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Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, I. Divisor-sum functions and Euler-type functions

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Abstract. We introduce the notion of cross-convolution of arithmetical functions as a special case of Narkiewicz's regular convolution. We give asymptotic formulae for the summatory functions of certain generalized divisor-sum functions and Euler-type functions related to cross-convolutions and to arbitrary sets of positive integers. These formulae generalize and unify many known results concerning the corresponding usual and unitary functions.

1. Introduction

There is a well-known analogy between properties of certain special arithmetical functions and their unitary analogues such as $\sigma(n)$ and $\sigma^*(n)$, $\tau(n)$ and $\tau^*(n)$, $\phi(n)$ and $\phi^*(n)$, representing the sum of divisors and the sum of unitary divisors of n, the number of divisors and the number of unitary divisors of n, the Euler function and its unitary analogue, respectively, see [Co60a], [Co60b], [McC86], [Siv89].

Common generalizations of these functions are given in terms of A-convolutions of arithmetical functions defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d),$$

where A is a mapping from the set $\mathbb{N}$ of positive integers to the set of subsets of $\mathbb{N}$ such that $A(n) \subseteq D(n)$ for each n, $D(n)$ denoting the set of all (positive) divisors of n.

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W. Narkiewicz [Nar63] defined an A-convolution to be regular if

(a) the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, has an inverse μ_A with respect to the A-convolution and $\mu(p^a) \in \{-1, 0\}$ for every prime power p^a $(a \geq 1)$.

It can be proved, see [Nar63], that an A-convolution is regular if and only if

(i) $A(mn) = \{de : d \in A(m),\ e \in A(n)\}$ for every $m, n \in \mathbb{N}, (m, n)=1$,

(ii) for every prime power p^a $(a \geq 1)$ there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, \ldots, p^{it}\}$ for every $i \in \{0, 1, \ldots, a/t\}$.

For example, the Dirichlet convolution D, where $D(n)=\{d\in\mathbb{N} : d|n\}$, and the unitary convolution U, where $U(n) = \{d \in \mathbb{N} : d|n, (d, n/d)=1\}$, are regular.

In this paper we consider regular A-convolutions, see also [McC86], [Sit78]. For $d, n \in \mathbb{N}$, the number d is said to be an A-divisor of n if $d \in A(n)$. Let $\sigma_{A,s}(n)$ denote the sum of s-powers of the A-divisors of n and let $\sigma_{A,1}(n) \equiv \sigma_A(n)$ be the sum of A-divisors of n. Furthermore, for $k \in \mathbb{N}$ let $\phi_{A,k}(n)$ denote the number of integers $x \pmod{n^k}$ such that $(x, n^k)_{A,k} = 1$, where $(a, b)_{A,k}$ stands for the greatest k-th power divisor of a which belongs to $A(b)$. The function $\phi_{A,k}(n)$ was defined by V. Sita Ramaiah [Sit78], for $k = 1$ the function $\phi_{A,1}(n) \equiv \phi_A(n)$ was studied by P. J. McCarthy [McC68].

For $A = D$ and for $A = U$ we obtain the functions $\sigma_s(n), \sigma(n), \phi_k(n)$ and their unitary analogues $\sigma_s^*(n), \sigma^*(n), \phi_k^*(n)$, respectively. The function $\phi_k(n)$ was investigated by E. Cohen [Co49], [Co56], $\phi_1(n) \equiv \phi(n)$ is the classical Euler function and $\phi_k^*(n)$ was defined by K. Nageswara Rao [Nag66].

Although asymptotic formulae concerning the usual and the unitary functions of above have been investigated by several authors, common generalizations of such formulae seem not to have appeared in the literature.

Narkiewicz's [Nar63] paper includes certain asymptotic formulae for arithmetical functions defined by regular A-convolutions, but with an additional condition on A, which is not satisfied neither for $A = D$ nor for $A = U$.

The aim of this paper is to establish asymptotic formulae for the summatory functions of $\sigma_{A,s}, \sigma_A(n), \phi_{A,k}(n)$, in fact we will deduce asymptotic formulae concerning more general functions, if A is a *cross-convolution* to be defined in Section 2, which generalize and unify the corresponding known results concerning the usual and the unitary functions.

Our method is elementary and applies some standard arguments. It is used in [T95] for various types of arithmetical functions. We plan to continue our investigations in forthcoming papers.

2. Preliminaries

If A is a regular convolution, then the generalized Möbius function μ_A is the multiplicative function such that, for all prime powers p^a $(a \geq 1)$,

$$\mu_A(p^a) = \begin{cases} -1, & \text{if } t_A(p^a) = a, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For $k \in \mathbb{N}$, let $A_k(n) = \{d \in \mathbb{N} : d^k \in A(n^k)\}$. It is known that an A_k-convolution is regular whenever the A-convolution is regular.

Let S be a subset of $\mathbb{N}$ and let ρ_S denote the characteristic function of S, that is $\rho_S(n) = 1$ if $n \in S$, and $\rho_S(n) = 0$ if $n \notin S$. The generalized Möbius function $\mu_{S,A}$ is defined by

$$\mu_{S,A} *_A I = \rho_S, \tag{2}$$

where $I(n) = 1$ for all $n \in \mathbb{N}$, see [H88]. If $S = \{1\}$, then $\mu_{S,A} = \mu_A$, and if $A = D$, then $\mu_A = \mu$, the classical Möbius function. If $A = U$, then $\mu_U = \mu^*$ is the Liouville function.

We say that S is multiplicative if its characteristic function ρ_S is multiplicative, i.e. $1 \in S$ and $mn \in S$ if and only if $m \in S,\ n \in S$ for every $m, n \in \mathbb{N}$ with $(m, n) = 1$.

By (2), (1) and by Möbius inversion we immediately have the following statements, see [TH96]:

Lemma 1. *The function $\mu_{S,A}$ is multiplicative if and only if S is multiplicative, and in this case*

$$\mu_{S,A}(n) = \prod_{p^a || n} \Big(\rho_S(p^a) - \rho_S(p^{a-t}) \Big)$$

for every $n \in \mathbb{N}$, where $t = t_A(p^a)$ is the type of p^a with respect to A and $p^a || n$ means $p^a | n$ and $p^{a+1} \nmid n$. If S is multiplicative, then $\mu_{S,A}(n) \in \{-1, 0, 1\}$ for every $n \in \mathbb{N}$.

Remark 1. If S is not multiplicative, then the function $\mu_{S,A}$ can take other values too. Moreover, it can be unbounded, as it is shown in the following example. Let $A = D$ and let $S = \mathbb{P}$ be the set of the primes. Then

$$\mu_{\mathbb{P}}(n) = \sum_{d|n} \rho_S(d)\mu(n/d) = \sum_{\substack{d|n \\ d\in\mathbb{P}}} \mu(n/d),$$

for every $n \in \mathbb{N}$, and for $n = p_1p_2\dots p_r$, where $p_1, p_2, \dots, p_r$ are distinct primes, we get

$$\mu_{\mathbb{P}}(p_1p_2\dots p_r) = \sum_{i=1}^{r} \mu(p_1\dots p_{i-1}p_{i+1}\dots p_r) = (-1)^{r-1}r.$$

Lemma 2. *For every subset S and for every regular convolution A we have $|\mu_{S,A}(n)| \le \tau(n)$ for every $n \in \mathbb{N}$, and $\mu_{S,A}(n) = O(n^{\varepsilon})$ for every $\varepsilon > 0$.*

If A is a regular convolution, $S \subseteq \mathbb{N}$ and $s \in \mathbb{R}$, let $\sigma_{S,A,s}$ be the function defined by

$$\sigma_{S,A,s}(n) = \sum_{\substack{d\in A(n) \\ n/d\in S}} d^s.$$

We have $\sigma_{S,A,s} = \rho_S *_A E_s$, where $E_s(n) = n^s$ for every $n \in \mathbb{N}$ and it follows that for S multiplicative $\sigma_{S,A,s}$ is multiplicative too. If $A = D$ and $S = \{n^k : n \in \mathbb{N}\}$ where $k \in \mathbb{N}$, then we obtain the classical function of L. Gegenbauer.

Furthermore, for a regular convolution A, for $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ let $\phi_{S,A,k}(n)$ denote the number of integers $x \pmod{n^k}$ such that $((x, n^k)_{A,k})^{1/k} \in S$. This function was introduced by P. Haukkanen [H88] and one has

$$\phi_{S,A,k} = \mu_{S,A_k} *_{A_k} E_k. \tag{3}$$

The function $\phi_{S,D,1}$ was introduced by E. Cohen [Co59]. If $S = \{n^h : n \in \mathbb{N}\}$, where $h \in \mathbb{N}$ and $A = D$, $k = 1$ we have the function of E. Cohen [Co60c], which reduces to the function b of S. Sivaramakrishnan [Siv79] if $h = 2$. Let $m \in \mathbb{N}, m \ge 2$. If $S = Q_m$ is the set of m-free integers (i.e. integers not divisible by the m-th power of any integer > 1) and $A = D$, $k = 1$ we obtain the function ϕ_m of V. L. Klee [K48]. If $S = Q_m^*$ is the set of unitarily m-free integers (i.e. integers not divisible unitarily by the

m-th power of any integer > 1) and $A = U, k = 1$ we have the function φ_m^* of D. SURYANARAYANA [Sur72].

For $S = \{1\}$ we get the function $\phi_{A,k}$ defined in the Introduction. For other particular cases of the function $\phi_{S,A,k}$we refer to the papers and books given in the bibliography.

Let A be a regular convolution. We say that A is a *cross-convolution* if for every prime p we have either $A(p^a) = \{1, p, p^2, \dots, p^a\} = D(p^a)$ or $A(p^a) = \{1, p^a\} = U(p^a)$ for every $a \in \mathbb{N}$. Let P and Q be the set of the primes of the first and of the second kind of above, respectively, where $P \cup Q = \mathbb{P}$ is the set of all primes. For $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ we obtain the unitary convolution U.

Furthermore, let $(P) = \{1\} \cup \{n \in \mathbb{N} :$ each prime factor of n belongs to $P\}, (Q) = \{1\} \cup \{n \in \mathbb{N} :$ each prime factor of n belongs to $Q\}$. It is clear that every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P), n_Q \in (Q)$ and $(n_P, n_Q) = 1$. If A is a cross-convolution, then $A(n) = \{d \in \mathbb{N} : d|n, (d, n/d) \in (P)\}$ for every $n \in \mathbb{N}$.

Remark 2. According to [Sit78], Theorem 3.3, the following statements are equivalent:

(i) A is a cross-convolution,

(ii) for every prime p, π_p is either $\{0, 1, 2, 3, \dots\}$ or $\{0, 1\}, \{0, 2\}, \{0, 3\}, \dots$,

(iii) $A = A_k$ for every $k \in \mathbb{N}$,

(iv) $d \in A(n)$ if and only if $d^k \in A(n^k)$ for all $n, k \in \mathbb{N}$.

Remark 3. If A is a cross-convolution, then $\sigma_A(n) = \sigma(n_P)\sigma^*(n_Q)$ and $\phi_A(n) = \phi(n_P)\phi^*(n_Q)$ for every $n \in \mathbb{N}$. Similar identities are valid also for the generalized functions discussed in this paper.

For an arithmetical function f let $D(f, z)$, $D_P(f, z)$ and $D_Q(f, z)$ denote the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^z}, \quad \sum_{\substack{n=1 \\ n \in (P)}}^{\infty} \frac{f(n)}{n^z} \quad \text{and} \quad \sum_{\substack{n=1 \\ n \in (Q)}}^{\infty} \frac{f(n)}{n^z},$$

respectively. If $f(n) = I(n) = 1$ for each $n \in \mathbb{N}$, then $D(I, z) = \zeta(z)$ is the Riemann zeta function and let $D_P(I, z) = \zeta_P(z), D_Q(I, z) = \zeta_Q(z)$. The next assertions follow by the Euler product formula.

Lemma 3. *If the function f is multiplicative and if $D(f,z)$ is absolutely convergent, then*

$$D_P(f,z) = \prod_{p\in P}\Big(1+\frac{f(p)}{p^z}+\frac{f(p^2)}{p^{2z}}+\dots\Big),$$
$$D_Q(f,z) = \prod_{p\in Q}\Big(1+\frac{f(p)}{p^z}+\frac{f(p^2)}{p^{2z}}+\dots\Big),$$
$$D_P(f,z)D_Q(f,z) = D(f,z).$$

If in addition f is completely multiplicative, then

$$D_P(f,z) = \prod_{p\in P}\Big(1-\frac{f(p)}{p^z}\Big)^{-1},$$
$$D_Q(f,z) = \prod_{p\in Q}\Big(1-\frac{f(p)}{p^z}\Big)^{-1}.$$

If $z\in\mathbb{C}, \operatorname{Re} z>1$, then

$$\zeta_P(z) = \prod_{p\in P}\Big(1-\frac{1}{p^z}\Big)^{-1},$$
$$\zeta_Q(z) = \prod_{p\in Q}\Big(1-\frac{1}{p^z}\Big)^{-1},$$
$$\zeta_P(z)\zeta_Q(z) = \zeta(z)$$
$$D_P(\mu,z) = 1/\zeta_P(z),$$
$$D_Q(\mu,z) = 1/\zeta_Q(z).$$

We need the following well-known estimates:

Lemma 4.

$$\sum_{n\le x} n^{-s} = \begin{cases} O(x^{1-s}), & 0<s<1,\\ O(\log x), & s=1,\\ O(1), & s>1, \end{cases} \tag{4}$$

$$\sum_{n>x} n^{-s} = O(x^{1-s}), \quad s>1. \tag{5}$$

Lemma 5 (see [T89], Lemma 5 and Lemma 8)**.**

$$\sum_{n\le x}\frac{\tau(n)}{n^s}=\begin{cases}O(x^{1-s}\log x), & 0<s<1,\\ O(\log^2 x), & s=1,\\ O(1), & s>1.\end{cases} \tag{6}$$

$$\sum_{n\le x}\frac{\tau^2(n)}{n^s}=\begin{cases}O(x^{1-s}\log^3 x), & 0<s<1,\\ O(\log^4 x), & s=1,\\ O(1), & s>1.\end{cases} \tag{7}$$

$$\sum_{n>x}\frac{\tau(n)}{n^s}=O(x^{1-s}\log x),\quad s>1. \tag{8}$$

Lemma 6 (see [Ch67], Lemma 2.3)**.** *If $s\ge 0$ and $a\in\mathbb{N}$, then*

$$\sum_{\substack{n\le x\\(n,a)=1}} n^s=\frac{\phi(a)x^{s+1}}{a(s+1)}+O(x^s\tau(a)).$$

The following is a key-lemma of our treatment, see also [TH96].

Lemma 7. *If A is a cross-convolution, $s\ge 0$ and $a\in\mathbb{N}$, then*

$$\sum_{\substack{n\le x\\(n,a)\in(P)}} n^s=\frac{\phi(a_Q)x^{s+1}}{a_Q(s+1)}+O(A_a^{(s)}(x,Q)),$$

where $A_a^{(s)}(x,Q)=x^s$ or $x^s\tau(a)$, according as Q is finite or Q is infinite.

Proof. Observe that $(n,a)\in(P)$ if and only if $(n,\gamma(a_Q))=1$, where $\gamma(m)$ denotes the product of distinct prime factors of m. Hence

$$\sum_{\substack{n\le x\\(n,a)\in(P)}} n^s=\sum_{\substack{n\le x\\(n,\gamma(a_Q))=1}} n^s=\frac{\phi(\gamma(a_Q))x^{s+1}}{\gamma(a_Q)(s+1)}+O(x^s\tau(\gamma(a_Q))),$$

by Lemma 6. Here $\phi(\gamma(a_Q))/\gamma(a_Q)=\phi(a_Q)/a_Q$ and if Q is finite, then $\tau(\gamma(a_Q))\le\tau(\prod_{p\in Q}p)=C$, a constant, which completes the proof.

Remark 4. We have $A_a^{(s)}(x,Q)=O(x^sa^\varepsilon)$ for every Q and for every $\varepsilon>0$.

Lemma 8. *If A is a cross-convolution and g is an arithmetical function such that $g(n) = O(n^{\varepsilon})$ for every $\varepsilon > 0$ and if $s > 0$, then the series*

$$\sum_{n=1}^{\infty} \frac{g(n)\phi(n_Q)}{n^{s+1}n_Q}$$

is absolutely convergent. Let $S_s(g)$ denote the sum of the series. If in addition g is multiplicative, then

$$S_s(g) = D_P(g, s+1) \prod_{p \in Q} \left(1 + \left(1 - \frac{1}{p}\right) \sum_{a=1}^{\infty} \frac{g(p^a)}{p^{a(s+1)}}\right),$$

and if g is completely multiplicative, then

$$S_s(g) = \frac{D(g, s+1)}{D_Q(g, s+2)}.$$

Proof. The absolute convergence of the series follows at once by

$$\frac{g(n)\phi(n_Q)}{n^{s+1}n_Q} = O\left(\frac{n^{\varepsilon}}{n^{s+1}}\right) = O\left(\frac{1}{n^{s+1-\varepsilon}}\right),$$

where ε is chosen such that $\varepsilon < s$. If g is multiplicative, then the general term is multiplicative and the series can be expanded into an infinite product of Euler-type and we use Lemma 3.

Lemma 9. *If A is a cross-convolution and $s > 0$, then the series*

$$\sum_{n=1}^{\infty} \frac{\phi(n_Q)}{n^{s+1}n_Q}$$

is absolutely convergent and its sum is $\zeta(s+1)/\zeta_Q(s+2)$.

Proof. Apply Lemma 8 for the completely multiplicative function $g = I$.

Lemma 10. *If A is a cross-convolution, $S \subseteq \mathbb{N}$ and $s > 0$, then the series*

$$\sum_{n=1}^{\infty} \frac{\mu_{S,A}(n)\phi(n_Q)}{n^{s+1}n_Q}$$

is absolutely convergent and for $S = \{1\}$ its sum is

$$\frac{1}{\zeta_P(s+1)} \prod_{p \in Q} \left(1 - \frac{p-1}{p(p^{s+1}-1)}\right).$$

PROOF. It follows from Lemma 2 that $g = \mu_{S,A}$ satisfies the condition of Lemma 8. For $S = \{1\}$ we obtain by Lemma 3

$$\begin{aligned}
S_s(\mu_A) &= D_P(\mu_A, s+1) D_Q(\mu_A, s+1) \\
&= D_P(\mu, s+1) \prod_{p\in Q} \Big(1 + \Big(1 - \frac{1}{p}\Big) \sum_{a=1}^{\infty} \frac{\mu^*(p^a)}{p^{a(s+1)}}\Big) \\
&= D_P(\mu, s+1) \prod_{p\in Q} \Big(1 - \Big(1 - \frac{1}{p}\Big) \sum_{a=1}^{\infty} \frac{1}{p^{a(s+1)}}\Big) \\
&= \frac{1}{\zeta_P(s+1)} \prod_{p\in Q} \Big(1 - \frac{p-1}{p^{s+2}} \Big(1 - \frac{1}{p^{s+1}}\Big)^{-1}\Big) \\
&= \frac{1}{\zeta_P(s+1)} \prod_{p\in Q} \Big(1 - \frac{p-1}{p(p^{s+1}-1}\Big).
\end{aligned}$$

3. Asymptotic formulae

Theorem 1. *If A is a cross-convolution, g is an arithmetical function such that $g(n) = O(\tau(n)), s > 0$ and $f = g *_A E_s$, $E_s(n) = n^s$ for every $n \in \mathbb{N}$, then*

$$\sum_{n\le x} f(n) = \frac{S_s(g)}{s+1} x^{s+1} + O(B_s(g, x, Q)),$$

where $S_s(g)$ is given by Lemma 8 and $B_s(g, x, Q) = x^s$ ($s > 1$), $x \log^4 x$ ($s = 1$, g unbounded and Q infinite), $x \log^3 x$ ($s < 1$, g unbounded and Q infinite), $x \log^2 x$ ($s = 1$, g unbounded and Q finite or $s = 1$, g bounded and Q infinite), $x \log x$ ($s < 1$, g unbounded and Q finite or $s = 1$, g bounded and Q finite or $s < 1$, g bounded and Q infinite), x ($s < 1$, g bounded and Q finite).

PROOF. Using Lemma 7 we deduce

$$\sum_{n\le x} f(n) = \sum_{\substack{de=n\le x \\ (d,e)\in(P)}} g(d) e^s = \sum_{d\le x} g(d) \sum_{\substack{e\le x/d \\ (e,d)\in(P)}} e^s$$

$$= \sum_{d\le x} g(d)\left(\frac{\phi(d_Q)}{(s+1)d_Q}(x/d)^{s+1} + O(A_d^{(s)}(x/d,Q))\right)$$

$$= \frac{x^{s+1}}{s+1}\sum_{n=1}^{\infty}\frac{g(d)\phi(d_Q)}{d^{s+1}d_Q} - \frac{x^{s+1}}{s+1}\sum_{d>x}\frac{g(d)\phi(d_Q)}{d^{s+1}d_Q} + \sum_{d\le x} g(d)O(A_d^{(s)}(x/d,Q))$$

$$\equiv T_1(x) - T_2(x) + T_3(x),$$

where the first term is $T_1(x) = \frac{x^{s+1}}{s+1}S_s(g)$ by Lemma 8, and for the second term we have

$$T_2(x) = O\Big(x^{s+1}\sum_{d>x}\frac{\tau(d)}{d^{s+1}}\Big) = O\Big(x^{s+1}\frac{\log x}{x^s}\Big) = O(x\log x),$$

applying (8). If g is bounded, then

$$T_2(x) = O\Big(x^{s+1}\sum_{d>x}\frac{1}{d^{s+1}}\Big) = O(x^{s+1}x^{-s}) = O(x)$$

by (5). The third term above can be evaluated as follows:

$$T_3(x) = O\Big(\sum_{d\le x}\tau(d)A_d^{(s)}(x/d,Q)\Big),$$

now if Q is finite, then

$$T_3(x) = O\Big(\sum_{d\le x}\tau(d)(x/d)^s\Big) = O\Big(x^s\sum_{d\le x}\frac{\tau(d)}{d^s}\Big)$$

$$= \begin{cases} O(x^s), & \text{if } s>1,\\ O(x\log^2 x), & \text{if } s=1,\\ O(x^s x^{1-s}\log x) = O(x\log x), & \text{if } s<1, \end{cases}$$

by (6). If Q is infinite, then

$$T_3(x) = O\Big(\sum_{d\le x}\tau^2(d)(x/d)^s\Big) = O\Big(x^s\sum_{d\le x}\frac{\tau^2(d)}{d^s}\Big)$$

$$= \begin{cases} O(x^s), & \text{if } s>1,\\ O(x\log^4 x), & \text{if } s=1,\\ O(x^s x^{1-s}\log^3 x) = O(x\log^3 x), & \text{if } s<1, \end{cases}$$

by (7). If g is bounded, then

$$T_3(x) = O\Big(\sum_{d\le x} A_d^{(s)}(x/d, Q)\Big),$$

where for Q finite one obtains

$$T_3(x)=O\Big(\sum_{d\le x}(x/d)^s\Big)=O\Big(x^s\sum_{d\le x}\frac{1}{d^s}\Big)=\begin{cases} O(x^s), & \text{if } s>1,\\ O(x\log x), & \text{if } s=1,\\ O(x^s x^{1-s})=O(x), & \text{if } s<1,\end{cases}$$

using (4), and for Q infinite

$$T_3(x) = O\Big(\sum_{d\le x}\tau(d)(x/d)^s\Big) = O\Big(x^s\sum_{d\le x}\frac{\tau(d)}{d^s}\Big),$$

and we apply again (6). The theorem follows upon combining the results of evaluating $T_1(x)$, $T_2(x)$ and $T_3(x)$.

Theorem 2. *If A is a cross-convolution, $S \subseteq \mathbb{N}$ and $s > 0$, then for the function $\sigma_{S,A,s}$ we have*

$$\sum_{n\le x}\sigma_{S,A,s}(n) = \frac{\alpha_{S,s}x^{s+1}}{s+1} + O(C_s(x,Q)),$$

where

$$\alpha_{S,s} = \sum_{\substack{n=1\\ n\in S}}^{\infty}\frac{\phi(n_Q)}{n^{s+1}n_Q}$$

and $C_s(x,Q) = x^s$ ($s>1$), $x\log^2 x$ ($s=1$ and Q infinite), $x\log x$ ($s=1$ and Q finite or $s<1$ and Q infinite), x ($s<1$ and Q finite).

Proof. This is a consequence of Theorem 1 applied for $g = \rho_S$, the characteristic function of the subset S.

For different choices of the subset S we obtain various formulae. We present some particular cases.

Theorem 3. *If A is a cross-convolution, $S = Q_k$ is the set of k-free integers, where $k \in \mathbb{N}, k \geq 2$, and $s > 0$, then*

$$\sum_{n \leq x} \sigma_{Q_k,A,s}(n) = \frac{\zeta(s+1)}{(s+1)\zeta_P(k(s+1))} \times \prod_{p \in Q} \Big(1 - \frac{1}{p^{s+2}} - \frac{1}{p^{k(s+1)}} + \frac{1}{p^{k(s+1)+1}}\Big) x^{s+1} + O(C_s(x,Q)),$$

where $C_s(x,Q)$ is given in Theorem 2.

Proof. We have

$$\alpha_{Q_k,s} = \sum_{\substack{n=1 \\ n \in Q_k}}^{\infty} \frac{\phi(n_Q)}{n^{s+1} n_Q}$$

$$= \prod_{p \in P} \Big(1 + \frac{1}{p^{s+1}} + \ldots + \frac{1}{p^{(k-1)(s+1)}}\Big) \prod_{p \in Q} \Big(1 + \frac{\phi(p)}{p^{s+1}p} + \frac{\phi(p^2)}{p^{2(s+1)}p^2} + \ldots + \frac{\phi(p^{k-1})}{p^{(k-1)(s+1)}p^{k-1}}\Big)$$

$$= \prod_{p \in P} \Big(\Big(1 - \frac{1}{p^{k(s+1)}}\Big)\Big(1 - \frac{1}{p^{s+1}}\Big)^{-1}\Big) \times \prod_{p \in Q} \Big(1 + \frac{1}{p^{s+1}}\Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{1}{p^{s+1}} + \ldots + \frac{1}{p^{(k-2)(s+1)}}\Big)\Big)$$

$$= \frac{\zeta_P(s+1)}{\zeta_P(k(s+1))} \prod_{p \in Q} \Big(1 + \frac{1}{p^{s+1}}\Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{p^{(k-1)(s+1)}}\Big)\Big(1 - \frac{1}{p^{s+1}}\Big)^{-1}\Big)$$

$$= \frac{\zeta_P(s+1)}{\zeta_P(k(s+1))} \zeta_Q(s+1) \prod_{p \in Q} \Big(1 - \frac{1}{p^{s+1}} + \frac{1}{p^{s+1}}\Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{p^{(k-1)(s+1)}}\Big)\Big)$$

$$= \frac{\zeta(s+1)}{\zeta_P(k(s+1))} \prod_{p \in Q} \Big(1 - \frac{1}{p^{s+2}} - \frac{1}{p^{k(s+1)}} + \frac{1}{p^{k(s+1)+1}}\Big).$$

For $A = D$ ($Q = \emptyset$) and for $s = 1$ this result is due to E. Cohen [Co60c], Theorem 3.1.

Theorem 4. *If A is a cross-convolution, $S = S_k$ is the set of k-th powers of the positive integers with $k \in \mathbb{N}$ and $s > 0$, then*

$$\sum_{n \leq x} \sigma_{S_k,A,s}(n) = \frac{\zeta(k(s+1))}{(s+1)\zeta_Q(k(s+1)+1)} x^{s+1} + O(C_s(x,Q)),$$

where $C_s(x,Q)$ is given in Theorem 2.

PROOF. We get

$$\alpha_{S,s} = \sum_{n=m^k=1}^{\infty} \frac{\phi(n_Q)}{n^{s+1}n_Q} = \sum_{m=1}^{\infty} \frac{\phi((m^k)_Q)}{(m^k)_Q m^{k(s+1)}} = \sum_{m=1}^{\infty} \frac{\phi(m_Q)}{m_Q m^{k(s+1)}}$$
$$= \frac{\zeta(k(s+1))}{\zeta_Q(k(s+1)+1)},$$

by Lemma 9.

Theorem 5. *If A is a cross-convolution, $S=\mathbb{P}$ is the set of primes and $s>0$, then*

$$\sum_{n\le x} \sigma_{\mathbb{P},A,s}(n) = \frac{x^{s+1}}{s+1}\Big(\sum_{p\in\mathbb{P}} \frac{1}{p^{s+1}} - \sum_{p\in Q} \frac{1}{p^{s+2}}\Big) + O(C_s(x,Q)),$$

where $C_s(x,Q)$ is defined in Theorem 2.

PROOF. In this case we have

$$\alpha_{\mathbb{P},s} = \sum_{\substack{n=1\\ n\in\mathbb{P}}}^{\infty} \frac{\phi(n_Q)}{n^{s+1}n_Q} = \sum_{p\in P} \frac{1}{p^{s+1}} + \sum_{p\in Q} \frac{\phi(p)}{p^{s+2}} = \sum_{p\in\mathbb{P}} \frac{1}{p^{s+1}} - \sum_{p\in Q} \frac{1}{p^{s+2}}.$$

Theorem 6. *If A is a cross-convolution, $S=\{m\}$ with $m\in\mathbb{N}$ and $s>0$, then*

$$\sum_{n\le x} \sigma_{S,A,s}(n) = \frac{x^{s+1}}{(s+1)m^{s+1}} \prod_{\substack{p\mid m\\ p\in Q}} \left(1-\frac{1}{p}\right) + O(C_s(x,Q)),$$

$C_s(x,Q)$ being defined in Theorem 2.

Theorem 7. *If A is a cross-convolution and $s>0$, then*

$$\sum_{n\le x} \sigma_{A,s}(n) = \frac{\zeta(s+1)x^{s+1}}{(s+1)\zeta_Q(s+2)} + O(C_s(x,Q)),$$
$$\sum_{n\le x} \sigma_{A}(n) = \frac{\pi^2 x^2}{12\zeta_Q(3)} + O(C(x,Q)),$$

where $C(x,Q)\equiv C_1(x,Q) = x\log x$ (Q finite), $x\log^2 x$ (Q infinite).

PROOF. Apply Theorem 4 for $k=1$.

In the unitary case ($Q = \mathbb{P}$) these formulae were proved by E. Cohen [Co60a], Corollary 4.1.1 for $s = 1$, and by J. Chidambaraswamy [Chi67], Corollary A/(ii) for $s \geq 1$.

In what follows we deduce asymptotic formulae for certain functions of Euler-type.

Theorem 8. *If A is a cross-convolution, $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, then*

$$\sum_{n \leq x} \phi_{S,A,k}(n) = \frac{\beta_{S,k} x^{k+1}}{k+1} + O(C_k(S, x, Q)),$$

where

$$\beta_{S,k} = \sum_{n=1}^{\infty} \frac{\mu_{S,A}(n)\phi(n_Q)}{n^{k+1} n_Q}$$

and $C_k(S, x, Q) = x^k$ $(k > 1)$, $x \log^4 x$ $(k = 1$ and Q infinite$)$, $x \log^2 x$ $(k = 1$ and Q finite or $k = 1, Q$ infinite and S multiplicative$)$, $x \log x$ $(k = 1, Q$ finite and S multiplicative$)$.

Proof. From (3) and Remark 2 we have $\phi_{S,A,k} = \mu_{S,A} *_A E_k$. Now apply Theorem 1 for $g = \mu_{S,A}$ and use Lemmas 1, 2 and 10.

Theorem 9. *$(A = D)$ If $S \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, then for the function $\phi_{S,D,k} \equiv \phi_{S,k}$ we have*

$$\sum_{n \leq x} \phi_{S,k}(n) = \frac{\zeta_S(k+1)}{(k+1)\zeta(k+1)} x^{k+1} + O(C_k(S, x)),$$

where $\zeta_S(z) = D(\rho_S, z), C_k(S, x) = x^k$ $(k > 1)$, $x \log^2 x$ $(k = 1$ and S not multiplicative$)$, $x \log x$ $(k = 1$ and S multiplicative$)$.

Proof. Apply Theorem 8 and use that by (2) we have $D(\mu_S, z) = D(\rho_S, z) D(\mu, z) = \zeta_S(z)/\zeta(z), \operatorname{Re} z > 1$.

For $k = 1$ this result was given in [ST90], Theorem 2 and it is cited in [MSC96], page 33, with the remainder term $O(x \log^2 x)$ for every S, and for $k = 1$ and $S = \{n^k : n \in \mathbb{N}\}$ we have the formula due to E. Cohen [Co60c], Theorem 4.1.

Theorem 10. *If b is the function of R. Sivaramakrishnan, then*

$$\sum_{n\le x} b(n) = \frac{\pi^2}{30}x^2 + O(x\log x).$$

PROOF. Using Theorem 9 for S, the set of squares and $k = 1$, we obtain

$$\frac{\zeta_S(k+1)}{(k+1)\zeta(k+1)} = \frac{\zeta_S(2)}{2\zeta(2)} = \frac{\zeta(4)}{2\zeta(2)} = \frac{\pi^2}{30}.$$

From Theorem 8 one can deduce the known asymptotic formulae for the function ϕ_m of Klee, see U. V. SATYANARAYANA, K. PATTABHIRAMASASTRY [SP65], for the function φ_m^* of D. SURYANARAYANA, see [Sur72], Theorem 3.2, and for other particular functions investigated in the literature.

Theorem 11. *If $k \in \mathbb{N}$, then*

$$\sum_{n\le x} \phi_{A,k}(n) = \frac{\beta_k}{k+1}x^{k+1} + O(C_k(x,Q)),$$

where

$$\beta_k = \frac{1}{\zeta_P(k+1)}\prod_{p\in Q}\Big(1 - \frac{p-1}{p(p^{k+1}-1)}\Big)$$

and $C_k(x,Q) = x^k$ $(k > 1)$, $x\log^2 x$ $(k = 1$ and Q infinite$)$, $x\log x$ $(k = 1$ and Q finite$)$.

PROOF. Apply Theorem 8 with $S = \{1\}$ and Lemma 10.

For $A = D$ and $k = 1$ we have the classical formula of F. Mertens, for $A = U$ it was established by E. COHEN [C60a], Corollary 4.4.2 for $k = 1$, and [C61], Corollary 3.1.2 for $k > 1$.

Remark 5. An asymptotic formula for an even more general Euler function, including all the above functions is given in [TH96].

The formulae obtained in this paper are very general, the remainder terms can be improved for particular choices of S, A, s and k. As an example, we have the following result, see Theorem 7:

Theorem 12. *If A is a cross-convolution, then*

$$\sum_{n\le x}\sigma_A(n)=\frac{\pi^2x^2}{12\zeta_Q(3)}+O(D(x,Q)),$$

where $D(x,Q)=x\log^{2/3}x$ (Q finite), $x\log^{5/3}x$ (Q infinite).

Proof. We have the identity

$$\sigma_{AP}(n)=\sum_{d^2e=n}h(d)\sigma(e),\quad n\in\mathbb{N},$$

where the multiplicative function h is defined by

$$h(p^a)=\begin{cases}-p, & \text{if } p\in Q,\ a=1,\\ 0, & \text{otherwise,}\end{cases}$$

for every prime power p^a $(a\ge 1)$, see [SS73].

Using now the following result of A. Walfisz [W63]

$$\sum_{n\le x}\sigma(n)=\frac{\pi^2x^2}{12}+O(x\log^{2/3}x),$$

we get

$$\begin{aligned}\sum_{n\le x}\sigma_A(n)&=\sum_{d^2e=n\le x}h(d)\sigma(e)=\sum_{d\le\sqrt{x}}h(d)\sum_{e\le x/d^2}\sigma(e)\\
&=\sum_{d\le\sqrt{x}}h(d)\Big(\frac{\pi^2x^2}{12d^4}+O\Big(\frac{x}{d^2}\log^{2/3}\frac{x}{d^2}\Big)\Big)\\
&=\frac{\pi^2x^2}{12}\sum_{d\le\sqrt{x}}\frac{h(d)}{d^4}+O\Big(x\log^{2/3}x\sum_{d\le\sqrt{x}}\frac{|h(d)|}{d^2}\Big)\\
&=\frac{\pi^2x^2}{12}\sum_{d=1}^{\infty}\frac{h(d)}{d^4}+O\Big(x^2\sum_{d>\sqrt{x}}\frac{1}{d^3}\Big)+O\Big(x\log^{2/3}x\sum_{\substack{d\le\sqrt{x}\\ d\in(Q)}}\frac{1}{d}\Big)\\
&=\frac{\pi^2x^2}{12}\prod_{p\in Q}\Big(1-\frac{1}{p^3}\Big)+O(x)+O(D(x,Q))\\
&=\frac{\pi^2x^2}{12\zeta_Q(3)}+O(D(x,Q)),\end{aligned}$$

applying (5), (4) and Lemma 3 and using that for Q finite

$$\sum_{\substack{d=1\\ d\in(Q)}}^{\infty}\frac{1}{d}=\prod_{p\in Q}\Big(1-\frac{1}{p}\Big)^{-1}\qquad\text{is a constant.}$$

In the unitary case this formula was established by R. Sita Rama Chandra Rao and D. Suryanarayana [SS73], formula (1.4).

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