

On a conjecture concerning additive arithmetical functions II

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Abstract. If f is completely additive and $f(2n + A) - f(n)$ is monotonic from some number on, then $f(n) = c \log n$.

In 1946 ERDÖS [2] proved the following theorem:

Theorem 1 (Erdős). *If a real valued additive function f is monotonically increasing, then $f(n) = c \log n$.*

As a possible generalization of this result I proposed the following conjecture:

Conjecture. *Let f be an additive function. If $f(an + b) - f(cn + d)$ is monotonic from some number on, then $f(n) = c \log n$ for all n coprime to $ac(ad - bc)$.*

If f is bounded, then $f(an + b) - f(cn + d)$ is convergent and the conjecture is true by a theorem of ELLIOTT [1]. In [3] we proved some special cases of the conjecture, including the following theorem:

Theorem 2. *Let f be an additive function and let a be an integer. If $f(n + a) - f(n)$ is monotonic or it is of constant sign from some number on, then $f(n) = c \log n$ for all n coprime to a . If f is completely additive, then $f(n) = c \log n$ for all n .*

Here we prove the conjecture in a further special case:

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Theorem. *Let f be a completely additive function. If*

$$(1) \quad f(2n + A) - f(n)$$

is monotonic from some number on, then $f(n) = c \log n$ for all n .

PROOF of the Theorem. We may assume that A is odd in (1). (Otherwise by Theorem 2 $f(n) = c \log n$ for all n .)

We consider now that (1) is monotonically increasing. We put $z = n \left(n + \frac{3+A}{2} \right)$. By comparing the value of (1) at the pairs of numbers (z, n) we have

$$(2) \quad f(2z + A) - f(n) - f\left(n + \frac{A+3}{2}\right) > f(2n + A) - f(n).$$

By connecting the pair $(2z + A, n + \frac{3-A}{2})$ we obtain

$$(3) \quad f(4z + 3A) - f(2z + A) > f(2n + 3) - f\left(n + \frac{3-A}{2}\right).$$

We observe that $4z + 3A = 4n^2 + 2(A+3)n + 3A = (2n + A)(2n + 3)$, therefore adding the rows (2) and (3) we obtain

$$f\left(n + \frac{A-3}{2}\right) - f\left(n + \frac{A+3}{2}\right) > 0.$$

Applying Theorem 2, we have $f(n) = c \log n$.

Background of the proof: We put $z = (n + a_1)(n + a_2)$. We apply (1) for the pair (z, N_1) (we determine N_1 later). So we have

$$(4) \quad f(2z + A) - f(n + a_1) - f(n + a_2) > f(2N_1 + A) - f(N_1).$$

To cancel the $f(2z + A)$ term by addition, we apply (1) for the pair $(2z + A, N_2)$ (we determine N_2 later). So we have

$$(5) \quad f(4z + 3A) - f(2z + A) > f(2N_2 + A) - f(N_2).$$

If we had $4z + 3A = 4n^2 + bn + c = (2n + A_1)(2n + A_2)$ with integers A_1 and A_2 , and also $2n + A_1 = 2N_1 + A$ and $2n + A_2 = 2N_2 + A$, then we could cancel some terms. We can do it only if A_1 and A_2 are odd. Then adding the rows (4) and (5), we have

$$-f(n + a_1) - f(n + a_2) > -f\left(n + \frac{A_1 - A}{2}\right) - f\left(n + \frac{A_2 - A}{2}\right).$$

If $a_1 = \frac{A_1 - A}{2}$ and $a_2 \neq \frac{A_2 - A}{2}$, then we arrive at a special case of Theorem 2.

By solving the equation $4z + 3A = 4(n + a_1)(n + a_2) + 3A = 0$, we get $A_{1,2} = a_1 + a_2 \pm \sqrt{(a_1 - a_2)^2 - 3A}$. To satisfy the condition $a_1 = \frac{A_1 - A}{2}$, we need the choice $a_2 = \frac{A+1}{2} + 1 + a_1$. Therefore $A_1 = 2a_1 + A$ and $A_2 = 2a_1 + 3$ are odd integers and we arrive at

$$f(n + a_2 - A) - f(n + a_2) > 0,$$

i.e. by Theorem 2 $f(n) = c \log n$.

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