

A generalized Pexider equation

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In [2] the general solution of the functional equation

$$F(x) + G(y) = P(x+y) + Q(x/y), \quad x, y > 0,$$

was found among functions F, G, P and Q mapping the positive real numbers into an additive abelian group H in which division by 2 is uniquely defined. By this we mean that for every $x \in H$ there exists a unique $y \in H$ such that $x = 2y$, in which case we write $y = x/2$. In this paper we study the functional equation

$$(1) \quad f(x) + g(y) = p(x+y) + q(xy)$$

which is a generalized Pexider equation (see [1]). Using different techniques than those employed in [2] we will prove

Theorem 1. *Let $(H, +)$ be an abelian group in which division by 2 is uniquely defined and suppose $f, g, p, q: (0, \infty) \rightarrow H$. Then (1) holds for all $x, y > 0$ if and only if $f(x) = \alpha(x^2) + a(x) + m(x) + b$, $g(x) = \alpha(x^2) + a(x) + m(x) + c$, $p(x) = \alpha(x^2) + a(x) + \beta$ and $q(x) = -2\alpha(x) + m(x) + \gamma$ for all $x > 0$ where b, c, β and γ are constants in H with $b + c = \beta + \gamma$ and where $\alpha, a, m: (0, \infty) \rightarrow H$ such that for all $x, y > 0$, $\alpha(x+y) = \alpha(x) + \alpha(y)$, $a(x+y) = a(x) + a(y)$ and $m(xy) = m(x) + m(y)$.*

If p and q satisfy the Pexider equations $p(x+y) = p_1(x) + p_2(y)$, $q(xy) = q_1(x) + q_2(y)$ for $x, y > 0$ then clearly $p(x+y) + q(xy) = f(x) + g(y)$ where $f = p_1 + q_1$ and $g = p_2 + q_2$. But Theorem 1 shows that not every solution of (1) can be obtained in this way. Thus (1) may be thought of as a generalized Pexider equation.

If (1) is assumed to hold for all *real* x and y , then putting $y=0$ shows that f and p differ by a constant. Putting $x=0$ shows that g and p differ by a constant. Thus, for an appropriate Q , $p(x) + p(y) = p(x+y) + Q(xy)$ for all real x and y . This equation was studied by ECSEDI and HOSSZÚ [5].

Equation $\left(\frac{x^2}{4}\right)$ is also equivalent to what might be called a generalized Jensen equation (see [1]). For if (1) holds for $x, y > 0$ and if we replace x by $x/2$, y by $y/2$ and let $R(x) = q\left(\frac{x^2}{4}\right)$ we find that

$$p\left(\frac{x+y}{2}\right) + R(\sqrt{xy}) = \varphi(x) + \psi(y)$$

where $\varphi(x)=f(x/2)$ and $\psi(y)=g(y/2)$. Interchanging x and y we see that φ and ψ differ by a constant and so we have

$$p\left(\frac{x+y}{2}\right) + R(\sqrt{xy}) = T(x) + T(y).$$

Putting $x=y$ we find $2T(x)=p(x)+R(x)$ so that

$$2p\left(\frac{x+y}{2}\right) + 2R(\sqrt{xy}) = p(x) + p(y) + R(x) + R(y)$$

which is a generalized Jensen equation.

In order to get some feeling for theorem 1 and its proof let us consider the "regular" solutions of (1). Suppose f, g, p and q are real valued, twice differentiable on $(0, \infty)$ and satisfy (1) for all $x, y > 0$. Differentiating (1) with respect to x and then differentiating the resulting equation with respect to y we conclude that

$$p''(x+y) + q'(xy) + xyq''(xy) = 0 \quad \text{for all } x, y > 0.$$

From Lemma 1 below we deduce that p'' is constant and hence $p(x) = \alpha x^2 + ax + \beta$ for all $x > 0$ where α, a and β are real constants. Similarly, since $q'(x) + xq''(x) = -2\alpha$ for $x > 0$, we find that $q(x) = -2\alpha x + k \log x + \gamma$, $x > 0$, where k and γ are real constants. Having found p and q it is easy to determine f and g .

To prove Theorem 1 we determine p using results of Djoković [3] concerning a difference analogue of the differential equation $p'''(x) = 0$. We then use a theorem of Daróczy, Lajkó and Székelyhidi [4] to find q .

Lemma 1. *If S is a set and $F, G: (0, \infty) \rightarrow S$ such that $F(x+y) = G(xy)$ for all $x, y > 0$ then F and G are constant.*

PROOF. Fix $k > 0$ arbitrarily. Then $F\left(x + \frac{k}{x}\right) = G(k)$ for all $x > 0$. If $t \geq 2\sqrt{k}$ then there is an $x > 0$ such that $x + \frac{k}{x} = t$. Hence $F(t) = G(k)$ for $t \geq 2\sqrt{k}$, i.e. F is constant on $(2\sqrt{k}, \infty)$ for any $k > 0$. Thus F is constant on $(0, \infty)$ and hence so is G .

We will also need

Lemma 2. *If S is a set and $F, G: (0, \infty) \rightarrow S$ such that $F(2x+y) = G(x+2y)$ for all $x, y > 0$ then F and G are constant.*

PROOF. For $0 < \frac{v}{2} < u < 2v$ put $x = \frac{2u-v}{3} > 0$ and $y = \frac{2v-u}{3} > 0$ so that $2x+y = u$ and $x+2y = v$. Thus $F(u) = G(v)$ for $\frac{v}{2} < u < 2v$ and hence F is constant on $\left(\frac{v}{2}, 2v\right)$ for any $v > 0$. It follows that F is constant on $(0, \infty)$ and hence G is as well.

Notation. $(H, +)$ denotes an abelian group in which division by 2 is uniquely defined. If $f: (0, \infty) \rightarrow H$ and $t > 0$ we let $\Delta_t f(x) = f(x+t) - f(x)$ and $\delta_t f(x) = f(tx) - f(x)$ for all $x > 0$.

PROOF OF THEOREM 1. Suppose $f, g, p, q: (0, \infty) \rightarrow H$ and (1) holds for all $x, y > 0$. Then

$$p(2x+y) + q(2xy) = f(2x) + g(y)$$

and

$$p(x+2y) + q(2xy) = f(x) + g(2y)$$

so that

$$(2) \quad p(2x+y) - p(x+2y) = \varphi(x) + \psi(y) \quad \text{for all } x, y > 0$$

where $\varphi(x) = f(2x) - f(x)$ and $\psi(y) = g(y) - g(2y)$ for all $x, y > 0$.

Replace x by $x+r$ in (2) and subtract (2) from the resulting equation to get

$$(3) \quad \Delta_{2r} p(2x+y) - \Delta_r p(x+2y) = \Delta_r \varphi(x) \quad \text{for } x, y, r > 0.$$

Replace y by $y+s$ in (3) and subtract to get

$$(4) \quad \Delta_s \Delta_{2r} p(2x+y) = \Delta_{2s} \Delta_r p(x+2y) \quad \text{for } x, y, r, s > 0.$$

Thinking of r and s as parameters and using Lemma 2 we conclude from (4) that $\Delta_s \Delta_{2r} p(2x+y)$ depends only on s and r . Hence we may write $\Delta_s \Delta_r p(x) = C(r, s)$ for $x, r, s > 0$, and then conclude that

$$(5) \quad \Delta_t \Delta_s \Delta_r p(x) = 0 \quad \text{for all } x, r, s, t > 0.$$

Now from [3] it follows that

$$(6) \quad p(x) = A(x, x) + a(x) + \beta, \quad x > 0$$

where $A: (0, \infty) \times (0, \infty) \rightarrow H$ is symmetric and biadditive, $a: (0, \infty) \rightarrow H$ is additive and β is a constant in H .

Using (1), (6), the symmetry and biadditivity of A and the additivity of a we find that

$$q(tx) = f(t) + f(x) - A(t, t) - 2A(x, t) - A(x, x) - a(t) - a(x) - \beta$$

so that

$$(7) \quad \delta_t q(x) = q(tx) - q(x) = \varphi(t) + B(x, t) \quad \text{for } x, t > 0$$

where $\varphi(t) = f(t) - f(1) - A(t, t) + A(1, 1) - a(t) + a(1)$ and $B(x, t) = -2A(x, t) + 2A(x, 1)$ for all $x, t > 0$. Notice that $B(x+y, t) = B(x, t) + B(y, t)$ for all $x, y, t > 0$ and so from (7) we conclude that

$$(8) \quad 2\delta_t q\left(\frac{x+y}{2}\right) = \delta_t q(x) + \delta_t q(y) \quad \text{for all } x, y, t > 0.$$

It follows from theorem 1 of [4] that, because of (8),

$$(9) \quad q(x) = m(x) + J(x) \quad \text{for } x > 0$$

where $m, J: (0, \infty) \rightarrow H$ such that

$$(10) \quad m(xy) = m(x) + m(y)$$

and

$$(11) \quad 2J\left(\frac{x+y}{2}\right) = J(x) + J(y) \quad \text{for all } x, y > 0.$$

Now it is well known that since J satisfies (11) there must exist an additive function $a_1: (0, \infty) \rightarrow R$ and a constant $\gamma \in H$ such that $J(x) = a_1(x) + \gamma$ for all $x > 0$. It is notationally convenient to let $a_1(x) = -2\alpha(x)$ so that (9) may be written

$$(12) \quad q(x) = m(x) - 2\alpha(x) + \gamma, \quad x > 0,$$

where $\alpha: (0, \infty) \rightarrow H$ such that

$$(13) \quad \alpha(x+y) = \alpha(x) + \alpha(y) \quad \text{for all } x, y > 0.$$

It remains to show that $A(x, x) = \alpha(x^2)$ for $x > 0$ and to determine f and g . Interchange x and y in (1) and deduce that

$$(14) \quad p(x+y) + q(xy) = h(x) + h(y), \quad x, y > 0$$

where

$$(15) \quad h(x) = \frac{f(x) + g(x)}{2} \quad \text{for } x > 0.$$

From (14), (6), (10), (12), (13), the biadditivity and symmetry of A and the additivity of a we find that

$$A(x, x) + 2A(x, y) + A(y, y) + a(x) + a(y) + \beta + m(x) + m(y) - 2\alpha(xy) + \gamma = h(x) + h(y)$$

or

$$(16) \quad A(x, y) - \alpha(xy) = k(x) + k(y) \quad \text{for all } x, y > 0$$

where

$$(17) \quad 2k(x) = h(x) - A(x, x) - a(x) - m(x) - \frac{\beta + \gamma}{2} \quad \text{for } x > 0.$$

From (16) it follows that

$$(18) \quad A(x, 1) - \alpha(x) = k(x) + k(1)$$

and

$$(19) \quad A(x, x) - \alpha(x^2) = 2k(x) \quad \text{for } x > 0.$$

From (18) and (19) we find

$$(20) \quad A(x, x) - \alpha(x^2) = 2A(x, 1) - 2\alpha(x) - 2k(1), \quad x > 0.$$

Replacing x by $2x$ in (20), using additivity and dividing by 2 we find

$$(21) \quad 2A(x, x) - 2\alpha(x^2) = 2A(x, 1) - 2\alpha(x) - k(1).$$

Subtracting (20) from (21) gives

$$(22) \quad A(x, x) - \alpha(x^2) = k(1).$$

From (19) and (22) it follows that $2k(x) = k(1)$ for all $x \equiv 0$ and hence

$$(23) \quad k(x) = 0 \quad \text{for all } x > 0.$$

Now (19) and (23) imply $A(x, x) = \alpha(x^2)$ so, from (6),

$$(24) \quad p(x) = \alpha(x^2) + a(x) + \beta \quad \text{for } x > 0.$$

From (17) and (23) it follows that

$$(25) \quad h(x) = \alpha(x^2) + a(x) + m(x) + \frac{\beta + \gamma}{2} \quad \text{for } x > 0.$$

But from (1) and (14) we find

$$f(x) + g(y) = h(x) + h(y) \quad \text{for } x, y > 0$$

from which it follows that f (and g) differ from h by a constant. Thus we have

$$(26) \quad f(x) = \alpha(x^2) + a(x) + m(x) + b$$

and

$$(27) \quad g(x) = \alpha(x^2) + a(x) + m(x) + c \quad \text{for } x > 0$$

where b and c are constants.

From (1), (12), (24), (26) and (27) it follows that $\beta + \gamma = b + c$.

An easy calculation shows that the converse is true. This completes the proof of Theorem 1.

Theorem 2. Suppose $f, g, p, q: (0, \infty) \rightarrow \mathbb{R}$ (the real numbers) such that (1) holds for all $x, y > 0$. If there exist Lebesgue measurable subsets S and T of $(0, \infty)$ of positive Lebesgue measure such that p is bounded on S and q is bounded on T then there exist real constants $b, c, \beta, \gamma, \varrho, \sigma$ and τ such that $f(x) = \varrho x^2 + \sigma x + \tau \log x + b$, $g(x) = \varrho x^2 + \sigma x + \tau \log x + c$, $p(x) = \varrho x^2 + \sigma x + \beta$ and $q(x) = -2\varrho x + \tau \log x + \gamma$ for all $x > 0$.

PROOF. From (5) we have $\Delta_t^3 p(x) = 0$ for all $x, t > 0$ and p is bounded on a set of positive measure so, from a theorem of KEMPERMAN [6], it follows that

$$p(x) = \varrho x^2 + \sigma x + \beta \quad \text{for all } x > 0$$

where ϱ, σ and β are real constants. Hence we conclude from (12) that m is bounded on a set of positive measure and satisfies (10). It follows that $m(x) = \tau \log x$ for all $x > 0$ where τ is a real constant. To see this let $M(t) = m(e^t)$ for t real so that $M(s+t) = M(s) + M(t)$ for all real s and t and M is bounded on a set of positive measure. Thus (see [6]) $M(t) = \tau t$ for all real t where τ is a real constant. Therefore $m(x) = M(\log x) = \tau \log x$ for all $x > 0$. This completes the proof.

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