On extensions of syntopogenous spaces

By KÁLMÁN MATOLCSY (Debrecen)

Introduction

Theory of *syntopogenous spaces* was worked up in [1]. The present paper will use both the terminology and the notations of this monograph.

For the purpose of the construction of certain *extensions* of syntopogenous spaces, in § 1 the notion of an *inductor* will be defined; this is a monotone mapping (in the sense of [4]) having some additional properties.

In view of its character similar to a *strict extension* of a topological space ([3], ch. 6), in § 2 an extension of a given syntopogenous space will be said to be *tight*, if it can be induced by a special inductor belonging to the trace filters of this extension. The concept of a tight extension can be identified with that of an extension introduced in [6].

§ 3 contains a method to look for a continuous extension of continuous real-valued functions, with the help of which one can get another definition of tight extensions.

Finally, in § 4 we shall consider various conditions on an extension $(E', \mathscr{S}', g)$ of a syntopogenous space $[E, \mathscr{S}]$, under which $g(E)$ is $\mathscr{S}'^b$-, $\mathscr{S}'^s$- and $\mathscr{S}'^{sb}$-dense respectively.

These later conditions characterize also the subspaces of a *double compactification* [5] and of a *completion* [1] of $[E, \mathscr{S}]$, provided a simple separation axiom is satisfied.

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1. Inductors

First of all let us recall the following notions introduced by Á. CSÁSZÁR ([4], (2.1), (2.3), (3.1), (3.19) and (3.20)).

Let E, E' be two sets, $\mathfrak{D}\subset 2^E$, finally suppose that $h: \mathfrak{D}\to 2^{E'}$ is a *monotone mapping*, i.e. $D_1, D_2\in\mathfrak{D}$, $D_1\subset D_2$ imply $h(D_1)\subset h(D_2)$. If $<$ is a semi-topogenous order on E, then

$$h(<) = \bigcup\{<_{h(A), h(B)}\colon A, B\in\mathfrak{D}, A<B\}$$

is a semi-topogenous order on E' (we put $h(<) = <_{\emptyset, E'}$, whenever the family $\{\dots\}$ is empty).

When $\mathscr{A}$ is an order family on E, one can define an order family on E' by the following equality:

$$h(\mathscr{A}) = \{h(<)^q\colon\ <\in\mathscr{A}\}.$$

If $\mathfrak{D}$ is a separator for a syntopogenous structure $\mathscr{S}$ on E (i.e. $<\in\mathscr{S}$, $A<B$ imply the existence of a set $D\in\mathfrak{D}$ such that $A\subset D\subset B$), then $h(\mathscr{S})$ is a syntopogenous structure on E', too.

We prove that any monotone mapping h has an extension H onto 2^E, which preserves certain important properties of the original mapping.

(1.1) Theorem. *If $h\colon \mathfrak{D}\to 2^{E'}$ is a monotone mapping, then there exists a monotone mapping $H\colon 2^E\to 2^{E'}$ such that*

(1.1.1) $H|\mathfrak{D}=h$;

(1.1.2) *if $D_1, D_2\in\mathfrak{D}$ implies $D_1\cap D_2\in\mathfrak{D}$ and $h(D_1)\cap h(D_2)=h(D_1\cap D_2)$, then $H(A)\cap H(B)=H(A\cap B)$ for any $A, B\subset E$;*

(1.1.3) *if $\mathfrak{D}$ is a separator for the syntopogenous structure $\mathscr{S}$ on E, then $H(\mathscr{S})\sim h(\mathscr{S})$;*

(1.1.4) *if $s\colon 2^E\to 2^{E'}$ is also a monotone mapping such that $s|\mathfrak{D}=h$, then $H(A)\subset s(A)$ for each $A\subset E$.*

Proof. We define the mapping H as follows:

$$H(A) = \{x'\in E'\colon\ x'\in h(D),\ D\subset A \text{ for some } D\in\mathfrak{D}\}.$$

Then it is clear that H is monotone.

(1.1.1): $D\in\mathfrak{D}$ implies $h(D)\subset H(D)$, and for $x'\in H(D)$ there exists $D_1\in\mathfrak{D}$ such that $D_1\subset D, x'\in h(D_1)$. Thus we get $x'\in h(D)$, since h is monotone.

(1.1.2): $A, B\subset E$ obviously implies $H(A\cap B)\subset H(A)\cap H(B)$. Conversely, if $x'\in H(A)\cap H(B)$, then for suitable $D_1, D_2\in\mathfrak{D}$ we have $D_1\subset A$, $D_2\subset B$, $x'\in h(D_1)\cap h(D_2)$. Because of $D_1\cap D_2\in\mathfrak{D}$ and $h(D_1)\cap h(D_2)=h(D_1\cap D_2)$, from $D_1\cap D_2\subset A\cap B$ we deduce $x'\in H(A\cap B)$.

(1.1.3): Obviously $h(\mathscr{S})\prec H(\mathscr{S})$. Conversely, let $<$ be an element of $\mathscr{S}$, $<_1\in\mathscr{S}$, $<\sqsubset<_1^3$. Then $A'H(<)B'$ $(A'\neq\emptyset,\ B'\neq E')$ implies $A<B$, $A'\subset H(A)$ and $H(B)\subset B'$. If $D_1, D_2\in\mathfrak{D}$ such that $A\subset D_1<_1 D_2\subset B$, then $H(A)\subset \subset H(D_1)=h(D_1)$ and $h(D_2)=H(D_2)\subset H(B)$, so that $A'h(<_1)B'$. We got $H(<)\sqsubset h(<_1)$, therefore $H(\mathscr{S})\prec h(\mathscr{S})$ is clear.

(1.1.4): From $x'\in H(A)$ the inclusions $x'\in h(D)=s(D)\subset s(A)$ follow for some $A\supset D\in\mathfrak{D}$. ∎

In consequence of the theorem, in this paper it will be sufficient to use monotone mappings defined on whole 2^E.

For the sake of further applications, we consider a special class of monotone mappings. Let g be a mapping of a syntopogenous space $[E,\mathscr{S}]$ into a set E'. A monotone mapping $h\colon 2^E\to 2^{E'}$ is an *inductor subordinated to $\mathscr{S}$ and g*, if the following condition is satisfied:

(I_1) $<\in\mathscr{S}$ *and* $A<B$ *imply* $g(A)\subset h(B)$ *and* $h(A)\subset E'-g(E-B)$.

(1.2) Theorem. *If h is an inductor subordinated to the syntopogenous structure $\mathcal{S}$ on E and to the mapping $g: E\to E'$, then $g^{-1}(h(\mathcal{S}))\sim\mathcal{S}$. The set $g(E)$ is dense in $[E', h(\mathcal{S})]$, provided the following condition is fulfilled:*

(I_2) *If $A_j < B_j$ $(1\leqq j\leqq n)$ for some natural number n and $<\in\mathcal{S}$, then $\bigcap_{j=1}^{n} B_j=\emptyset$ implies*

$$\bigcap_{j=1}^{n} h(A_j) = \emptyset.$$

PROOF. Let $<$ and $<_1$ be elements of $\mathcal{S}$, $< \subset <_1^3$. Then because of (I_1) $< \subset g^{-1}(h(<_1))$ and $g^{-1}(h(<))\subset <_1$. Indeed, $A<B$ implies $A<_1 C<_1 D<_1 B$, thus

$$g(A)\subset h(C)h(<_1)h(D)\subset E'-g(E-B),$$

i.e. $Ag^{-1}(h(<_1))B$. Conversely, if $g(A)h(<)E'-g(E-B)$, where $A, B\subset E$, there exists $A_0<B_0$ such that $g(A)\subset h(A_0)$ and $h(B_0)\subset E'-g(E-B)$. Putting $A_0<_1C<_1D<_1B_0$, we have $h(A_0)\subset E'-g(E-C)$ and $g(D)\subset h(B_0)$. From this we deduce $g(A)\subset E'-g(E-C)$ and $g(D)\subset E'-g(E-B)$, so that $A\subset C<_1 D\subset B$.

If $<\in\mathcal{S}$ and $x' h(<)^q V'$, one can easily prove that

$$x'\in\bigcap_{j=1}^{n} h(A_j) \quad\text{and}\quad \bigcap_{j=1}^{n} h(B_j)\subset V',$$

where $A_j<B_j$ $(1\leqq j\leqq n)$ for some natural number n. Put $<_1\in\mathcal{S}$, $< \subset <_1^2$ and $A_j<_1C_j<_1B_j$ for a suitable $C_j\subset E$ $(1\leqq j\leqq n)$. With an application of (I_2) we get $\bigcap_{j=1}^{n} C_j\neq\emptyset$, thus in view of ($I_1$) $\emptyset\neq\bigcap_{j=1}^{n} g(C_j)\subset V'$, consequently $g(E)\cap V'\neq\emptyset$. ∎

We say that $(E', \mathcal{S}', g)$ is an *extension* of the syntopogenous space $[E, \mathcal{S}]$, if $[E', \mathcal{S}']$ is a syntopogenous space, and g is an isomorphism of $[E, \mathcal{S}]$ onto a dense subspace of $[E', \mathcal{S}']$. (Cf. [2], p. 238.) We can easily show that $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$, iff g is an injection of E into E' such that $g(E)$ is dense in the syntopogenous space $[E', \mathcal{S}']$ and $g^{-1}(\mathcal{S}')\sim\mathcal{S}$ (cf. [1], (10.14)).

It is also obvious that if $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$, then $(E', \mathcal{S}'^t, g)$ is an extension of $[E, \mathcal{S}^t]$, and $(E', \mathcal{S}'^{tp}, g)$ is that of $[E, \mathcal{S}^{tp}]$.

(1.3) Corollary. *Let $[E, \mathcal{S}]$ be a syntopogenous space, g be an injection of E into a set E', finally let h be an inductor subordinated to $\mathcal{S}$ and g satisfying* (I_2). *Then $(E', h(\mathcal{S}), g)$ is an extension of $[E, \mathcal{S}]$.* ∎

In particular, if $g: E\to E$ is the identity of the set $E=E'$, and h is the inductor subordinated to $\mathcal{S}$ and g defined by $h(A)=A$ for $A\subset E$, then $h(\mathcal{S})\sim\mathcal{S}$.

2. Tight extensions

Let $[E, \mathcal{S}]$ be a syntopogenous space. A filter base $\mathfrak{r}$ in E is *round* in $[E, \mathcal{S}]$, if for any $R\in\mathfrak{r}$ there exist $R_1\in\mathfrak{r}$ and $<\in\mathcal{S}$ such that $R_1<R$ (in this case $\mathfrak{r}$ will be called $\mathcal{S}$*-round*, too). If $\mathfrak{r}$ is an arbitrary filter base in E, then

$$\mathcal{S}(\mathfrak{r}) = \{X\subset E:\ R<X,\ <\in\mathcal{S},\ R\in\mathfrak{r}\}$$

is an $\mathscr{S}$-round filter in E, which is called the *neighbourhood filter* of x in $\mathscr{S}$. (We use the more simple notation $\mathscr{S}(x)$ instead of $\mathscr{S}(\{\{x\}\})$ for $x\in E$.)

Putting a mapping $g: E\to E'$ and a filter $\mathfrak{f}'$ in E', every element of which has a non empty intersection with $g(E)$, by $g^{-1}(\mathfrak{f}')$ we shall mean the filter generated in E by the filter base $\{g^{-1}(F'): F'\in\mathfrak{f}'\}$. If $(E', \mathscr{S}', g)$ is an extension of the syntopogenous space $[E, \mathscr{S}]$, then the filter $\mathscr{S}'(x')$ satisfies the condition mentioned above for any $x'\in E'$, and the filters $g^{-1}(\mathscr{S}'(x'))$ $(x'\in E')$ are called the *trace filters* of this extension.

(2.1) Lemma. *If* $(E', \mathscr{S}', g)$ *is an extension of* $[E, \mathscr{S}]$, *then the trace filters are* $\mathscr{S}$*-round, and we have*

$$g^{-1}(\mathscr{S}'(g(x))) = \mathscr{S}(x)$$

for $x\in E$. ∎

Conversely, let $\mathfrak{f}(x')$ be a filter in E for every $x'\in E'$. Then the equality (cf. [6])

$$h(A) = \{x'\in E':\ A\in\mathfrak{f}(x')\} \quad (A\subset E)$$

defines a monotone mapping $h: 2^E\to 2^{E'}$, which will be called the *monotone mapping belonging to the filters* $\mathfrak{f}(x')$.

The following statement is obvious:

(2.2) Lemma. *If* h *is the monotone mapping belonging to the filters* $\mathfrak{f}(x')$, *then* $h(\emptyset)=\emptyset$ *and* $h(A)\cap h(B)=h(A\cap B)$ *for* $A, B\subset E$, *consequently* h *has property* (I_2), *too.* ∎

(2.3) Theorem. *Let us suppose that* g *is a mapping of the syntopogenous space* $[E, \mathscr{S}]$ *into a set* E', *and* $\mathfrak{f}(x')$ *is a filter in* E *for any* $x'\in E'$. *Then the monotone mapping* h *belonging to these filters is an inductor subordinated to* $\mathscr{S}$ *and* g *if, and only if*

$$\mathscr{S}(\mathfrak{f}(g(x))) = \mathscr{S}(x) \tag{2.3.1}$$

holds for any $x\in E$. *In this case with the notation* $\mathscr{S}'=h(\mathscr{S})$, *we have the equality*

$$g^{-1}(\mathscr{S}'(x')) = \mathscr{S}(\mathfrak{f}(x')) \tag{2.3.2}$$

for any $x'\in E'$, *consequently, if* $\mathfrak{f}(x')$ *is* $\mathscr{S}$*-round, then* $g^{-1}(\mathscr{S}'(x'))$ *agrees with* $\mathfrak{f}(x')$.

Proof. Suppose that (2.3.1) is satisfied, and put $A<B$, where $<\in\mathscr{S}$. Then $x\in A$ implies $B\in\mathscr{S}(x)\subset\mathfrak{f}(g(x))$, therefore $g(x)\in h(B)$. Let x' be in $h(A)$. If $x'=g(x)$, $x\in E$, then $A\in\mathfrak{f}(g(x))$ implies $B\in\mathscr{S}(x)$, thus $x\in B$. On the basis of this $x'\in E'-g(E-B)$. Conversely, let us assume that (I_1) holds for h. If $B\in\mathscr{S}(x)$, $x\in E$, then $x<A<B$ for an order $<\in\mathscr{S}$, therefore $g(x)\in h(A)$, that is $A\in\mathfrak{f}(g(x))$, thus $B\in\mathscr{S}(\mathfrak{f}(g(x)))$. If $B\in\mathscr{S}(\mathfrak{f}(g(x)))$, then $A<C<B$ for a set $A\in\mathfrak{f}(g(x))$ and a suitable $<\in\mathscr{S}$. $h(A)\subset E'-g(E-C)$ and $g(x)\in h(A)$ imply $x\in C$, so that $B\in\mathscr{S}(x)$.

We prove the equality $g^{-1}(\mathscr{S}'(x'))=\mathscr{S}(\mathfrak{f}(x'))$. Assume $g^{-1}(X')\subset X$, where $x'h(<)^q X'$ for some $<\in\mathscr{S}$. Then

$$x'\in\bigcap_{j=1}^{n} h(A_j) \quad\text{and}\quad \bigcap_{j=1}^{n} h(B_j)\subset X',$$

where $A_j < B_j$ $(1 \leqq j \leqq n)$ for some natural number n. If $<_1 \in \mathcal{S}$, $< \sqsubset <_1^2$ and $A_j <_1 C_j <_1 B_j$ $(1 \leqq j \leqq n)$, then with

$$A = \bigcap_{j=1}^{n} A_j, \quad C = \bigcap_{j=1}^{n} C_j \quad \text{and} \quad B = \bigcap_{j=1}^{n} B_j$$

we have $A \in \mathfrak{f}(x')$ (cf. (2.2)), $A <_1 C$ and $C \subset g^{-1}(h(B)) \subset g^{-1}(X') \subset X$ (cf. (I_1)), thus $X \in \mathcal{S}(\mathfrak{f}(x'))$. On the other hand, if $X \in \mathcal{S}(\mathfrak{f}(x'))$, then $A < X$ for a suitable $A \in \mathfrak{f}(x')$ and $< \in \mathcal{S}$. If $<_1 \in \mathcal{S}$, $< \sqsubset <_1^2$ and $A <_1 C <_1 X$, one can show that $x' h(<_1) h(C)$ and $g^{-1}(h(C)) \subset X$ (cf. (I_1)), therefore $X \in g^{-1}(\mathcal{S}'(x'))$. Finally, if $\mathfrak{f}(x')$ is $\mathcal{S}$-round, then obviously $\mathcal{S}(\mathfrak{f}(x')) = \mathfrak{f}(x')$. ∎

(2.4) Corollary. *If g is an injection of the syntopogenous space $[E, \mathcal{S}]$ into a set E', and $\mathfrak{z}(x')$ is an $\mathcal{S}$-round filter in E for $x' \in E'$ such that $\mathfrak{z}(g(x)) = \mathcal{S}(x)$ for $x \in E$, then denoting by s the monotone mapping belonging to the filters $\mathfrak{z}(x')$, $(E', s(\mathcal{S}), g)$ is an extension of $[E, \mathcal{S}]$, the trace filters of which are identical with the filters $\mathfrak{z}(x')$.* ∎

(2.5) Corollary. *Under the conditions of* (2.4) *$(E', s(\mathcal{S})^{tp}, g)$ is a strict extension* (see [3], ch. 6) *of the topological space $[E, \mathcal{S}^{tp}]$ belonging to the filters $\mathfrak{z}(x')$.*

Proof. As it can be easily seen, $V' \subset E'$ is an $s(\mathcal{S})$-neighbourhood of a point $x' \in E'$, iff there is a set $V \in \mathfrak{z}(x')$ such that $s(V) \subset V'$. Because of the $\mathcal{S}$-roundness of $\mathfrak{z}(x')$, the set V can be $\mathcal{S}^{tp}$-open. ∎

In view of (2.5) an extension $(E', \mathcal{S}, g)$ of the syntopogenous space $[E, \mathcal{S}]$ will be called *tight*, if for the monotone mapping s belonging to the trace filters of this extension the equivalence $s(\mathcal{S}) \sim \mathcal{S}'$ is valid. We shall give two characterizations of tight extensions (see (2.10) and (3.7)). The first of these demands the following generalization of the properties of the monotone mapping belonging to a family of filters.

A monotone mapping h will be said to be $(\mathcal{S}, \cap)$*-preserving* (or $(\mathcal{S}, \cup)$-*preserving*), if $h(\emptyset) = \emptyset$, and for an arbitrary set I of indices, $< \in \mathcal{S}$, $A_i < B_i$ $(i \in I)$ imply $\bigcap_{i \in I} h(A_i) \subset h(\bigcap_{i \in I} B_i)$ (or $h(\bigcup_{i \in I} A_i) \subset \bigcup_{i \in I} h(B_i)$). If this condition is satisfied by h only for finite sets I of indices, then it will be called *finitely* $(\mathcal{S}, \cap)$-preserving (or *finitely* $(\mathcal{S}, \cup)$-preserving).

The finite case can be reduced as follows:

(2.6) Lemma. *A monotone mapping h is finitely $(\mathcal{S}, \cap)$-preserving (or finitely $(\mathcal{S}, \cup)$-preserving), iff $h(\emptyset) = \emptyset$, and $A_1 < B_1$, $A_2 < B_2$ imply $h(A_1) \cap h(A_2) \subset h(B_1 \cap B_2)$ (or $h(A_1 \cup A_2) \subset h(B_1) \cup h(B_2)$) for any $< \in \mathcal{S}$.*

Proof. One of the parts of this statement is trivial. We verify the other part with an inductive proof. Let m be a natural number, and let us suppose that $A_j < B_j$ $(1 \leqq j \leqq n \leqq m)$ implies $\bigcap_{j=1}^{n} h(A_j) \subset h(\bigcap_{j=1}^{n} B_j)$ for any $< \in \mathcal{S}$, finally put $n = m+1$. If $< \in \mathcal{S}$, $A_j < B_j$ $(1 \leqq j \leqq n)$, then there exists $<_1 \in \mathcal{S}$ such that $< \sqsubset <_1^2$. Assume $A_j <_1 C_j <_1 B_j$ $(1 \leqq j \leqq n)$. In this case

$$\bigcap_{j=1}^{m} h(A_j) \subset h\left(\bigcap_{j=1}^{m} C_j\right), \quad \bigcap_{j=1}^{m} C_j <_1 \bigcap_{j=1}^{m} B_j$$

and $A_n <_1 B_n$, hence

$$\bigcap_{j=1}^{n} h(A_j) \subset h\left(\bigcap_{j=1}^{m} C_j\right) \cap h(A_n) \subset h\left(\bigcap_{j=1}^{n} B_j\right).$$

The statement concerning the finitely $(\mathscr{S}, \cup)$-preserving case is totally similar. ∎

In our terminology lemma (2.2) can be formulated so that the monotone mapping belonging to a given family of filters is always finitely $(\mathscr{D}_E, \cap)$-preserving, where $\mathscr{D}_E = \{\subset\}$. But it is obviously finitely $(\mathscr{S}, \cap)$-preserving for any syntopogenous structure $\mathscr{S}$ on E, and in general we can state:

(2.7) Lemma. *If $\mathscr{S}_1 < \mathscr{S}_2$ and h is a (finitely) $(\mathscr{S}_2, \cap)$- or $(\mathscr{S}_2, \cup)$-preserving monotone mapping, then it is (finitely) $(\mathscr{S}_1, \cap)$- or $(\mathscr{S}_1, \cup)$-preserving, too.* ∎

(2.8) Lemma. *Any finitely $(\mathscr{S}, \cap)$-preserving monotone mapping has property* (I_2).

PROOF. In fact, if $< \in S$, and $A_j < B_j$ $(1 \leqq j \leqq n)$, then $\bigcap_{j=1}^{n} h(A_j) \subset h\left(\bigcap_{j=1}^{n} B_j\right) = h(\emptyset) = \emptyset$, provided $\bigcap_{j=1}^{n} B_j = \emptyset$. ∎

(2.9) Lemma. *Let h_1 and h_2 be two monotone mappings. If for any $< \in \mathscr{S}$, $A < B$ implies $h_1(A) \subset h_2(B)$ and $h_2(A) \subset h_1(B)$, then $h_1(\mathscr{S}) \sim h_2(\mathscr{S})$.*

PROOF. Obviously, for $< \sqsubset <_1^3$, $<$, $<_1 \in \mathscr{S}$, we have $h_1(<) \sqsubset h_2(<_1)$ and $h_2(<) \sqsubset h_1(<_1)$, therefore $h_1(\mathscr{S})$ and $h_2(\mathscr{S})$ are equivalent. ∎

(2.10) Theorem. *An extension $(E', \mathscr{S}', g)$ of a syntopogenous space $[E, \mathscr{S}]$ is tight, iff there exists a finitely $(\mathscr{S}, \cap)$-preserving inductor h subordinated to $\mathscr{S}$ and g such that $\mathscr{S}' \sim h(\mathscr{S})$.*

PROOF. If $(E', \mathscr{S}', g)$ is a tight extension, then $\mathscr{S}' \sim s(\mathscr{S})$, where s is the monotone mapping belonging to the trace filters, which is a finitely $(\mathscr{S}, \cap)$-preserving inductor subordinated to $\mathscr{S}$ and g. After this let h be a finitely $(\mathscr{S}, \cap)$-preserving inductor subordinated to $\mathscr{S}$ and g such that $\mathscr{S}' \sim h(\mathscr{S})$, and let $\mathfrak{z}(x')$ denote the trace filter of $x' \in E'$. First of all we show that for $X \subset E$: $s(X) \subset h(X)$, and $< \in \mathscr{S}$, $A < B$ imply $h(A) \subset s(B)$. In fact, $x' \in s(X)$ means that $X \in \mathfrak{z}(x')$, and from this $g^{-1}(X') \subset X$, $x' <' X'$ follows for some $<' \in \mathscr{S}'$. Let $<$ be an element of $\mathscr{S}$ such that $<' \sqsubset h(<)^q$. Then $A_j < B_j$ $(1 \leqq j \leqq n)$,

$$x' \in \bigcap_{j=1}^{n} h(A_j), \quad \bigcap_{j=1}^{n} h(B_j) \subset X'$$

hold for a suitable natural number n. If $<_1 \in \mathscr{S}$, $< \sqsubset <_1^2$ and $A_j <_1 C_j <_1 B_j$ $(1 \leqq j \leqq n)$, then using (I_1), we have

$$\bigcap_{j=1}^{n} C_j \subset \bigcap_{j=1}^{n} g^{-1}(h(B_j)) \subset X$$

and

$$x' \in \bigcap_{j=1}^{n} h(A_j) \subset h\left(\bigcap_{j=1}^{n} C_j\right) \subset h(X).$$

On the other hand, put $<\mathscr{S}$ and $A<B$. If $<\subset<_1^2$, $<_1\in\mathscr{S}$, $A<_1C<_1B$ and $<'\in\mathscr{S}'$ such that $h(<_1)^q\subset<'$, then $x'\in h(A)$ implies $x'<'h(C)$, and because of (I_1) $g^{-1}(h(C))\subset B$. Thus $B\in\mathfrak{z}(x')$, i.e. $x'\in s(B)$. Finally lemma (2.9) gives that $\mathscr{S}'\sim h(\mathscr{S})\sim s(\mathscr{S})$, hence the extension in question is tight. ∎

3. Extension of functions

Let $\mathbf{R}$ be the real line, $\overline{\mathbf{R}}=\mathbf{R}\cup\{-\infty,+\infty\}$, and let $\mathscr{I}=\{<_\varepsilon:\varepsilon>0\}$ be the natural syntopogenous structure on $\mathbf{R}$ defined in [1]. Throughout in this § it will be assumed that $g: E\to E'$ is a mapping, $\mathscr{S}$ is a syntopogenous structure on E, and $\mathfrak{z}(x')$ is a round filter in $[E,\mathscr{S}]$ for any $x'\in E'$, in particular $\mathfrak{z}(g(x))=\mathscr{S}(x)$, whenever $x\in E$.

If $f: E\to\mathbf{R}$ is a real-valued function, we can define a function $f^*: E'\to\overline{\mathbf{R}}$ by the following formula:

$$f^*(x')=\inf\{\sup f(A):\ A\in\mathfrak{z}(x')\}\quad (x'\in E').$$

(3.1) Lemma. *If $f^*(x')\in\mathbf{R}$, then this is the smallest of all numbers $p\in\mathbf{R}$ such that $f(\mathfrak{z}(x'))\to p\ (\mathscr{I})$.*

Proof. In fact, for a filter base $\mathfrak{r}$ in $\mathbf{R}$ the condition $\mathfrak{r}\to p\ (\mathscr{I})$ is equivalent to the inequality

$$\inf\{\sup R:\ R\in\mathfrak{r}\}\leqq p. \ ∎$$

(3.2) Lemma. *We have the following properties of f^*:*

(3.2.1) *If f is bounded, then f^* is bounded, too.*

(3.2.2) *$f\leqq f^*\circ g$, and if f is $(\mathscr{S}^{tp},\mathscr{I})$-continuous, then here equality stands.*

Proof. (3.2.1) is obvious. (3.2.2): $A\in\mathfrak{z}(g(x))=\mathscr{S}(x)$ implies $x\in A$, therefore $f(x)\leqq\sup f(A)$, and from this we get $f(x)\leqq f^*(g(x))$. If f is $(\mathscr{S}^{tp},\mathscr{I})$-continuous, then $f(\mathfrak{z}(g(x)))=f(\mathscr{S}(x))\to f(x)\ (\mathscr{I})$, and we have $f^*(g(x))\leqq f(x)$ (cf. (3.1)). ∎

(3.3) Lemma. *Let s be the monotone mapping belonging to the filters $\mathfrak{z}(x')$, and suppose that f is bounded. Then we have $f^{*-1}(<_\varepsilon)\subset s(f^{-1}(<_{\varepsilon/4}))$ and $s(f^{-1}(<_\varepsilon))\subset f^{*-1}(<_\varepsilon)$, for each real number $\varepsilon>0$.*

Proof. Assume $A'f^{*-1}(<_\varepsilon)B'$. This implies the existence of a $p\in\mathbf{R}$ such that $A'\subset f^{*-1}((-\infty,p])$ and $f^{*-1}((-\infty,p+\varepsilon))\subset B'$. If $A=f^{-1}((-\infty,p+\varepsilon/4])$, $B=f^{-1}((-\infty,p+\varepsilon/2))$, and $x'\in A'$, then $X\subset A$ for some $X\in\mathfrak{z}(x')$, thus $A\in\mathfrak{z}(x')$. If $B\in\mathfrak{z}(x')$, then $f^*(x')\leqq p+\varepsilon/2$, therefore $x'\in B'$. We got $A's(f^{-1}(<_{\varepsilon/4}))B'$, because $Af^{-1}(<_{\varepsilon/4})B$.

Conversely, suppose $A's(f^{-1}(<_\varepsilon))B'$. There exist $A,B\subset E$ such that $Af^{-1}(<_\varepsilon)B$, $A'\subset s(A)$ and $s(B)\subset B'$. Let p be the supremum of $f(A)$. $x'\in A'$ implies $A\in\mathfrak{z}(x')$, thus $f^*(x')\leqq p$. If for an $x'\in E'$ the inequality $f^*(x')<p+\varepsilon$ holds, then we can take a set $X\subset E$ satisfying the conditions $X\in\mathfrak{z}(x')$ and $\sup f(X)<p+\varepsilon$, thus $X\subset f^{-1}((-\infty,p+\varepsilon))\subset B$. Consequently $B\in\mathfrak{z}(x')$, therefore $x'\in B'$. This means that $A'f^{*-1}(<_\varepsilon)B'$. ∎

(3.4) Theorem. *Let s be the monotone mapping belonging to the filters $\mathfrak{z}(x')$. Then for any bounded $(\mathcal{S}, \mathcal{I})$-continuous function f there exists a bounded $(s(\mathcal{S}), \mathcal{I})$-continuous function f' such that $f = f' \circ g$.* ■

(3.5) Corollary. *If $(E', \mathcal{S}', g)$ is a tight extension of the syntopogenous space $[E, \mathcal{S}]$, then every boounded $(\mathcal{S}, \mathcal{I})$-continuous function f has a bounded $(\mathcal{S}', \mathcal{I})$ continuous extension f' onto E', i.e. for which $f = f' \circ g$.* ■

(3.6) Corollary. *If $(E', \mathcal{T}', g)$ is an extension of the topological space $[E, \mathcal{T}]$ such that $\mathcal{T}'$ is a topology, then any bounded $(\mathcal{T}, \mathcal{I})$-continuous function f has a bounded $(\mathcal{T}', \mathcal{I})$-continuous extension f' onto E', i.e. for which $f = f' \circ g$.*

PROOF. Let s be the monotone mapping belonging to the trace filters of this extension. Then $s(\mathcal{T})^p$ is coarser than $\mathcal{T}'$ (see (2.5)), therefore every $(s(\mathcal{T})^p, \mathcal{I})$-continuous function is $(\mathcal{T}', \mathcal{I})$-continuous, too. ■

Supposing that Φ is an arbitrary functional structure on E (see ch. 12 of [1]), we have a functional structure Φ^* on E' determined as follows:

$$\Phi^* = \{\varphi^* \colon \varphi \in \Phi\},$$

where

$$\varphi^* = \{f^* \colon f \in \varphi\}.$$

(3.7) Theorem. *Let s denote the monotone mapping belonging to the filters $\mathfrak{z}(x')$. If $\mathcal{S} \sim \mathcal{S}_\Phi$ for an ordering structure Φ on E such that*

(3.7.1) $$\varphi_i \in \Phi \ (1 \leqq i \leqq n) \ \textit{implies} \ \bigcup_{i=1}^{n} \varphi_i \subset \varphi \ \textit{for some} \ \varphi \in \Phi,$$

then $s(\mathcal{S}) \sim \mathcal{S}_{\Phi^*}$.

PROOF. If $\{<_i : i \in I\}$ is a family of semi-topogenous orders on E, then

(3.7.2) $$s\Big(\bigcup_{i \in I} <_i\Big) = \bigcup_{i \in I} s(<_i)$$

Using the terminology of ch. 12 of [1], on the basis of (3.3) and (3.7.2) we can state $<_{\varphi^*, \varepsilon} = \big(\bigcup_{f \in \varphi} f^{*-1}(<_\varepsilon)\big)^q \subset \big(\bigcup_{f \in \varphi} s(f^{-1}(<_{\varepsilon/4}))\big)^q = s\big(\bigcup_{f \in \varphi} f^{-1}(<_{\varepsilon/4})\big)^q = s(<_{\varphi, \varepsilon/4})^q$. From this $\mathcal{S}_{\varphi^*} < s(\mathcal{S}_\varphi) < s(\mathcal{S})$, therefore $\mathcal{S}_{\varphi^*} < s(\mathcal{S})$.

Conversely, $s(<_{\varphi, \varepsilon})^q = s\big(\bigcup_{f \in \varphi} f^{-1}(<_\varepsilon)\big)^q = \big(\bigcup_{f \in \varphi} s(f^{-1}(<_\varepsilon))\big)^q \subset \big(\bigcup_{f \in \varphi} f^{*-1}(<_\varepsilon)\big)^q = {<_{\varphi^*, \varepsilon}}$, that is $s(\mathcal{S}_\varphi) < \mathcal{S}_{\varphi^*}$. From (3.7.1) $\mathcal{S} \sim \mathcal{S}_\Phi \sim \bigcap_{\varphi \in \Phi} \mathcal{S}_\varphi$ follows, thus $s(\mathcal{S}) \sim s\big(\bigcap_{\varphi \in \Phi} \mathcal{S}_\varphi\big) = \bigcap_{\varphi \in \Phi} s(\mathcal{S}_\varphi) < \bigcap_{\varphi \in \Phi} \mathcal{S}_{\varphi^*} \subset \bigvee_{\varphi \in \Phi} \mathcal{S}_{\varphi^*} = \mathcal{S}_{\Phi^*}$. ■

(3.7.) shows that if an ordering structure Φ satisfying (3.7.1) compatible with $\mathcal{S}$ is known, then $s(\mathcal{S})$ can be determined without an effective using of s, namely Φ^* is compatible with $s(\mathcal{S})$. This gives a new definition of tight extensions, since such a compatible Φ can be always found for $\mathcal{S}$ (see (12,37) and (12.29) of [1]).

Let us observe that the finiteness of $f^*(x')$ is not guaranteed in the general case, therefore a point $x' \in E'$ will be called *finite from below (from above)*, iff $-\infty < f^*(x')$ $(f^*(x') < +\infty)$ for any $(\mathcal{S}, \mathcal{I})$-continuous function f on E. Otherwise x' is *infinite* from below (or above).

For the sake of a characterization of points finite from below or above, let us consider the following notion. A countable system $\mathfrak{B} = \{B_n : n = 0, 1, \ldots\}$ will be said to be *decreasing* in the syntopogenous space $[E, \mathcal{S}]$ (or simply in $\mathcal{S}$), if there is an order $<\in\mathcal{S}$ such that $B_{n+1}<B_n$ for any natural number n. For two systems of sets $\mathfrak{A}$ and $\mathfrak{B}$, we shall use the notation $\mathfrak{A}(\cap)\mathfrak{B}=\{A\cap B: A\in\mathfrak{A}, B\in\mathfrak{B}\}$. Now we can state:

(3.8) Theorem. *A point $x'\in E'$ is finite from below (from above), iff for any decreasing system $\mathfrak{B}$ in $\mathcal{S}$ (in $\mathcal{S}^c$), $\mathfrak{B}\subset\mathfrak{z}(x')$ $(\emptyset\notin\mathfrak{B}(\cap)\mathfrak{z}(x'))$ implies $\cap\mathfrak{B}\neq\emptyset$.*

The proof of the theorem is based upon the following lemma:

(3.9) Lemma. *Suppose that $\mathfrak{B}$ is a decreasing system in a syntopogenous space $[E, \mathcal{S}]$, and $\cap\mathfrak{B}=\emptyset$. Then there exists an $(\mathcal{S}, \mathcal{I})$-continuous function f on E such that for any natural number n, $x\in B_n$ implies $f(x)\leqq -n$.*

PROOF. Let $<$ denote an order of $\mathcal{S}$ such that for any natural number n the inequality $B_{n+1}<B_n$ holds. If $\{<_n: n=0, 1, \dots\}\subset\mathcal{S}$ for which $<_0=<$ and $<_n\subset<_{n+1}^2$ $(n=0, 1, \dots)$, then for each n there exists a function $t_n: E\to[0, 1]$ such that

$$\varepsilon>\frac{1}{2^m} \quad \text{implies} \quad t_n^{-1}(<_\varepsilon)\subset<_{m+1}, \tag{3.9.1}$$

$t_n(B_{n+1})=\{0\}$ and $t_n(E-B_n)=\{1\}$ (see [1], (12.41)). Putting $f_n=t_n-(n+1)$ for any natural number n, f_n also satisfies (3.9.1). We define a function f as follows:

$$f(x)=\begin{cases} 0 & \text{for} \quad x\in E-B_0 \\ f_n(x) & \text{for} \quad x\in B_n-B_{n+1} \end{cases} \quad (x\in E).$$

Because of $E=E-\bigcap\limits_{n=0}^{\infty} B_n=(E-B_0)\cup\left(\bigcup\limits_{n=0}^{\infty}(B_n-B_{n+1})\right)$, this definition is possible and unambigous. If $x\in B_n$, then $x\in B_m-B_{m+1}$ for some $m\geqq n$, hence $f(x)=f_m(x)=$ $=t_m(x)-(m+1)\leqq 1-(m+1)=-m\leqq -n$. We show that if r is a real number such that $-(n+1)\leqq r<-n$, then

$$f^{-1}((-\infty, r])\subset f_n^{-1}((-\infty, r]) \tag{3.9.2}$$

and

$$f_n^{-1}((-\infty, r))\subset f^{-1}((-\infty, r]). \tag{3.9.3}$$

In fact, suppose $f(x)\leqq r$. If $x\notin B_n$, then $f(x)=f_m(x)$ for some $m<n$, thus $0=$ $=-n+n>r+n\geqq f(x)+n\geqq f(x)+m+1=t_m(x)$, which is impossible by $t_m(E)\subset[0, 1]$, therefore $x\in B_n$. If $x\in B_{n+1}$, then $t_n(x)=0$, hence $f_n(x)=t_n(x)-(n+1)=$ $=-(n+1)\leqq r$. If $x\notin B_{n+1}$, then $x\in B_n-B_{n+1}$, that is $f_n(x)=f(x)\leqq r$.

Conversely, suppose $f_n(x)<r$. Then $t_n(x)<r+n+1<1$, so that $x\in B_n$. If $x\notin B_{n+1}$, then $x\in B_n-B_{n+1}$, and $f(x)=f_n(x)<r$. If $x\in B_{n+1}$, then for some $k\geqq n+1$ we have $f(x)=f_k(x)=t_k(x)-(k+1)\leqq 1-(k+1)=-k\leqq -(n+1)\leqq r$.

Further we shall verify the $(\mathcal{S}, \mathcal{I})$-continuity of f. Let ε be a positive real number; without loss of generality we can assume that $\varepsilon<1$. If $Xf^{-1}(<_\varepsilon)Y$, then $X\subset f^{-1}((-\infty, p]), f^{-1}((-\infty, p+\varepsilon))\subset Y$ for a suitable $p\in\mathbf{R}$. If $p\geqq 0$, then $Y=E$. Suppose $p<0$, and let n_0 denote the greatest natural number such that $p<-n_0$. The nobviously $-(n_0+1)\leqq p$, and $p+\varepsilon<-(n_0-1)$. If $p+\varepsilon/3<-n_0$, then by (3.9.2)—(3.9.3) $X\subset f_{n_0}^{-1}((-\infty, p])$ and $f_{n_0}^{-1}((-\infty, p+\varepsilon/3))\subset Y$. If $-n_0\leqq p+\varepsilon/3$, then

$X\subset f^{-1}_{n_0-1}((-\infty, p+\varepsilon/3])$ and $f^{-1}_{n_0-1}((-\infty, p+2\varepsilon/3))\subset Y$, and we get

$$Xf^{-1}_{n_0}(<_{\varepsilon/3})Y \quad\text{or}\quad Xf^{-1}_{n_0-1}(<_{\varepsilon/3})Y.$$

Both in the one and in the other case, we have $X<_{m+1}Y$, where m is a natural number such that $\varepsilon/3>\frac{1}{2^m}$. From this we deduce $f^{-1}(<_\varepsilon)\sqsubset <_{m+1}$, and this means that f is $(\mathcal{S},\mathcal{I})$-continuous. ∎

Proof of (3.8). If $x'\in E'$ is infinite from below, then for an $(\mathcal{S},\mathcal{I})$-continuous function f on E, $f^*(x')=-\infty$. Then for any natural number n, there is $A\in\mathfrak{z}(x')$ such that $\sup f(A)\leqq -n$. In this case $\mathfrak{B}=\{f^{-1}((-\infty, -n]): n=0, 1, \dots\}$ is a decreasing system in $[E,\mathcal{S}]$, $\mathfrak{B}\subset\mathfrak{z}(x')$, and obviously $\bigcap\mathfrak{B}=\emptyset$. Conversely, if $\mathfrak{B}=\{B_n: n=0, 1, \dots\}$ is a decreasing system in $[E,\mathcal{S}]$, for which $\mathfrak{B}\subset\mathfrak{z}(x')$ and $\bigcap\mathfrak{B}=\emptyset$, then $f^*(x')\leqq\inf\{\sup f(B_n): n=0, 1, \dots\}=-\infty$, where f is the $(\mathcal{S},\mathcal{I})$-continuous function constructed in (3.9).

Suppose that $x'\in E'$ is infinite from above. Then $f^*(x')=+\infty$ for an $(\mathcal{S},\mathcal{I})$-continuous function f, and this means $\sup f(A)=+\infty$ for every $A\in\mathfrak{z}(x')$, hence

$$\mathfrak{B}=\{f^{-1}((n, +\infty)): n=0, 1, \dots\}$$

is a decreasing system in $\mathcal{S}^c$ such that $\emptyset\notin\mathfrak{B}(\bigcap)\mathfrak{z}(x')$ and clearly $\bigcap\mathfrak{B}=\emptyset$. Conversely, let us assume that $\mathfrak{B}=\{B_n: n=0, 1, \dots\}$ is a decreasing system in $\mathcal{S}^c$ such that $\emptyset\notin\mathfrak{B}(\bigcap)\mathfrak{z}(x')$ and $\bigcap\mathfrak{B}=\emptyset$. Then by (3.9) there exists an $(\mathcal{S}^c,\mathcal{I})$-continuous function f_0 such that $f_0(x)\leqq -n$, whenever $x\in B_n$. In this case $f=-f_0$ is an $(\mathcal{S},\mathcal{I})$-continuous function, for which in every $A\in\mathfrak{z}(x')$ a point x lies with $f(x)\geqq n$. This shows that $\sup f(A)=+\infty$ for each $A\in\mathfrak{z}(x')$, so that $f^*(x')=+\infty$. ∎

(3.10) Theorem. *If $\mathcal{S}$ is symmetrical, and $\mathfrak{z}(x')$ is compressed in $\mathcal{S}$, then the point $x'\in E'$ is finite from above, iff it is one from below.*

Proof. Suppose that there is a decreasing system $\mathfrak{B}$ in $\mathcal{S}$ such that $\emptyset\notin\mathfrak{z}(x')(\bigcap)\mathfrak{B}$ and $\bigcap\mathfrak{B}=\emptyset$. Then denoting by $<$ an order of $\mathcal{S}$ such that $B_{n+1}<B_n$ for any $B_n\in\mathfrak{B}$, we get an other $<_1\in\mathcal{S}$, for which $<\sqsubset<_1^2$, so that $B_{n+1}<_1C_n<_1B_n$ for suitable sets C_n $(n=0, 1, \dots)$. $C_n\cap A\neq\emptyset$ for each $A\in\mathfrak{z}(x')$, hence $B_n\in\mathfrak{z}(x')$, that is $\mathfrak{B}\subset\mathfrak{z}(x')$.

On the other hand, if $\mathfrak{B}\subset\mathfrak{z}(x')$ for a decreasing system $\mathfrak{B}$ in $\mathcal{S}$, then $\emptyset\notin\mathfrak{z}(x')(\bigcap)\mathfrak{B}$ is clear. This shows that the definitions of points finite from below and that of finite from above are equivalent (see (3.8)). ∎

(3.11) Theorem. *If $\mathcal{S}=\mathcal{T}$ is a topogenous structure, then $x'\in E'$ is finite from below, iff $\mathfrak{z}(x')$ is strongly centred, that is $A_n\in\mathfrak{z}(x')$ $(n=0, 1, \dots)$ implies $\bigcap_{n=0}^{\infty} A_n\neq\emptyset$.*

Proof. If $\mathfrak{z}(x')$ is strongly centred, then $\bigcap\mathfrak{B}\neq\emptyset$ for any decreasing system $\mathfrak{B}$ in $\mathcal{T}$ contained in $\mathfrak{z}(x')$, therefore by (3.8) x' is finite from below. Conversely, suppose that $\mathfrak{z}(x')$ is not strongly centred, then there is a countable sequence $\{A_n: n=0, 1, \dots\}\subset\mathfrak{z}(x')$ with $\bigcap_{n=0}^{\infty} A_n=\emptyset$. We construct a system $\{B_n: n=0, 1, \dots\}\subset\subset\mathfrak{z}(x')$ for which $B_{n+1}<B_n$ $(n=0, 1, \dots$ and $\mathcal{T}=\{<\})$, further $\bigcap_{n=0}^{\infty} B_n=\emptyset$. Assume that for some fixed natural number n, $0<m\leqq n$ implies the existence of a set $B_m\in\mathfrak{z}(x')$ such that $B_m<B_{m-1}$ and $B_m\subset A_m$, where $B_0=A_0$. $\mathfrak{z}(x')$ is an $\mathcal{T}$-round filter, therefore we can find a set $C\in\mathfrak{z}(x')$, for which $C<B_n$. Put $B_{n+1}=C\cap A_{n+1}$. Continuing this procedure, we arrive at the demanded system. By (3.8) this means that x' cannot be finite from below. ∎

Applying theorems (3.10) and (3.11) in that case, when $\mathcal{T}$ is the finest symmetrical topogenous structure inducing a given completely regular (or in the terminology of [1] *uniformizable*) topology on E, we get a well-known property of the Čech—Stone compactification, namely in that a point is *finite* in the sense of [3] (finite from below or equivalently from above in our terminology), iff the corresponding trace filter is strongly centred (see ch. 6.4.e of [3]).

4. Conditions of density of $g(E)$ in $[E', \mathcal{S}'^k]$

Let ^{k} be an ordinary operator in the sense of [1]. If $(E', \mathcal{S}', g)$ is an extension of the syntopogenous space $[E, \mathcal{S}]$, and $g(E)$ is dense in $[E', \mathcal{S}'^k]$, then $(E', \mathcal{S}'^k, g)$ is obviously an extension of $[E, \mathcal{S}^k]$. In order that this principle be applicable for making syntopogenous extensions (with $\mathcal{S}'=\mathcal{S}'^k$ and $\mathcal{S}=\mathcal{S}^k$), we need to know the conditions of the density of $g(E)$ in $[E', \mathcal{S}'^k]$. We shall consider only the operators $^k={}^b, {}^s$ and ^{sb}, which are the most particular cases, at the same time they are the most important ones.

Let $(E', \mathcal{S}', g)$ denote a tight extension of $[E, \mathcal{S}]$. For the sake of the formulation of a condition of the density of $g(E)$ in $[E', \mathcal{S}'^b]$, we shall generalize the notion of a Corson filter base of a uniform space (see [3], ch. 8.2.c). A filter base $\mathfrak{r}$ will be said to be *Corson* in the syntopogenous space $[E, \mathcal{S}]$, iff for an arbitrary set I of indices, $<\in\mathcal{S}$, $R_i\in\mathfrak{r}$ and $R_i<B_i$ $(i\in I)$ imply $\bigcap_{i\in I} B_i\neq\emptyset$.

Let $\mathfrak{P}(<)$ be, for $<\in\mathcal{S}$, the family of all sets $P\subset E$ such that $A<B$, $P\cap A\neq\emptyset$ imply $P\subset B$ (see [1], p. 220). If $<$ is symmetrical biperfect, and U is the symmetrical reflexive relation associated with that, then $P\in\mathfrak{P}(<)$, iff $(x, y)\in U$ for $x, y\in P$.

(4.1) Lemma. *If for every $<\in\mathcal{S}$ there exists a member of $\mathfrak{P}(<)$, which intersects any element of a filter base $\mathfrak{r}$ in E, then $\mathfrak{r}$ is Corson in $[E, \mathcal{S}]$. The converse is also true, provided $\mathcal{S}$ is a symmetrical syntopology.*

Proof. If $<\in\mathcal{S}$ and $P\in\mathfrak{P}(<)$, then for any set I of indices, $R_i\in\mathfrak{r}$, $R_i<B_i$, $P\cap R_i\neq\emptyset$ $(i\in I)$ imply $\emptyset\neq P\subset\bigcap_{i\in I} B_i$. Conversely, suppose that $\mathcal{S}$ is a symmetrical syntopology, and let $\mathcal{U}$ be the uniformity associated with $\mathcal{S}$. If $\mathfrak{r}$ is Corson in $[E, \mathcal{S}]$, and $U\in\mathcal{U}$ is associated with an arbitrary $<\in\mathcal{S}$, then there exists $U_1\in\mathcal{U}$ such that $U_1^2\subset U$. It is obvious that we have a point $x\in\bigcap\{U_1(R): R\in\mathfrak{r}\}$. Then $P=U_1(x)$ is in $\mathfrak{P}(<)$, since from the symmetricity of U_1 the relation $(y, z)\in U$ follows for any $y, z\in P$. Finally if $R\in\mathfrak{r}$, then there is a point $y\in R$ such that $(y, x)\in U_1$, therefore $y\in P\cap R$. ∎

As a simple corollary of (3.8), we can state:

(4.2) Lemma. *If $\mathfrak{z}(x')$ is a Corson filter in $[E, \mathcal{S}]$, then the point x' is finite from below.* ▮

(4.3) Theorem. *Let $(E', \mathcal{S}', g)$ be a tight extension of the syntopogenous space $[E, \mathcal{S}]$ with the trace filter $\mathfrak{z}(x')$ for $x' \in E'$. Then the following statements are equivalent:*

(4.3.1) *$g(E)$ is dense in $[E', \mathcal{S}'^b]$.*

(4.3.2) *For every $x' \in E'$, $\mathfrak{z}(x')$ is Corson in $[E, \mathcal{S}]$.*

(4.3.3) *If $\{f_i: i \in I\}$ is an arbitrary $(\mathcal{S}, \mathcal{J})$-continuous family of functions on E, then*

$$\inf_{x \in E} \sup_{i \in I} f_i(x) \leqq \inf_{x' \in E'} \sup_{i \in I} f_i^*(x').$$

PROOF. (4.3.1)⇒(4.3.2). Suppose $A_i \in \mathfrak{z}(x')$ and $A_i < B_i$ for a given $< \in \mathcal{S}$ and for an arbitrary set I of indices. Let us consider an order $<' \in \mathcal{S}'$ such that $s(<)^q \sqsubset <'$, where s is the monotone mapping belonging to the trace filters. Then $x' <' s(B_i)$ for $i \in I$, therefore $x' <'^b \bigcap_{i \in I} s(B_i)$. From this we can deduce $\emptyset \neq g^{-1}(\bigcap_{i \in I} s(B_i)) \subset \bigcap_{i \in I} B_i$.

(4.3.2)⇒(4.3.3). Let us introduce the notations

$$q = \inf_{x \in E} \sup_{i \in I} f_i(x) \quad \text{and} \quad p = \inf_{x' \in E'} \sup_{i \in I} f_i^*(x)'.$$

Suppose $-\infty < p < +\infty$ and let ε denote an arbitrary positive real number. Then there is a point $x' \in E'$, for which $\sup_{i \in I} f_i^*(x') < p + \varepsilon/2$, consequently a set $A_i \in \mathfrak{z}(x')$ can be found with the property $\sup f(A_i) \leqq p + \varepsilon/2$ for every $i \in I$. This implies $A_i \subset f_i^{-1}((-\infty, p+\varepsilon/2]) f_i^{-1}(<_{\varepsilon/2}) f_i^{-1}((-\infty, p+\varepsilon))$. Because of the $(\mathcal{S}, \mathcal{J})$-continuity of the family of functions in question, one can choose an order $<$ of $\mathcal{S}$ such that $f_i^{-1}(<_{\varepsilon/2}) \sqsubset <$ for each $i \in I$, hence we have

$$A_i < f_i^{-1}((-\infty, p+\varepsilon)) \quad (i \in I).$$

From this we get a point $x \in \bigcap_{i \in I} f_i^{-1}((-\infty, p+\varepsilon))$, for which $\sup_{i \in I} f_i(x) \leqq p + \varepsilon$ is obvious. This shows that $q \leqq p + \varepsilon$ for any $\varepsilon > 0$, consequently $q \leqq p$. If $p = -\infty$, then with a similar train of thought $q = -\infty$ can be deduced. Finally, if $p = +\infty$, the inequality is clear.

(4.3.3)⇒(4.3.1). Suppose that Φ is an ordering structure compatible with $\mathcal{S}$ (see [1], (12.37)). Then $\mathcal{S}' \sim \mathcal{S}_{\Phi^*}$ (cf. (3.7)), hence for an arbitrary $<' \in \mathcal{S}'$ there exist real numbers $\varepsilon_1, \ldots, \varepsilon_n > 0$ and ordering families $\varphi_1, \ldots, \varphi_n \in \Phi$ such that

$$<' \sqsubset \left(\bigcup_{j=1}^{n} <_{\varphi_j^*, \varepsilon_j} \right)^q \sqsubset \left(\bigcup_{j=1}^{n} \left(\bigcup_{f \in \varphi_j} f^{*-1}(<_\varepsilon) \right)^q \right)^q =$$

$$= \left(\bigcup_{j=1}^{n} \bigcup_{f \in \varphi_j} f^{*-1}(<_\varepsilon) \right)^q = \left(\bigcup_{f \in \varphi} f^{*-1}(<_\varepsilon) \right)^q,$$

where $\varepsilon=\min\{\varepsilon_1, \dots, \varepsilon_n\}$ and $\varphi=\bigcup_{j=1} \varphi_j$ (cf. [1], (3.25)). It can be easily seen that $<'^b \subset (\bigcup_{f\in\varphi} f^{*-1}(<_\varepsilon))^{qb}=(\bigcup_{f\in\varphi} f^{*-1}(<_\varepsilon))^b$ (see [1], (5.22)). Suppose $x' <'^b V'$ for a point $x'\in E'$, then $x' f_i^{*-1}(<_\varepsilon) B_i'$ and $\bigcap_{i\in I} B_i' \subset V'$ holds $(f_i\in\varphi, i\in I)$. Since for any constant $c\in \mathbf{R}$ and for any function f, we have $(f+c)^*=f^*+c$, in view of axiom (F_2) of [1], the function f_i can be chosen so that $f_i^*(x')=0$ $(i\in I)$. Then $f_i^{*-1}\cdot((-\infty, \varepsilon))\subset B_i'$ for each $i\in I$. As a subfamily of the union of a finite number of $(\mathscr{S}, \mathscr{I})$-continuous families of functions, the family $\{f_i: i\in I\}$ is also $(\mathscr{S}, \mathscr{I})$-continuous (see [1], (12.33)). $\sup_{i\in I} f_i^*(x')=0$, therefore $\inf_{x\in E}\sup_{i\in I} f_i(x)\leqq 0$. By (3.2.2) we can state, for a suitable $x\in E$,

$$x\in \bigcap_{i\in I} f_i^{-1}((-\infty, \varepsilon)) = \bigcap_{i\in I} g^{-1}\big(f_i^{*-1}((-\infty, \varepsilon))\big) \subset g^{-1}(V'),$$

namely $g(x)\in g(E)\cap V'$. ∎

(4.4) Theorem. *$(E', \mathscr{S}', g)$ is a tight extension of the syntopogenous space $[E, \mathscr{S}]$ such that $g(E)$ is dense in $[E', \mathscr{S}'^b]$, iff $\mathscr{S}'\sim h(\mathscr{S})$ for a finitely $(\mathscr{S}, \cap)$ preserving inductor h subordinated to $\mathscr{S}$ and g with the following property:*

(I_3) If $<\in\mathscr{S}$, and $A_i<B_i$ $(i\in I)$ for an arbitrary set I of indices, then $\bigcap_{i\in I} B_i=\emptyset$ implies $\bigcap_{i\in I} h(A_i)=\emptyset$.

Proof. If $(E', \mathscr{S}', g)$ is a tight extension, then the monotone mapping s satisfies these conditions (see (2.2) and (4.3.2)). Conversely, suppose that $h(\mathscr{S})\sim\mathscr{S}'$ for a finitely $(\mathscr{S}, \cap)$-preserving inductor h subordinated to $\mathscr{S}$ and g satisfying (I_3). Then by (2.10) $(E', \mathscr{S}', g)$ is tight. If $<'\in\mathscr{S}'$ and $<\in\mathscr{S}$ such that $<'\subset h(<)^q$, then $<'^b\subset h(<)^{qb}=h(<)^b$, therefore $x'<'^b V'$ implies $x'\in\bigcap_{i\in I} h(A_i)$, $\bigcap_{i\in I} h(B_i)\subset V'$, where $A_i<B_i$ $(i\in I)$. If $<\subset<_1^2$, $<_1\in\mathscr{S}$ and $A_i<_1 C_i<_1 B_i$, then from (I_3) and (I_1) we can deduce

$$\emptyset\neq g\big(\bigcap_{i\in I} C_i\big)\subset\bigcap_{i\in I} g(C_i)\subset\bigcap_{i\in I} h(B_i)\subset V',$$

consequently $g(E)$ is dense in $[E', \mathscr{S}'^b]$. ∎

Let E, E' be two sets. By the *dual* of a mapping $h: 2^E\to 2^{E'}$ we shall mean the mapping $h': 2^E\to 2^{E'}$ determined by the following formula

$$h'(X) = E'-h(E-X) \quad (X\subset E)$$

(cf. [4], (2.12)). If h is monotone, then h' is also monotone. Let $<$ be a semi-topogenous order on E, then one can define a semi-topogenous order $h^*(<)$ on E' by

$$h^*(<)=\bigcup\{_{h(A),h(B)}: A<B\}.$$

If $\mathfrak{f}(x')$ is a filter in E for each $x'\in E'$, and h is the monotone mapping belonging to these, then for $X\subset E$ we have

$$h'(X)=\{x'\in E': \emptyset\notin\mathfrak{f}(x')(\cap)\{X\}\}$$

(cf. [5], (4)). In fact,

$$x' \in E' - h'(X) \Leftrightarrow E - X \in \mathfrak{f}(x') \Leftrightarrow \emptyset \in \mathfrak{f}(x')(\cap)\{X\}.$$

In this particular case $h^*(<)$ is topogenous, provided $<$ is one. If each filter $\mathfrak{f}(x')$ is compressed in the syntopogenous structure $\mathscr{S}$ on E, then

$$h^*(\mathscr{S}) = \{h^*(<):\ < \in \mathscr{S}\}$$

is a syntopogenous structure on E', for which

$$h^*(\mathscr{S}) \sim h(\mathscr{S}) \tag{4.5}$$

(cf. [5], (10)).

(4.6) Theorem. *Let* $(E', \mathscr{S}', g)$ *be an extension of the syntopogenous space* $[E, \mathscr{S}]$. *Then the following statements are equivalent:*

(4.6.1) $g(E)$ *is dense in* $[E', \mathscr{S}'^s]$.

(4.6.2) *For each* $x' \in E'$ *there exists a compressed filter* $\mathfrak{f}(x')$ *in* $[E, \mathscr{S}]$ *such that* $\mathscr{S}(\mathfrak{f}(g(x))) = \mathscr{S}(x)$ *for* $x \in E$, *and* $h^*(\mathscr{S}) \sim \mathscr{S}'$, *where* h *is the monotone mapping belonging to the filters* $\mathfrak{f}(x')$.

(4.6.3) $(E', \mathscr{S}'\ g)$ *is a tight extension of* $[E, \mathscr{S}]$, *and for any* $x' \in E'$ *there exists a compressed filter* $\mathfrak{f}(x')$ *in* $[E, \mathscr{S}]$ *such that the trace filters* $\mathfrak{z}(x')$ *of the extension agree with the filters* $\mathscr{S}(\mathfrak{f}(x'))$ $(x' \in E')$.

(4.6.4) $\mathscr{S}' \sim h(\mathscr{S})$ *for a both finitely* $(\mathscr{S}, \cap)$*-preserving and finitely* $(\mathscr{S}, \cup)$*-preserving inductor* h *subordinated to* $\mathscr{S}$ *and* g.

(4.6.5) $(E', \mathscr{S}', g)$ *is a tight extension of* $[E, \mathscr{S}]$, *and for any bounded* $(\mathscr{S}, \mathscr{I})$*-continuous function* f *there exists a unique bounded* $(\mathscr{S}', \mathscr{I})$*-continuous extension* f' *onto* E', *i.e. for which* $f' \circ g = f$.

(4.6.6) $(E', \mathscr{S}', g)$ *is a tight extension of* $[E, \mathscr{S}]$, *and*

$$\min\{f_1^*, f_2^*\} = (\min\{f_1, f_2\})^*$$

for any bounded $(\mathscr{S}, \mathscr{I})$*-continuous function* f_1, f_2 *on* E.

Remark. If under condition (4.6.1) $[E', \mathscr{S}']$ is relatively separated with respect to $g(E)$, then it can be embedded into the double compactification of $[E, \mathscr{S})$ (cf. [5], moreover [1], (16.45)).

Proof of (4.6). We shall prove the following implications:

$$\begin{array}{ccccc} (4.6.3) & \Rightarrow & (4.6.4) & \Leftarrow & (4.6.6) \\ \Uparrow & & \Downarrow & & \Uparrow \\ (4.6.2) & \Leftarrow & (4.6.1) & \Rightarrow & (4.6.5) \end{array}$$

(4.6.1)$\Rightarrow$(4.6.2). Let $\mathfrak{f}(x')$ be equal to $g^{-1}(\mathscr{S}'^s(x'))$ for each $x' \in E'$. Then $\mathfrak{f}(x')$ is compressed in $[E, \mathscr{S}]$ (cf. [1], (15.45), (15.52) and (15.53)). $g^{-1}(\mathscr{S}'^s) = g^{-1}(\mathscr{S}')^s \sim \mathscr{S}^s$ implies $\mathfrak{f}(g(x)) = \mathscr{S}^s(x)$ for $x \in E$. But $\mathscr{S}(x) \subset \mathscr{S}^s(x)$ is trivial, and owing to $\mathscr{S} < \mathscr{S}^2$ we have $\mathscr{S}(x) \subset \mathscr{S}(\mathscr{S}(x)) \subset \mathscr{S}(\mathscr{S}^s(x)) \subset \mathscr{S}(x)$, so that $\mathscr{S}(x) = \mathscr{S}(\mathscr{S}^s(x)) = \mathscr{S}(\mathfrak{f}(g(x)))$.

If $<' \in \mathcal{S}'$, $<'_1 \in \mathcal{S}'$ such that $<' \sqsubset <'^3_1$, and $< \in \mathcal{S}$, for which $g^{-1}(<'_1) \sqsubset <$, then $A' <' B'$ and $A' <'_1 C' <'_1 D' <'_1 B'$ imply $A' \subset h(g^{-1}(C'))$ and $h'(g^{-1}(D')) \subset B'$ where h is the monotone mapping belonging to the filters $\mathfrak{f}(x')$. Because of $g^{-1}(C') < g^{-1}(D')$ we get $A' h^*(<) B'$, thus $<' \sqsubset h^*(<)$.

If $< \in \mathcal{S}$, $<'$, $<'_1 \in \mathcal{S}'$ such that $< \sqsubset g^{-1}(<')$ and $<' \sqsubset <'^3_1$, then $A' h^*(<) B'$ implies the existence of sets $A, B \subset E$, for which $A < B$, $A' \subset h(A)$ and $h'(B) \subset B'$. If $g(A) <'_1 C' <'_1 D' <'_1 E' - g(E-B)$, then we have $A' \subset C'$ and $D' \subset B'$, hence $A' <'_1 B'$. With this $h^*(<) \sqsubset <'_1$.

(4.6.2)$\Rightarrow$(4.6.3). Let us consider the monotone mapping h belonging to the filters $\mathfrak{f}(x')$. Then by (4.5) $\mathcal{S}' \sim h^*(\mathcal{S}) \sim h(\mathcal{S})$, therefore in view of (2.2), (2.3) and (2.10) $(E', \mathcal{S}', g)$ is a tight extension, the trace filters of which agree with the filters $\mathcal{S}(\mathfrak{f}(x'))$ (see (2.3.2)).

(4.6.3)$\Rightarrow$(4.6.4). $(E', \mathcal{S}', g)$ is induced by the monotone mapping s belonging to the trace filters $\mathfrak{z}(x')$ of this extension, which is obviously finitely $(\mathcal{S}, \cap)$-preserving. Let $\mathfrak{f}(x')$ be a compressed filter in $[E, \mathcal{S}]$ for each $x' \in E'$, and put $\mathfrak{z}(x') = \mathcal{S}(\mathfrak{f}(x'))$. If $< \in \mathcal{S}$, $A_1 < B_1$, $A_2 < B_2$, $<_1 \in \mathcal{S}$, $< \sqsubset <^2_1$ and $A_i <_1 C_i <_1 B_i$ $(i=1, 2)$, then $A_1 \cup A_2 \in \mathfrak{z}(x')$ implies $C_1 \in \mathfrak{f}(x')$ or $C_2 \in \mathfrak{f}(x')$. From this we get $B_1 \in \mathfrak{z}(x')$ or $B_2 \in \mathfrak{z}(x')$, hence s is finitely $(\mathcal{S}, \cup)$-preserving by (2.6).

(4.6.4)$\Rightarrow$(4.6.1). Let h be a both finitely $(\mathcal{S}, \cap)$- and finitely $(\mathcal{S}, \uplus)$-preserving inductor subordinated to $\mathcal{S}$ and g, further suppose $h(\mathcal{S}) \sim \mathcal{S}'$. If $x' \in E'$, $<' \in \mathcal{S}'$ and $x' <'^s V'$, then there exists $< \in \mathcal{S}$ such that $<' \sqsubset h(<)^q$, therefore we have $<'^s \sqsubset h(<)^{qs} = h(<)^s = <''^q$, where $<'' = h(<) \sqcup h(<)^c$. We can find a natural number n such that $x' <'' B'_j$ $(1 \leq j \leq n)$, $\bigcap_{j=1}^{n} B'_j \subset V'$ for some sets $B'_j \subset E'$.

We decompose the set of indices into two sets as follows: $j \in I_1$, iff $x' h(<) B'_j$, and $j \in I_2$ otherwise. In this way $j \in I_1$ implies the existence of A_j, $B_j \subset E$, for which $A_j < B_j$ and $x' \in h(A_j)$, $h(B_j) \subset B'_j$, moreover $j \in I_2$ implies the existence of $C_j, D_j \subset E$, with $C_j < D_j$ and $E' - B'_j \subset h(C_j)$, $h(D_j) \subset E' - x'$ (namely in such a case $x' h(<)^c B'_j$). If $<_1 \in \mathcal{S}$, $< \sqsubset <^2_1$ and in addition $A_j <_1 X_j <_1 B_j$ for $j \in I_1$, $C_j <_1 Y_j <_1 D_j$ for $j \in I_2$, then in view of (I_1) we get the inclusions

$$\bigcap_{j \in I_1} X_j \subset \bigcap_{j \in I_1} g^{-1}(B'_j) \quad \text{and} \quad E - \bigcup_{j \in I_2} Y_j \subset \bigcap_{j \in I_2} g^{-1}(B'_j).$$

We can see that $\emptyset = V' \cap g(E)$ is impossible, because in this case

$$\bigcap_{j \in I_1} X_j \subset \bigcup_{j \in I_2} Y_j,$$

and

$$x' \in \bigcap_{j \in I_1} h(A_j) \subset h\Big(\bigcap_{j \in I_1} X_j\Big) \subset h\Big(\bigcup_{j \in I_2} Y_j\Big) \subset \bigcup_{j \in I_2} h(D_j),$$

but this fact contradicts the choice of the sets D_j $(j \in I_2)$.

(4.6.1)$\Rightarrow$(4.6.5). The proof of this implication is based upon the extension theorem (16.45) of [1].

(4.6.5)$\Rightarrow$(4.6.6). We know that under condition (4.6.5) the $(\mathcal{S}, \mathcal{J})$-continuity of f_1, f_2 implies the $(\mathcal{S}', \mathcal{J})$-continuoty of f_1^*, f_2^*. We can easily show that both min $\{f_1^*, f_2^*\}$ and (min $\{f_1, f_2\})^*$ are $(\mathscr{S}', \mathscr{I})$-continuous extensions of the $(\mathscr{S}, \mathscr{I})$-continuous function min $\{f_1, f_2\}$, therefore these are equal.

$(4.6.6)\Rightarrow(4.6.4)$. We prove that the monotone mapping s belonging to the trace filters $\mathfrak{z}(x')$ of this extension is finitely $(\mathscr{S}, \cup)$-preserving. In view of [1], (12.10) and (12.27), we have an ordering structure Φ on E inducing $\mathscr{S}$ such that $\varphi_j\in\Phi$ $(1\leqq j\leqq n)$ implies the existence of a $\varphi\in\Phi$ fulfilling $\bigcup_{j=0}^{n}\varphi_j\subset\varphi$. In this case $\mathscr{S}\sim\mathscr{S}_\Phi\sim\bigcup_{\varphi\in\Phi}\mathscr{S}_\varphi$. Put $<\in\mathscr{S}$, $A_1<B_1$ and $A_2<B_2$. There exist $\varepsilon>0$ and $\varphi\in\Phi$ such that $<\subset<_{\varphi,\varepsilon}$. Then $A_j\subset f_j^{-j}((-\infty,0])$, $f_j^{-j}((-\infty,\varepsilon))\subset B_j$ for some $f_j\in\varphi$ $(j=1,2)$. If $f_0=\min\{f_1,f_2\}$ and $x'\in h(A_1\cup A_2)$, then

$$A_1\cup A_2\subset\bigcup_{j=1}^{2}f_j^{-1}((-\infty,0])=f_0^{-1}((-\infty,0]),$$

hence $f_0^{-1}((-\infty,0])\in\mathfrak{z}(x')$. But in this case, because of the assumption concerning f_0: min $\{f_1^*,f_2^*\}(x')=f_0^*(x')\leqq 0$. For example put $f_1^*(x')\leqq 0$, then there exists $X\in\mathfrak{z}(x')$ such that $\sup f_1(X)<\varepsilon$, thus $X\subset f_1^{-1}((-\infty,\varepsilon))\subset B_1$, so that $x'\in h(B_1)$.

The proof is complete. ∎

In the case of ${}^k={}^{sb}$ we have a theorem analogous with (4.6).

(4.7) Theorem. *For any extension $(E', \mathscr{S}', g)$ of a syntopogenous space $[E, \mathscr{S}]$ the following statements are equivalent:*

(4.7.1) *$g(E)$ is dense in $[E', \mathscr{S}'^{sb}]$.*

(4.7.2) *For every $x'\in E'$ there exists a Cauchy filter $\mathfrak{f}(x')$ in $[E,\mathscr{S}]$ such that $\mathscr{S}(\mathfrak{f}(g(x)))=\mathscr{S}(x)$ for $x\in E$, and $h^*(\mathscr{S})\sim\mathscr{S}'$, where h is the monotone mapping belonging to the filters $\mathfrak{f}(x')$.*

(4.7.3) *$(E', \mathscr{S}', g)$ is a tight extension of $[E,\mathscr{S}]$, and for each $x'\in E'$ there exists a Cauchy filter $\mathfrak{f}(x')$ in $[E,\mathscr{S}]$ such that the trace filters of this extension agree with the filters $\mathscr{S}(\mathfrak{f}(x'))$ $(x'\in E')$.*

(4.7.4) *$\mathscr{S}'\sim h(\mathscr{S})$ for a both $(\mathscr{S},\cap)$- and $(\mathscr{S},\cup)$-preserving inductor h subordinated to $\mathscr{S}$ and g.* ∎

Remark. If under condition (4.7.1) $\mathscr{S}'$ is relatively separated with respect to $g(E)$, then it can be embedded into the *completion* of $[E,\mathscr{S}]$. This is a consequence of theorem (16.30) of [1].

PROOF. We prove only $(3.7.3)\Rightarrow(4.7.4)$, because the verification of $(4.7.1)\Rightarrow\Rightarrow(4.7.2)\Rightarrow(4.7.3)\Rightarrow$ and $(4.7.4)\Rightarrow(4.7.1)$ is closely similar to that of the corresponding part of (4.6) $(4.7.3)\Rightarrow(4.7.4)$: Let h be the monotone mapping belonging to the filters $\mathfrak{f}(x')$. Since from our conditions $\mathscr{S}(\mathfrak{f}(g(x)))=\mathscr{S}(x)$ follows, h is an inductor subordinated to $\mathscr{S}$ and g by (2.3) We show $\mathscr{S}\sim h(\mathscr{S})$. In fact, assume that s is the monotone mapping belonging to the trace filters $\mathfrak{z}(x')$ of this extension. We have $s(A)\subset h(A)$ for any $A\subset E$. If $<\in\mathscr{S}$ and $A<B$, then $x'\in h(A)$ implies $B\in\mathscr{S}(\mathfrak{f}(x'))=\mathfrak{z}(x')$, that is $x'\in s(B)$. The extension in question is tight, therefore by (2.9) $\mathscr{S}'\sim s(\mathscr{S})\sim h(\mathscr{S})$. h is both $(\mathscr{S},\cap)$- and $(\mathscr{S},\cap)$-preserving. Indeed, $A_i<B_i$

$(i\in I)$, $x'\in\bigcap_{i\in I}h(A_i)$ imply $A_i\in\mathfrak{f}(x')$ for $i\in I$. If $P\in\mathfrak{P}(<)\cap\mathfrak{f}(x')$, then $\emptyset\neq A_i\cap P$ gives $P\subset B_i$ $(i\in I)$, thus $\bigcap_{i\in I}B_i\in\mathfrak{f}(x')$, i.e. $x'\in h(\bigcap_{i\in I}B_i)$. If $x'\in h(\bigcap_{i\in I}A_i)$, then $\bigcap_{i\in I}A_i\in\mathfrak{f}x')$, hence $P\cap A_i\neq\emptyset$ implies $P\subset B_i$ for some $i\in I$ and $P\in\mathfrak{P}(<)\cap\mathfrak{f}x')$. Thus $B_i\in\mathfrak{f}(x')$ and $x'\in\bigcap_{i\in I}h(B_i)$. ∎

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