

On the monotone classes of maximal limit-logic M

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1. Introduction

The notion of limit-logic was introduced by S. V. YABLONSKI [13] in 1958. The necessity of this notion arose in the study of finite-valued logics [12] and infinite-valued logics [14]. In particular, the infinite-valued logics contain as many as continuum functions, thus, working with these logics is very difficult. Therefore the necessity of a logic was recognized which contains countably many functions and can be regarded, from a certain point of view, as the model of all k -valued logics. Contrary to the many- and the infinite-valued logics, the model of limit-logics has many realizations, or rather there are continuum, pairwise non-isomorphic limit-logics [1]. In the first place the equivalent classes are examined, and, at the same time, the question arises with respect to every partial ordering, whether or not there is a maximal and a minimal element. The best known and simplest equivalence is isomorphism. From the references (6, 8, 13) it is well-known that there are as many as continuum, pairwise non-isomorphic limit-logics. It has been proved [1] that no maximal and minimal elements exist under the well-known algebraic ways of ordering. The partial ordering introduced in [1, 11] in a logical way also decomposes the set of limit-logics into equivalence classes, and under such a partial ordering there are already maximum and minimum limit-logics. Such a logic is maximal to the extent that it contains every other limit-logic, and is minimal if it is contained in every limit-logic. S. V. Yablonski [12] examined the pre-complete classes of the k -valued logic. These classes have been investigated by him and many others [4, 5, 7, 14, 18].

The aim of the paper is to examine certain classes studied by Yablonski in a maximal limit-logic M with help of the method given in [12]. In the k -valued logic there are finite pre-complete classes. In our paper we shall prove that in the limit-logic M there are continuum monotone pre-complete classes. We succeeded in generalizing Yablonski's concept of monotone pre-complete classes and we have reached the following results:

- a) in a limit-logic M the monotone function class belonging to a linear order r is pre-complete.
- b) a limit-logic M contains continuum pre-complete classes. Moreover, the number of the pre-complete monotone function classes with respect to linear order is continuum.

2. Elements

Definition 2.1 [4]. Let E_k be an arbitrary set of k elements. Denote by P_k^n ($n=0, 1, 2, \dots$) the set of functions with n variables, with the variables and values taken from the set E_k . The function set $P_k = \bigcup_{n=0}^{\infty} P_k^n$ will be called k -valued logic. Without loss of generality we suppose $E_k = \{0, 1, \dots, k-1\}$, $k \geq 2$.

Definition 2.2 If the finite set E_k in Definition 2.1 is replaced by the countably infinite set, $E_{\aleph_0}$, the function-set $P_{\aleph_0}$ will be called the infinite-valued logic. In this paper, $E_{\aleph_0}$ will always be the non-negative integers, that is

$$E_{\aleph_0} = \{0, 1, 2, \dots, n, n+1, \dots\}.$$

Definition 2.3 [6]. The subset P of the infinite-valued logic ($P_{\aleph_0}$) will be called limit-logic, if

- a) in the function-set P there are countably many functions,
- b) for every natural number k ($k \geq 2$) there exists a function-set A_k ($A_k \subseteq P$), which can be mapped onto the k -valued logic. Let P be a limit-logic, ε a subset of $E_{\aleph_0}$ ($\varepsilon \subseteq E_{\aleph_0}$) and $g(x_1, \dots, x_n) \in P$.

Definition 2.4. Put

$$g_{\varepsilon}(x_1, \dots, x_n) = \begin{cases} g(x_1, \dots, x_n) & \text{if } x_1, \dots, x_n \in \varepsilon^n (\varepsilon \times \dots \times \varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

The function $g_{\varepsilon}(x_1, \dots, x_n)$ will be called the restriction of the function $g(x_1, \dots, x_n)$ to ε .

Definition 2.5. We say that the function set A_k is the model of the k -valued logic on the set $\varepsilon_k = \{e_0, e_1, \dots, e_{k-1}\}$ ($k \geq 2$), ($\varepsilon_k \subset E_{\aleph_0}$), if in the set $[A_k]$ there is such a function $f(x_1, x_2)$ that

$$f_{\varepsilon_k}(x_1, x_2) = \begin{cases} e_{i+1} & \text{if } (x_1, x_2) \in \varepsilon_k \times \varepsilon_k \text{ and } \max(x_1, x_2) = e_i, \\ & \text{where } 0 \leq i \leq k-2; \\ e_0, & \text{if } (x_1, x_2) \in \varepsilon_k \times \varepsilon_k \text{ and } \max(x_1, x_2) = e_{k-1}. \end{cases}$$

It is easy to see that if A_k is the model of the k -valued logic on the set ε_k , then in the function set A_k there is at least one function-set, whose restriction to the set ε_k is isomorphic with the k -valued logic.

Definition 2.6 [2]. The system of the finite subsets ε_k will be called the domain of the function-set A and we denote it by T_A . The set $\varepsilon_k = \{e_0, e_1, \dots, e_{k-1}\}$ belongs to the domain T_A iff it is model of the k -valued logic on the set ε_k .

Definition 2.7 [2]. The limit-logic P is called increasing, if its domain T_P contains an increasing sequence of the finite sets

$$(\Pi = \{\varepsilon_2, \varepsilon_3, \dots, \varepsilon_k, \dots\}) \quad (\varepsilon_i \subset \varepsilon_{i+1} \quad (i = 2, 3, \dots)).$$

Theorem 2.1 [2]. The limit-logic is maximal if and only if it is increasing.

The notions of superposition, closure and pre-completeness are defined as in [12].

3. § The limit-logic M

We shall define a representation M of the maximal limit logics which we are going to examine in detail:

Define the function $\mu_k(x_1, x_2) \in P_{\aleph_0} (k \geq 2)$ as follows:

$$\mu_k(x_1, x_2) = \begin{cases} e, & \text{if } (x_1, x_2) \in \varepsilon_k \times \varepsilon_k \text{ and } \max(x_1, x_2) = e-1, \\ & \text{where } 1 \leq e-1 \leq k-1; \\ 1, & \text{if } (x_1, x_2) \in \varepsilon_k \times \varepsilon_k \text{ and } \max(x_1, x_2) = k \\ 0, & \text{otherwise.} \end{cases}$$

Let M_k denote the closure of the function-set

$$\{\mu_k(x_1, x_2)\} ([\{\mu_k(x_1, x_2)\}]) \quad \text{and} \quad M = \left[\bigcup_{k=2}^{\infty} M_k \right].$$

Remark 3.1. It is easy to see that M_k is isomorphic with P_k and M is a limit-logic.

Remark 3.2. As an illustration we shall give the functions $\mu_2(x_1, x_2)$, $\mu_3(x_1, x_2)$ and $\mu_k(x_1, x_2)$.

$x_2 \backslash x_1$	0	1	2	3	4	...
0	0	0	0	0	0	...
1	0	2	1			
2	0	1	1	:	:	
3	0	...	...	...	...	
4	0	...	0	...	0	...
:	:	:	:	:	:	
$\mu_2(x_1, x_2)$						

$\varepsilon_k = \{1, 2\}$

$x_2 \backslash x_1$	0	1	2	3	4	...
0	0	0	0	0	0	0 ...
1	0	2	3	1	:	:
2	0	3	3	1	0	...
3	0	1	1	1	:	:
4	0	...	...	...	...	...
:	0	...	0	...	0	...
:	:	:	:	:	:	
$\mu_3(x_1, x_2)$						

$\varepsilon_k = \{1, 2, 3\}$

$x_2 \backslash x_1$	0	1	2	3	4	...	$k-1$	k	$k+1$	...
0	0	0	0	0	0	...	0	0	0	...
1	0	2	3	4	5	...	k	1	0	...
2	0	3	3	4	5	...	k	1	0	...
3	0	4	4	4	5	...	k	1	0	...
4	0	5	5	5	5	...	k	1	0	...
...	...								...	...
...	...								...	...
...	...								...	...
$k-1$	0	k	k	k	k	...	k	1	0	...
k	0	1	1	1	1	...	1	1	0	...
$k+1$	0									
$\vdots$	0	...	...	0				0	...	...
$\vdots$	$\vdots$	...	...	$\vdots$	...	...	...	$\vdots$	...	...

$\mu_k(x_1, x_2)$

$\varepsilon_k = \{1, 2, \dots, k\}$

Theorem 3.1. *M is a limit-logic.*

Proof. By Definition 2.3 we have to show:

a) the function set M is countable, since $M = \left[\bigcup_{k=2}^{\infty} M_k \right]$, a countable union of countable sets,

b) for every natural number k ($k \geq 2$) there is a function subset in M which can be mapped onto the k -valued logic.

This statement follows from the fact that the functions $\mu_k(x_1, x_2)$ can be brought into one to one correspondance with the $W_k(x_1, x_2)$ Webb functions [5]. Furthermore, let $\varepsilon_k \subseteq E_{\aleph_0} \setminus 0$ be a subset, for which we suppose without loss of generality $\varepsilon_k = \{0, 1, \dots, k-1\}$.

Remark 3.3. The functions of the limit-logic have the property $f(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0$.

Theorem 3.2. *M is a maximal limit-logic.*

PROOF. The limit-logic M being maximal follows immediately from the fact that M is increasing (according to Definition 2.7), a necessary and sufficient condition by Theorem 2.1 for the limit-logic M to be maximal.

4. The monotone function classes of the maximal limit-logic M

In this paragraph we examine the monotone classes of the maximal limit-logic M . We prove that the monotone classes of the M are similar to the monotone classes of P_k and their cardinality is continuum. Moreover, in the limit-logic M are continuum monotone pre-complete classes, while in the k -valued logic there is but a finite number of them.

Remark 4.1. Of course we shall speak about the elementwise restriction M_{ε_γ} of the function class to ε_γ to. This means no restriction as to the validity of the theorems. We define the set ε_γ , starting from a function $f(x_1, \dots, x_n) \in M$, as follows:

1. Let α denote the maximum in the range of

$$f(x_1, \dots, x_n).$$

2. Let β be the largest value, in some n -tuple, on which the function is not equal to 0. ($f(\dots, \beta, \dots) \neq 0$).

3. Let $\gamma = \max(\alpha, \beta)$, $\varepsilon_\gamma = \{1, 2, \dots, \gamma\}$ ($\gamma \leq k$).

Definition 4.1. $\tilde{\alpha} \equiv \tilde{\beta}$, if $\alpha_i \equiv \beta_i$ where $\alpha_i, \beta_i \in E_k$ ($i=1, 2, \dots, n$). We say that the function $f(x_1, \dots, x_n) \in P_k(P_{\aleph_0})$ is monotone with respect to the linear ordering if for an arbitrary pair $\tilde{\alpha}, \tilde{\beta} \in E_k(E_{\aleph_0})$ $\tilde{\alpha} \equiv \tilde{\beta}$ implies $f(\tilde{\alpha}) \equiv f(\tilde{\beta})$.

Remark 4.2. Let $<$ denote the ordering $1 < 2 < 3 < \dots < n < n+1 < \dots$. Let be 0 a distinguished element in this ordering incomparable with any element of $E_{\aleph_0} \setminus 0$.

Denote by $\mathcal{M}_{\varepsilon_k}^r$ the set of the all monotone functions with respect to the order r over ε_k . Then let $\mathcal{M}^r = \bigcup_{k=2}^{\infty} \mathcal{M}_{\varepsilon_k}^r$.

We shall consider an arbitrary linear order and we shall prove, that the monotone functions with respect to this order are pre-complete, and different pre-complete classes belong to different linear orders.

Remark 4.3. Let $\mathcal{M}_{\varepsilon_k}^r$ be the set of all the functions $f(x_1, \dots, x_n)$ from the set $\mathcal{M}^r$ of monotone functions with respect to $<_r$ for which $f(x_1, \dots, e, \dots, x_n) = 0$ if $e \notin \varepsilon_k$ is true.

Theorem 4.1. *The function classes $\mathcal{M}_{\varepsilon_k}^r$ and $\mathcal{M}^r$ are closed.*

PROOF. Replace the variables by all the functions, which are monotone by $<_r$. Let

$$f_{\varepsilon_k}(x_1, x_2, \dots, x_n) = g_{\varepsilon_k}(g_{\varepsilon_k,1}(x_1, \dots, x_n), g_{\varepsilon_k,2}(x_1, \dots, x_n), \dots, g_{\varepsilon_k,m}(x_1, \dots, x_n))$$

where the functions $g_{\varepsilon_k}, g_{\varepsilon_k,1}, \dots, g_{\varepsilon_k,m}$ are monotone according to $<_r$. We shall show that the function f is monotone according to $<_r$. Let us consider two sequences $\tilde{\alpha}$ and $\tilde{\beta}$, so that $\tilde{\alpha} <_r \tilde{\beta}$. It is clear that by the condition

$$g_{\varepsilon_k,i}(\tilde{\alpha}) \leq_r g_{\varepsilon_k,i}(\tilde{\beta}), \quad (i = 1, 2, \dots, m)$$

holds, therefore the sequence

$$\{g_{\varepsilon_k,1}(\tilde{\alpha}), g_{\varepsilon_k,2}(\tilde{\alpha}), \dots, g_{\varepsilon_k,m}(\tilde{\alpha})\} \quad \text{and} \quad \{g_{\varepsilon_k,1}(\tilde{\beta}), g_{\varepsilon_k,2}(\tilde{\beta}), \dots, g_{\varepsilon_k,m}(\tilde{\beta})\}$$

are such that

$$\{g_{\varepsilon_k,1}(\tilde{\alpha}), g_{\varepsilon_k,2}(\tilde{\alpha}), \dots, g_{\varepsilon_k,m}(\tilde{\alpha})\} \leq_r \{g_{\varepsilon_k,1}(\tilde{\beta}), g_{\varepsilon_k,2}(\tilde{\beta}), \dots, g_{\varepsilon_k,m}(\tilde{\beta})\}.$$

From here we have, because of the monotony of the function g_{ε_k}

$$f_{\varepsilon_k}(\tilde{\alpha}) = g_{\varepsilon_k}(g_{\varepsilon_k,1}(\tilde{\alpha}), \dots, g_{\varepsilon_k,m}(\tilde{\alpha})) \leq_r g_{\varepsilon_k}(g_{\varepsilon_k,1}(\tilde{\beta}), \dots, g_{\varepsilon_k,m}(\tilde{\beta})) = f_{\varepsilon_k}(\tilde{\beta}).$$

So, this function class is invariant to superposition, therefore, it is a closed class. This completes the proof.

Denote by $C_{\varepsilon_k}^j(x)$ the next function:

$$C_{\varepsilon_k}^j \left| \begin{array}{c|c} x & 0 \ 1 \ \dots \ k \\ \hline 0 & j \ \dots \ j \end{array} \right| \begin{array}{c} k+1 \ \dots \\ 0 \ \dots \end{array}$$

Remark 4.4. If the order $<_r$ is the natural ordering $1 < 2 < 3 < \dots$ we get the usual class ($\mathcal{M}^1$) of monotone functions. The ideas of the proofs of the next few theorems and lemmas are due to Yablonski's paper [12].

Theorem 4.2. *The class of monotone functions is generated by the functions: $\max_{\varepsilon_k}(x_1, x_2)$, $\min_{\varepsilon_k}(x_1, x_2)$, $c_{\varepsilon_k}^j(x)$ ($j=1, 2, \dots, k$) and the $m_{\varepsilon_k}^i(x)$ ($i=2, \dots, k$) all of $\mathcal{M}_{\varepsilon_k}^1$, where*

$$m_{\varepsilon_k}^i(x) = \begin{cases} 1, & \text{if } 1 \leq x < i \text{ and } x \in \varepsilon_k; \\ k, & \text{if } k \leq x \leq i \text{ and } x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Define the function $Z_{\varepsilon_k}^{\tilde{\alpha}, \beta}(x_1, \dots, x_n)$ as follows:

$$Z_{\varepsilon_k}^{\tilde{\alpha}, \beta}(x_1, \dots, x_n) = \begin{cases} \beta, & \text{if } 1 \leq x < i \text{ and } x \in \varepsilon_k; \\ k, & \text{if } k \leq x \leq i \text{ and } x \in \varepsilon_k; \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \varepsilon_k$.

Obviously

$$Z_{\varepsilon_k}^{\tilde{\alpha}, \beta}(x_1, \dots, x_n) = \min_{\varepsilon_k}(\beta, m_{\varepsilon_k}^{\alpha_1}(x_1), \dots, m_{\varepsilon_k}^{\alpha_n}(x_n)).$$

Let $Z_{\varepsilon_k}(x_1, \dots, x_n)$ be an arbitrary monotone function of $\mathcal{M}_{\varepsilon_k}^1$ then

$$Z_{\varepsilon_k}(x_1, \dots, x_n) = \max_{\tilde{\alpha}} \{Z_{\varepsilon_k}^{\tilde{\alpha}, f(\tilde{\alpha})}(x_1, \dots, x_n)\}.$$

This completes the proof.

Lemma 4.1. *If the function $Z(x_1, \dots, x_n) \in M_k$ is not monotone, then with the help of the substitution of C^j we shall obtain the one-variable non-monotone function.*

PROOF. It will be sufficient to regard the elements from ε_k and so the proof is the same as in [12].

Definition 4.2. Let B be the sum of the sets $B_1, B_2, \dots, B_l$ so that: $B = B_1 + B_2 + \dots + B_l$, if

1. $B = B_1 \cup B_2 \cup \dots \cup B_l$.
2. B_i, B_j are pairwise disjoint, i.e. $B_i \cap B_j = \emptyset$, if $i \neq j$.

Lemma 4.1.1. *If the set is the direct product of two sets $\mathcal{C}$ and $\mathcal{D}$ which are non-empty and $\tilde{\gamma}^0 \in \mathcal{C}$, $\tilde{\delta}^0 \in \mathcal{D}$ then there are two elements $\tilde{\gamma} \in \mathcal{C}$ and $\tilde{\delta} \in \mathcal{D}$ such that they are next to each other by the ordering $<_r$. Moreover, if $\tilde{\gamma}^0 <_r \tilde{\delta}^0$, or, conversely, $\tilde{\gamma}^0 >_r \tilde{\delta}^0$, then $\tilde{\gamma}^0 \leq_r \tilde{\gamma} <_r \tilde{\delta} \leq_r \tilde{\delta}^0$ or $\tilde{\gamma}^0 \geq_r \tilde{\gamma} >_r \tilde{\delta} \geq_r \tilde{\delta}^0$, respectively.*

PROOF. See [12].

Theorem 4.3. *The class of monotone functions $\mathcal{M}^1$ is pre-complete in the limit-logic M , where $\mathcal{M}^1$ is the class of monotone functions belonging to an arbitrary linear order.*

PROOF. We prove that $[\{f(x_1, \dots, x_n)\} \cup \mathcal{M}^1] = M$, $f(x_1, \dots, x_n) \notin \mathcal{M}^1$. Let $k \geq \gamma$, where $\gamma = \max(\alpha, \beta)$; $\alpha = \max f(\tilde{x})$. Let β be the maximum of the values for which $f(\dots, \beta, \dots) \neq 0$. We show that the function $f_{\varepsilon_k}(x_1, \dots, x_n)$ and the set $\mathcal{M}_{\varepsilon_k}^1$ generate the logic $[\{\mu_k(x_1, x_2)\}] = M_k$, which is isomorphic to the k -valued logic P_k .

Let $f_{\varepsilon_k}(x_1, \dots, x_n) \notin \mathcal{M}_{\varepsilon_k}^1$. Then on account of Lemma 4.1 and with the help of the functions $C_{\varepsilon_k}^j$ we shall obtain the one-variable non-monotone function $g_{\varepsilon_k}(x)$. Suppose that over $x = t$

$$g_{\varepsilon_k}\{t\} > g_{\varepsilon_k}(t+1) \quad (g(0) = 0)$$

$$h_{\varepsilon_k}^1(x) = \begin{cases} t, & \text{if } x = 1; \\ t+1, & \text{if } x \neq 1 \text{ and } x \in \varepsilon_k; \\ 0 & \text{otherwise;} \end{cases}$$

$$h_{\varepsilon_k}^2(x) = \begin{cases} 1, & \text{if } x \leq g(t+1) \text{ and } x \in \varepsilon_k; \\ k, & \text{if } x > g(t+1) \text{ and } x \in \varepsilon_k; \\ 0, & \text{if } x \notin \varepsilon_k. \end{cases}$$

It is easy to see that $h_{\varepsilon_k}^1(x)$ and $h_{\varepsilon_k}^2(x)$ are monotone functions, that is, $h_{\varepsilon_k}^1(x)$, $h_{\varepsilon_k}^2(x) \in \mathcal{M}_{\varepsilon_k}^1$ and

$$h_{\varepsilon_k}^2(g_{\varepsilon_k}(h_{\varepsilon_k}^1(x))) = \begin{cases} k, & \text{if } x = 1; \\ 1, & \text{if } x \neq 1 \text{ and } x \in \varepsilon_k; \\ 0 & \text{otherwise,} \end{cases}$$

where so $h_{\varepsilon_k}^2(g_{\varepsilon_k}(h_{\varepsilon_k}^1(x))) \equiv j_{\varepsilon_k}^1$ where

$$j_{\varepsilon_k}^i(x) = \begin{cases} k, & \text{if } x = i \neq 0 \text{ and } x \in \varepsilon_k; \\ 1, & \text{if } x \neq i \text{ and } x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the following functions:

$$\varphi_{\varepsilon_k}^i(x) = \begin{cases} 1, & \text{if } x < i \text{ and } x \in \varepsilon_k; \\ k, & \text{if } x \equiv i \text{ and } x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi_{\varepsilon_k}^i(x) = \begin{cases} 1, & \text{if } x \leq i \text{ and } x \in \varepsilon_k; \\ 2, & \text{if } x > i \text{ and } x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\varphi_{\varepsilon_k}^i(x), \psi_{\varepsilon_k}^i(x) \in \mathcal{M}_{\varepsilon_k}^1$. Thus we have the following functions:

$$C_{\varepsilon_k}^j(x), \max_{\varepsilon_k}(x_1, x_2), \min_{\varepsilon_k}(x_1, x_2) \quad \text{and} \quad j_{\varepsilon_k}^i(x) \quad (i = 1, 2, \dots, k).$$

Lemma 4.3.1. *The set of functions $C_{\varepsilon_k}^j(x), \max_{\varepsilon_k}(x_1, x_2), \min_{\varepsilon_k}(x_1, x_2)$ is complete in $M_k(\{\mu_k(x_1, x_2)\}) = M_k$*

$$C_{\varepsilon_k}^j(x) = \begin{cases} j & \text{if } x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

$$j_{\varepsilon_k}^i(x) = \begin{cases} k, & \text{if } x = i \neq 0 \quad \text{and } x \in \varepsilon_k; \\ 1 & \text{if } x \neq i \quad \text{and } x \in \varepsilon_k; \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We shall obtain an arbitrary function from M_k from the given function system by induction on the number of the variables. Let $f(x_1, \dots, x_n) \in M_k$ be an arbitrary function.

1. If f is 0-ary, then the statement is evident, since the initial system contains j ($j=1, \dots, k$).

2. Suppose we have constructed every n -variable function of the $\mathcal{M}_{\varepsilon_k}^1$ by superposition.

We show next that we can produce an arbitrary function of $n+1$ variables of $\mathcal{M}_{\varepsilon_k}^1$ and from the functions which were given in Lemma 4.3.1.

Therefore, we assume

$$\max_{\varepsilon_k}(y_1, y_2, \dots, y_n) = \max_{\varepsilon_k} \{ \max_{\varepsilon_k} [\dots \max_{\varepsilon_k} (\max_{\varepsilon_k}(y_1, \dots, y_n)] y_n \}$$

(similarly for $\min_{\varepsilon_k}(y_1, y_2, \dots, y_n)$).

Then

$$\begin{aligned} & f_{\varepsilon_k}(x_1, \dots, x_n, x_{n+1}) = \\ & = \max_{\varepsilon_k} \{ \min_{\varepsilon_k} [j_{\varepsilon_k}^1(x_{n+1}), f(x_1, \dots, x_n, 1)] \min_{\varepsilon_k} [j_{\varepsilon_k}^2(x_{n+1}), f(x_1, \dots, x_n, 2)] \cdot \\ & \quad \cdot \min_{\varepsilon_k} [j_{\varepsilon_k}^k(x_{n+1}), f(x_1, \dots, x_n, k)] \} \quad (\varepsilon_k = 1, 2, \dots, k). \end{aligned}$$

Hence we obtain the statement.

It follows from the foregoing that this system is complete. So the theorem is proved.

From the Theorem 4.3 follows

Theorem 4.4. *In the limit-logic M the monotone function class $(\mathcal{M}^r)$ belonging to any linear order is pre-complete.*

PROOF. If $\mathcal{M}_{\varepsilon_k}^r$ is the dual of the function class $\mathcal{M}_{\varepsilon_k}^1$ by the permutation

$$s_{\varepsilon_k}(x) = \begin{pmatrix} a_1, a_2, \dots, a_k \\ 1, 2, \dots, k \end{pmatrix},$$

then the results concerning the class $\mathcal{M}_{\varepsilon_k}^1$ can be extended (because of the duality principle) to the function class $\mathcal{M}_{\varepsilon_k}^r$.

Corollary 4.1. *Every function class $\mathcal{M}_{\varepsilon_k}^r$ is pre-complete in M_k .*

Theorem 4.5. *The cardinality of the pre-complete monotone classes is continuum in the limit-logic M .*

PROOF. Since in M there are countably many functions, so in M there cannot be more pre-complete classes than continuum. That is

- a) the number of the different orders $\prec_r$ is continuum,
- and
- b) different $\mathcal{M}^r$ -s correspond to different orders.

Corollary 4.2. *In the limit-logic M there are as many as continuum pre-complete classes.*

Summary. In this paper we prove, that the maximal limit-logic to be examined contains as many as continuum monotone pre-complete classes.

We obtained the following results:

1. In the maximal limit-logic M any monotone function class which belongs to a linear order r is pre-complete.
2. The number of pre-complete classes of monotone functions with respect to the linear ordering is continuum.

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