1, \ldots, n, \tag{13}$$

and $\sigma_j\in U_S\cap\mathbf{Z}_G$ hold, where U_S denotes the group of S-units of $\mathbf{G}$ determined by $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s'}$.

Let h be an arbitrary but fixed integer with $1\leqq h\leqq k$, and let $\mathscr{I}_h$ be the set of indices j satisfying $l_j\in\mathscr{L}_h$. We may suppose without loss of generality that $h\in\mathscr{I}_h$. Let $j\in\mathscr{I}_h\setminus\{h\}$. Since $\mathscr{L}_h$ is connected, there exists a sequence $l_h=l_{j_1}, \ldots, l_{j_v}=l_j$ in $\mathscr{L}_h$ such that for each u, $1\leqq u\leqq v-1$,

$$\lambda'_{j_u} l_{j_u} - \lambda''_{j_{u+1}} l_{j_{u+1}} = \lambda_{j_u, j_{u+1}} l_{j_u, j_{u+1}} \tag{14}$$

with $l_{j_u, j_{u+1}}\in\mathscr{L}_h$ and with non-zero algebraic integers $\lambda'_{j_u}, \lambda''_{j_{u+1}}, \lambda_{j_u, j_{u+1}}\in\mathbf{G}$ of size $\leqq 2A^2$ (cf. [10], [11]). We define $\mathscr{N}$ as above with $N=b^f(4A^{n+1}P^{s'h_G})^g$. Consider the set $\mathscr{B}_h$ of integers of $\mathbf{G}$ consisting of 0 and $\lambda'_{j_u}\beta_{j_u}$, $\lambda''_{j_{u+1}}\beta_{j_{u+1}}$, $u=1, \ldots, v-1$, when j runs through $\mathscr{I}_h\setminus\{h\}$. We can choose the elements of $\mathscr{B}_h$ such that Card $\mathscr{B}_h\leqq 2n$. In view of (12) and (13) each non-zero element of $\mathscr{B}_h$ belongs to $\mathscr{N}(U_S\cap\mathbf{Z}_G)$. Further, for fixed h and $j\in\mathscr{I}_h\setminus\{h\}$, by (14), (13) and (12) we have

$$\lambda'_{j_u}\beta_{j_u} - \lambda''_{j_{u+1}}\beta_{j_{u+1}}\in\mathscr{N}(U_S\cap\mathbf{Z}_G), \quad u = 1, \ldots, v-1,$$

and, if $\lambda'_{j_{u+1}}\neq\lambda''_{j_{u+1}}$,

$$\lambda'_{j_{u+1}}\beta_{j_{u+1}} - \lambda''_{j_{u+1}}\beta_{j_{u+1}}\in\mathscr{N}(U_S\cap\mathbf{Z}_G)$$

for each u, $1\leqq u\leqq v-2$. Define the graph $\mathscr{G}_h=\mathscr{G}_h(\mathscr{B}_h, S, \mathscr{N})$ in the same way as in the above Lemma. Then it is easy to see that the complementary graph of $\mathscr{G}_h$ is triangular connected and so, by our Lemma,

$$\lambda_j^*\beta_j = \sigma_h^*\cdot\delta_j \quad \text{for all} \quad j\in\mathscr{I}_h \tag{15}$$

where $0\neq\lambda_j^*\in\mathbf{Z}_G$ with $|\overline{\lambda_j^*}|\leqq 2A^2$, $\sigma_h^*\in U_S\cap\mathbf{Z}_G$ and $\delta_j\in\mathbf{Z}_G$ satisfies

$$\max_{\substack{j\\ j\in\mathscr{I}_h}} |\overline{\delta_j}| < \exp\left\{(c_5(s+1))^{sf(2r+13)+20r+42} n P^g (\log P)^{sf+7}(1+n\log(Ab))\right\} = C_1. \tag{16}$$

Here, and below, $c_5, c_6, \ldots$ will denote effectively computable positive numbers which depend only on g, R_G and h_G.

From (15) we obtain

$$(\lambda_j^* a_0) l_j(\mathbf{x}) = \sigma_h^*\delta_j \quad \text{for all} \quad j\in\mathscr{I}_h.$$

By assumption x_l has the same value in any solution $\mathbf{x}\in\mathbf{G}^m$ of this equation

system. Therefore, it is easily seen that $x_i=\sigma_h^*\tau_h/\mu_h$ for $h=1, \ldots, k$, where τ_h, $\mu_h\in\mathbf{Z}_G$ and

$$\overline{|\mu_h|} \leqq (2mA^3)^m, \quad \overline{|\tau_h|} \leqq (2mA^3)^m C_1.$$

By putting $\mu_j=\mu_h$, $\tau_j=\tau_h$ for $j\in\mathcal{J}_h$ we get

$$(\vartheta_j a_0) l_j(\mathbf{x}) = \sigma_1^* \varrho_j, \quad j = 1, \ldots, n, \tag{17}$$

where $\vartheta_j=\mu_1\tau_j\lambda_j^*$ and $\varrho_j=\mu_j\tau_1\delta_j$. By virtue of $F\in\mathbf{Z}_L[x_1, \ldots, x_m]$ $\mathcal{L}$ consists of the conjugates of l_j over $\mathbf{L}, j=1, \ldots, n$. Since (2) has no non-trivial solution in $\mathbf{L}^m$, hence an argument of the proof of Lemma 2 in [13] shows that (2) has no non-trivial solution in $\mathbf{C}^m$ and $m\leqq n$. Thus $\mathbf{x}$ is the only solution of (17) in $\mathbf{K}^m$ and by Cramer's rule we get

$$x_i = \sigma_1^* v_i/v, \quad i = 1, \ldots, m, \tag{18}$$

where v_i, $v\in\mathbf{Z}_G$ and for each i

$$\max\left(\overline{|v|}, \overline{|v_i|}\right) \leqq \left(c_6(mA)^{7m}\right)^m C_1^{2m}.$$

In view of (18) $N_{G/Q}(\sigma_1^*)$ divides $N_{G/Q}(v)$ Norm $((x_1, \ldots, x_m))$ which implies

$$|N_{G/Q}(\sigma_1^*)| \leqq d^f \left(c_6(mA)^{7m}\right)^{mg} C_1^{2mg} = C_2. \tag{19}$$

So, by a well-known lemma (see e.g. [9]) there exists a unit ε in $\mathbf{G}$ such that $\sigma_1^*=\varepsilon\sigma'$, $\sigma'\in\mathbf{Z}_G$ and

$$\overline{|\sigma'|} \leqq c_7 C_2^{1/g}.$$

From (18) we get now $x_i=\varepsilon x_i'$ with $\mathbf{x}'=(x_1', \ldots, x_m')\in\mathbf{Z}_G^m$ and

$$\max_i \overline{|x_i'|} \leqq c_7(c_6 m^{7m} A^{7m})^{mg} C_1^{2mg} C_2^{1/g} = C_3.$$

Further

$$\overline{|a_0^n l_1(\mathbf{x}') \ldots l_n(\mathbf{x}')|} \leqq (mAC_3)^n. \tag{20}$$

On the other hand, by (1), (17), (19) and (16) we have

$$\prod_{j=1}^{t} |N_{K/Q}(\pi_j)|^{z_j} \leqq |N_{K/Q}(a_0^{n-1}\beta\pi_1^{z_1} \ldots \pi_t^{z_t})| \leqq |N_{G/Q}((\vartheta_1\beta_1) \ldots (\vartheta_n\beta_n))|^{1/f} =$$
$$= |N_{G/Q}(\sigma_1^*)|^{n/f} |N_{G/Q}(\varrho_1 \ldots \varrho_n)|^{1/f} \leqq C_2^{n/f} C_1^{3ng/f} = C_4$$

which yields (4). Now (1) gives

$$\overline{|\beta a_0^{n-1} \pi_1^{z_1} \ldots \pi_t^{z_t}|} \leqq \overline{|\beta|} A^{n-1} C_4^{t \log \mathcal{P}} = C_5. \tag{21}$$

Finally, from (1), (20) and (21) we obtain

$$\overline{|\varepsilon|}^n = \overline{|\varepsilon^n|} \leqq C_5(mAC_3)^{n(g-1)}$$

and so

$$\max_i \overline{|x_i|} \leqq C_5^{1/n}(mAC_3)^{g-1} \cdot C_3,$$

whence (3) follows.

PROOF OF THEOREM 2. By hypothesis there are at least three pairwise non-proportional linear factors in the factorization

$$F(x_1, x_2) = a_0(x_1+\alpha_1 x_2)\dots(x_1+\alpha_n x_2),$$

hence these factors form a connected system and the condition (2) is also satisfied. Since $\max_i |\overline{a_0\alpha_i}| < |\overline{a_0}| + |\overline{F}| \leqq 2|\overline{F}| \leqq A$, Theorem 2 immediately follows from Theorem 1.

PROOF OF THEOREM 3. Let $\mathbf{x} \in \mathbf{Z}_K^m$, $z_1, \dots, z_t \geqq 0$ be an arbitrary but fixed solution of (6) with $\operatorname{Norm}((x_1, \dots, x_m)) \leqq d$, and let l be the greatest integer for which $x_l \neq 0$. The case $l = 1$ being trivial, we suppose $l \geqq 2$. $\mathbf{x}, z_1, \dots, z_t$ satisfy the equation

$$F_l(\mathbf{x}) = a_0\big(N_{M_l/L}(x_1+\alpha_2 x_2+\dots+\alpha_l x_l)\big)^{n_l} = \beta\pi_1^{z_1}\dots\pi_t^{z_t},$$

where $\mathbf{M}_l = \mathbf{L}(\alpha_2, \dots, \alpha_l)$ and $n_l = [\mathbf{M}:\mathbf{M}_l]$. Apply Theorem 1 to this equation. Since $F_l(\mathbf{x}) \neq 0$ for all $\mathbf{0} \neq \mathbf{x} \in \mathbf{L}^l$, the condition (2) is fulfilled. Divide the linear factors of F_l into subsystems $\mathscr{L}_1, \dots, \mathscr{L}_k$ so that two linear forms belong to the same subsystem if and only if in these forms the coefficients of $x_1, \dots, x_{l-1}$ coincide. It is easily seen that $\mathscr{L}_1, \dots, \mathscr{L}_k$ satisfy all conditions of Theorem 1 and so, by Theorem 1, $\mathbf{x}, z_1, \dots, z_t$ satisfy (3) and (4).

PROOF OF THEOREM 4. Applying the above proof with $l = m$, the assertion easily follows.

PROOF OF THEOREM 5. Let $x_1, \dots, x_m \in \mathbf{Z}_K$, $z_1, \dots, z_t \geqq 0$ be an arbitrary but fixed solution of (7). If $D_{M/L}(\alpha_1) \neq 0$, the equation (7) satisfies all conditions of our Theorem 1 with $k = 1$ (cf. [10]) and (3) and (4) easily follow. Suppose now $D_{M/L}(\alpha_1) = 0$. As is known, there exist $a_2, \dots, a_m \in \mathbf{Z}_L$ with sizes $\leqq n^4$ such that $\mathbf{M} = \mathbf{L}(\alpha)$ for $\alpha = \alpha_1+\alpha_2 a_2+\dots+\alpha_m a_m$, that is $D_{M/L}(\alpha) \neq 0$. Write $x_1 = x_1^*$, $x_2 = a_2 x_1^* + x_2^*, \dots, x_m = a_m x_1^* + x_m^*$. Since the coefficient of $(x_1^*)^{n(n-1)}$ in $F(\mathbf{x}) = D_{M/L}(\alpha_1 x_1^* + \alpha_2(a_2 x_1^* + x_2^*) + \dots + \alpha_m(a_m x_1^* + x_m^*)) = D_{M/L}(\alpha x_1^* + \alpha_2 x_2^* + \dots \alpha_m x_m^*)$ is not zero, we may apply our Theorem 1 to (1) with the above $F(\mathbf{x})$ and we get bounds for $\max_i |\overline{x_i^*}|$ and $\prod_{j=1}^{t} |N_{K/Q}(\pi_j)|^{z_j}$. Finally, since $\max_i |\overline{x_i}| \leqq (n^4+1)\max_i |\overline{x_i^*}|$, we obtain (3) and (4) with a suitable c_1.

PROOF OF THEOREM 6. All solutions of (8) satisfy

$$D_{M/L}(\alpha_2 x_2+\dots+\alpha_n x_n) = D_{M/L}(1, \alpha_2, \dots, \alpha_n)\beta^2\pi_1^{2z_1}\dots\pi_t^{2z_t}.$$

By applying Theorem 5 we get (3) and (4) with $D_{M/L}(1, \alpha_2, \dots, \alpha_n)\beta^2$ in place of β.

PROOF OF THEOREM 7. Let $\mathbf{x} \in \mathbf{Z}_K^m$ with $F(\mathbf{x}) \neq 0$, $x_l \neq 0$ and $\operatorname{Norm}((x_1, \dots, x_m)) \leqq d$, and let

$$(F(\mathbf{x})) = \mathfrak{p}_1^{u_1}\dots\mathfrak{p}_s^{u_s} \tag{22}$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are distinct prime ideals in $\mathbf{K}$. In case $\omega(F(\mathbf{x})) = 0$ write $(F(\mathbf{x})) = \mathfrak{p}_1^{u_1}$ with $u_1 = 0$ and with a prime ideal $\mathfrak{p}_1$ lying above 2, and suppose $P = \operatorname{Norm}(\mathfrak{p}_1)$. Denote the degree and class number of $\mathbf{K}$ by k and h_K, respectively.

Put $\mathfrak{p}_i^{h_K}=(\pi_i)$ and $u_i=h_K z_i+r_i$ with $z_i, r_i\in\mathbf{Z}$, $0\leqq r_i<h_K$. By (22) $\mathfrak{p}_1^{r_1}\dots\mathfrak{p}_s^{r_s}=(\beta)$ is a principal ideal in $\mathbf{K}$ and we have $F(\mathbf{x})=\varepsilon\beta\pi_1^{z_1}\dots\pi_s^{z_s}$ with a suitable unit $\varepsilon\in\mathbf{K}$. By virtue of Lemma 3 of [9] we may suppose that $\overline{|\pi_i|}\leqq P^{h_K}c_8$, where $c_8=c_8(\mathbf{K})>0$ is effectively computable. Further, $\varepsilon\beta=\eta^{-n}\beta_1$ with a unit $\eta\in\mathbf{K}$ and with $\beta_1\in\mathbf{Z}_K$ satisfying $\overline{|\beta_1|}\leqq P^{sh_K}c_8^r$. It follows from (22) that

$$F(\eta\mathbf{x})=\beta_1\pi_1^{z_1}\dots\pi_s^{z_s}. \tag{23}$$

By applying Theorem 1 to (23) we get

$$\max_i \overline{|\eta x_i|}<\exp\left\{c_9(c_{10}(s+1))^{sf(2r+13)+20r+44}P^g(\log P)^{sf+g}\right\} \tag{24}$$

with effectively computable $c_9=c_9(F, \mathbf{G}, \mathbf{K}, d)$, $c_{10}=c_{10}(F, \mathbf{G}, \mathbf{K})$.

It is clear that $|N_{K/Q}(x_i)|\leqq\overline{|\eta x_i|}^g$. Further, by a well-known theorem $s\leqq$ $\leqq 2gP/\log P$. Thus, if N_0 is sufficiently large, then in view of (24) P is also sufficiently large and so $\omega(F(\mathbf{x}))>0$. Consequently, for sufficiently large N_0 (24) implies (9), whence (10) easily follows.

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Corrections to [10] and [14]

[10] p. 341 line −7, p. 343 lines 7 and 14 and p. 346 line −5. "$F(\mathbf{x}) \neq 0$" should be supposed.
p. 343, line 18. For "in L" read "in L such that $\pi_1, \ldots, \pi_t$ are not units".
p. 345, line 12. For "that" read "that $K = L(\alpha_2, \ldots, \alpha_m)$,".
p. 346, line 3. For "with" read "with $\mathrm{I}(\mathbf{x}) \neq 0$,".
p. 353, line −13. For "x_1" read "x".

[14] In Corollary 3 "$D_{K/\mathbf{Q}}(\alpha_1 x_1 + \ldots + \alpha_m x_m) \neq 0$", and in Corollary 5 "$D_{K/L}(\alpha_1 x_1 + \ldots + \alpha_m x_m) \neq 0$" should be supposed.
p. 50 line −2. For "$|p_1, \ldots, |p_s$" read "$|\ldots|_{p_1}, \ldots, |\ldots|_{p_s}$".
p. 52 line 5. For "$D_k^{31/2k}$" read "$D_K^{31/2k}$".
line −15. For "well as" read "well, as".
p. 56 line −11. For "indepenent" read "independent".
p. 57 line 4. For "c_{26}" read "c_{25}".
p. 58 line 1. For "$\mathscr{P}^{sn(h_L-1)}$" read "$\mathscr{P}^{s(h_L-1)}$".

Remark added in proof. In our recent paper "On certain graphs associated with an integral domain and their applications to diophantine problems" (to appear), some results of the present article have been extended to the case when $\mathbf{L}$ and $\mathbf{K}$ are fields of finite type over $\mathbf{Q}$, and $\mathbf{Z}_L$ and $\mathbf{Z}_K$ are replaced by subrings of $\mathbf{L}$ and $\mathbf{K}$ of finite type over $\mathbf{Z}$. We proved amond other things that if $\mathbf{L}$ is such a field and $F \in \mathbf{L}[x_1, \ldots, x_m]$ is a decomposable form having the properties specified in Theorem 1, 2, 3, 4, 5 or 6 of the present paper, then the equation $F(\mathbf{x}) = \beta$ $(0 \neq \beta \in \mathbf{L})$ has only finitely many solutions $\mathbf{x} = (x_1, \ldots, x_m)$ (with $x_l \neq 0$ (resp. $x_m \neq 0$) in the case of Theorem 1 (resp. of Theorem 4)) in any given integral domain finitely generated over $\mathbf{Z}$.

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