

Theorem 3. *In a non-flat $HR-F_n$ which is also a $WR-F_n$ its recurrence vector λ_m is necessarily independent of $\dot{x}^i$'s. Conversely, a non-flat $HR-F_n$ with recurrence vector being independent of $\dot{x}^i$'s becomes a $WR-F_n$ provided it also admits (2.8).*

Now we consider a manifold F_n admitting (2.3). Analogous to the terminology adopted by Misra [4], a manifold F_n admitting (2.3) will be called a W_{jk}^i -recurrent manifold. Similarly we may define a W_j^i -recurrent manifold admitting (2.4).

Differentiating (2.3) partially with respect to $\dot{x}^h$ and using (1.12) we get

$$(2.9) \quad \dot{\partial}_h \mathcal{B}_m W_{jk}^i = (\dot{\partial}_h \lambda_m) W_{jk}^i + \lambda_m W_{jkh}^i.$$

Interchange of the operators of partial and covariant differentiation by means of the commutation formula exhibited by (1.5) for the tensor-field W_{jk}^i , and using (1.12), the equation (2.9) reduces to

$$(2.10) \quad \mathcal{B}_m W_{jkh}^i + W_{jk}^r G_{rhm}^i - W_{rk}^i G_{jhm}^r - W_{jr}^i G_{khm}^r = (\dot{\partial}_h \lambda_m) W_{jk}^i + \lambda_m W_{jkh}^i.$$

In view of (1.15) and (2.10) it is clear that a W_{jk}^i -recurrent Finsler manifold will be a $WR-F_n$ if and only if there holds the identity

$$(2.11) \quad W_{jk}^r G_{rhm}^i = W_{rk}^i G_{jhm}^r + W_{jr}^i G_{khm}^r + (\dot{\partial}_h \lambda_m) W_{jk}^i.$$

In view of Theorem 2 and the above conclusion we have the

Theorem 4. *For a $HR-F_n$ to be a $WR-F_n$ (2.11) is a necessary and sufficient condition.*

Transvecting (2.11) with $\dot{x}^m$, and using (1.4) we have

$$(\dot{x}^m \dot{\partial}_h \lambda_m) W_{jk}^i = 0.$$

Thus, for a F_n admitting $W_{jk}^i \neq 0$ and (2.11), we have

$$\dot{x}^m \dot{\partial}_h \lambda_m = 0.$$

Putting $\lambda = \lambda_m \dot{x}^m$, the above relation reduces to

$$(2.12) \quad \dot{\lambda}_m = \dot{\partial}_m \lambda.$$

This result gives rise to the

Theorem 5. *If a W_{jk}^i -recurrent manifold F_n becomes a $WR-F_n$ its recurrence vector λ_m satisfies (2.12).*

Next, let us consider a W_j^i -recurrent manifold characterized by (2.4). Differentiating (2.4) partially with respect to $\dot{x}^k$ we have

$$(2.13) \quad \dot{\partial}_k \mathcal{B}_m W_j^i = (\dot{\partial}_k \lambda_m) W_j^i + \lambda_m \dot{\partial}_k W_j^i.$$

Applying (1.5) for W_j^i this equation reduces to

$$\mathcal{B}_m \dot{\partial}_k W_j^i + W_j^r G_{rkm}^i - W_r^i G_{jkm}^r = (\dot{\partial}_k \lambda_m) W_j^i + \lambda_m \dot{\partial}_k W_j^i.$$

Taking skew-symmetric part in k, j , multiplying by $1/3$ throughout the above equa-

tion, and using the symmetric property of G_{jkh}^i in its lower indices and (1.12), we get

$$\mathcal{B}_m W_{jk}^i + \frac{2}{3} W_{[j}^r G_{k]rm}^i = \frac{2}{3} W_{[j}^i \dot{\partial}_{k]} \lambda_m + \lambda_m W_{jk}^i.$$

Thus, for a W_j^i -recurrent Finsler manifold to be a W_{jk}^i -recurrent Finsler manifold the existence of relation

$$(2.14) \quad W_{[j}^r G_{k]rm}^i = W_{[j}^i \dot{\partial}_{k]} \lambda_m$$

is a necessary and sufficient condition. Therefore, we have the

Theorem 6. *A W_j^i -recurrent Finsler manifold becomes W_{jk}^i -recurrent if and only if it admits (2.14).*

§ 3. Possibilities for a $WR-F_n$ to be a $HR-F_n$. In this section we discuss the possibilities for a projective recurrent Finsler manifold F_n to be a recurrent Finsler manifold. Let us consider a projective recurrent Finsler manifold F_n admitting (1.15). Differentiating (1.9) covariantly with respect to x^m , and using (1.15) we have

$$\begin{aligned} \lambda_m W_{jkh}^i &= \mathcal{B}_m H_{jkh}^i - \frac{1}{n+1} \{ \delta_h^i \mathcal{B}_m H_{jkr}^r + \dot{x}^i \mathcal{B}_m \dot{\partial}_h H_{jkr}^r \} + \\ &+ \frac{2}{n^2-1} \{ n \mathcal{B}_m \dot{\partial}_h H_{[j} H_{k]} + \mathcal{B}_m H_{h[j} H_{k]} + \dot{x}^r \mathcal{B}_m \dot{\partial}_h H_{r[k]} \} \delta_{k]}^i. \end{aligned}$$

Applying the commutation formula (1.5), and (1.4) the above identity reduces to

$$\begin{aligned} (3.1) \quad \lambda_m W_{jkh}^i &= \mathcal{B}_m H_{jsh}^i - \frac{1}{n+1} \{ \delta_h^i \mathcal{B}_m H_{jkr}^r + \dot{x}^i (\dot{\partial}_h \mathcal{B}_m H_{jkr}^r + H_{skr}^r G_{hmj}^s + H_{jsr}^r G_{hmk}^s) \} + \\ &+ \frac{2}{n^2-1} \{ n \mathcal{B}_m \dot{\partial}_h H_{[j} H_{k]} + \mathcal{B}_m H_{h[j} H_{k]} + \dot{x}^r (\dot{\partial}_h \mathcal{B}_m H_{r[k]} + H_{rs} G_{hm[j}^s) \} \delta_{k]}^i. \end{aligned}$$

However, if the manifold under consideration also admits (1.14c) so that the tensor-fields $H_{jkr}^r, \dot{\partial}_h H_j$ are also recurrent*), the above identity simplifies to

$$\begin{aligned} (3.2) \quad \lambda_m H_{jkh}^i &= \mathcal{B}_m H_{jkh}^i - \frac{1}{n+1} \dot{x}^i \{ H_{jkr}^r (\dot{\partial}_h \lambda_m) + H_{skr}^r G_{hmj}^s + H_{jsr}^r G_{hmk}^s \} + \\ &+ \frac{2}{n^2-1} \dot{x}^r \{ (\dot{\partial}_h \lambda_m) H_{r[k]} + H_{rs} G_{hm[j}^s \} \delta_{k]}^i, \end{aligned}$$

where (1.9) is used. In view of (1.9) and (3.2) it is clear that the projective recurrent manifold F_n satisfying (1.14c) will be recurrent if and only if there holds the identity

$$\begin{aligned} (3.3) \quad &\dot{x}^i \{ H_{jkr}^r (\dot{\partial}_h \lambda_m) + H_{skr}^r G_{hmj}^s + H_{jsr}^r G_{hmk}^s \} - \\ &- \frac{2}{n-1} \dot{x}^r \{ (\dot{\partial}_h \lambda_m) H_{r[k]} + H_{rs} G_{hm[j}^s \} \delta_{k]}^i = 0. \end{aligned}$$

*) By definition, $\dot{\partial}_h H_j = H_{jh}$ [7].

Conversely, if a $WR-F_n$ is $HR-F_n$ so that there holds both the equations (1.15) and (1.13) it is then easily seen from (1.13) that it also satisfies (1.14c). Consequently (3.3) results from (3.2). Thus, we conclude the

Theorem 7. *A projective recurrent Finsler manifold F_n is recurrent if and only if there holds the identities (1.14c) and (3.3).*

Transvecting (3.2) with $\dot{x}^h$, using (1.4), (1.7a), and homogeneous property of λ_m we have

$$(3.4) \quad \mathcal{B}_m H_{jk}^i = \lambda_m H_{jk}^i.$$

Thus, a $WR-F_n$ satisfying (1.14c) is necessarily a H_{jk}^i -recurrent manifold. Conversely, when $WR-F_n$ is H_{jk}^i -recurrent there follows from (3.1) under transvection with $\dot{x}^h$ and making use of (1.2), (1.4), (1.7a), (1.10), and homogeneous properties of Berwald's curvature tensor and its associate tensors:

$$-\dot{x}^i (\mathcal{B}_m - \lambda_m) H_{jkr} + \frac{2}{n-1} \dot{x}^h (\mathcal{B}_m - \lambda_m) H_{h[j} \delta_{k]}^i = 0.$$

Multiplying it by $\dot{x}^k$, applying (1.8b), (1.8c) and the fact that H_j is also recurrent under the hypothesis we get

$$(3.5) \quad (\mathcal{B}_m - \lambda_m) \dot{x}^k H_{kj} = 0,$$

which proves that $\dot{x}^k H_{kj}$ is recurrent under the hypothesis and not H_{kj} . Thus, we have the

Theorem 8. *The condition expressed by (1.14c) is sufficient to reduce a $WR-F_n$ into a H_{jk}^i -recurrent manifold but not necessary.*

The second author [4] proved that a H_{jk}^i -recurrent Finsler manifold becomes a recurrent Finsler manifold if and only if

$$(3.6) \quad H_{jk}^i \dot{\partial}_h \lambda_m = H_{jk}^r G_{rhm}^i + 2H_{r[j}^i G_{k]hm}^r.$$

In view of this result we may conclude the

Theorem 9. *A $WR-F_n$ admitting (1.14c) is a $HR-F_n$ if and only if it admits (3.6).*

Transvecting (3.5) with $\dot{x}^k$, and using (1.7b) we have

$$\mathcal{B}_m H_j^i = \lambda_m H_j^i.$$

Thus, we conclude that in a projective recurrent manifold admitting (1.14c) the tensor H_j^i is also recurrent. Using the theorem [4]: An H_j^i -recurrent F_n admitting vanishing covariant derivative of the tensor-field G_{jkh}^i is an H -recurrent space, we conclude the

Theorem 10. *A projective recurrent Finsler manifold admitting (1.14c) and vanishing covariant derivative of the tensor-field G_{jkh}^i is a recurrent manifold.*

§ 4. Associate projective curvature tensor W_{jkhm} . Let us consider a manifold F_n admitting (1.15). Transvecting (1.15) with g_{ip} and using

$$(4.1) \quad W_{jkh}^i g_{ip} = W_{jkhp},$$

we have

$$(4.2) \quad \mathcal{B}_m W_{jkhp} - (\mathcal{B}_m g_{ip}) W_{jkh}^i = \lambda_m W_{jkhp}.$$

Thus, the associate projective curvature tensor W_{jkhp} is not, in general, recurrent in a $WR-F_n$. However, if the manifold is affinely connected so that there holds [7, pp. 80—81]

$$(4.3) \quad \mathcal{B}_m g_{ip} = 0,$$

it follows from (4.2) that

$$\mathcal{B}_m W_{jkhp} = \lambda_m W_{jkhp}.$$

Thus, we have the

Theorem 11. *In an affinely connected $WR-F_n$ the associate projective curvature tensor W_{jkhp} given by (4.1) is recurrent.*

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Appendix

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(Received September 27, 1976.)