A strong topology for the union of topological spaces

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0. Introduction

If $(X_i)_{i\in I}$ is a family of topological spaces then $X=U_{i\in I}X_i$ is usually considered to be equipped with the finest topology for which the identity mappings of all the spaces X_i into X are continuous.

In this paper, we shall study a finer topology on X, namely we are considering X to be equipped with the coarsest topology for which the identity mappings of all the spaces X_i into X are open.

In connection with this topology, we prove here a few immediate results which are needed as a basis for some topological considerations in the multiplier extensions of admissible vector modules [8].

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1. Final topologies

Notation 1.1. Let $(X_i)_{i\in I}$ be a family of topological spaces and Y be a set. Moreover, for each $i\in I$, let f_i be a mapping from X_i into Y, and suppose that $Y=U_{i\in I}f_i(X_i)$.

Theorem 1.2. *There exists a coarsest topology on Y for which all the mappings f_i are open. Moreover, this topology has the family $U_{i\in I}f_i(\mathcal{T}_i)$, where $\mathcal{T}_i$ denotes the topology of X_i, as a subbase.*

PROOF. Obvious.

Remark 1.3. This theorem does not requires that $Y=U_{i\in I}f_i(X_i)$.

Theorem 1.4. *The coarsest topology on Y for which all the mappings f_i are open is finer than the finest topology on Y for which all the mappings f_i are continuous.*

PROOF. Denote $\mathcal{T}$ (resp. $\mathcal{T}^*$) the coarsest (resp. finest) topology on Y for which all the mappings f_i are open (resp. continuous). If $V\in\mathcal{T}^*$, then we have

$$V=U_{i\in I}V\cap f_i(X_i)=U_{i\in I}f_i\big(f_i^{-1}(V)\big),$$

whence it is clear that $V\in\mathcal{T}$. Consequently, $\mathcal{T}^*\subset\mathcal{T}$.

Corollary 1.5. *Equip Y with the coarsest topology for which all the mappings f_i are open. If φ is a mapping from Y into a topological space Z such that $\varphi \circ f_i$ is continuous for all $i \in I$, then φ is continuous.*

PROOF. This follows immediately from Theorem 1.4.

2. Union spaces

Notation 2.1. Let $(X_i)_{i \in I}$ be a family of topological spaces, $X = U_{i \in I} X_i$, and equip X with the coarsest topology for which the identity mappings of all the spaces X_i into X are open.

Remark 2.2. By Theorem 1.2, the topology of X may also be described as the supremum of the topologies $\mathcal{T}_i \cup \{X\}$, where $\mathcal{T}_i$ denotes the topology of X_i.

Theorem 2.3. *For each $i \in I$, X_i is an open subset of X, and the restriction to X_i of the topology of X is finer than the original topology of X_i.*

PROOF. This is an immediate consequence of the definition of the topology of X.

Theorem 2.4. *For some $i \in I$, X_i is a subspace of X if and only if the identity mapping of X_i into X is continuous.*

PROOF. This follows at once from Theorem 2.3.

Theorem 2.5. *Suppose that there exists $i_0 \in I$ such that $X_{i_0} \subset X_i$ for all $i \in I$. Then the identity mapping of X_{i_0} into X is continuous if and only if the identity mapping of X_{i_0} into X_i is continuous for all $i \in I$.*

PROOF. This is quite obvious from Theorem 1.2.

Theorem 2.6. *Suppose that I is directed, and $X_i \subset X_j$ if $i \leqq j$. Then the following conditions are equivalent:*

(i) *the identity mappings of all the spaces X_i into X are continuous;*

(ii) *the identity mapping of X_i into X_j is continuous and open if $i \leqq j$.*

PROOF. Suppose that (i) holds. If $i, j \in I$ such that $i \leqq j$, and U is an open subset of X_i, then U is also open in X, and thus $U = U \cap X_j$ is also open in X_j. On the other hand, if V is open in X_j, then V is also open in X, and thus $V \cap X_i$ is open in X_i. This proves (ii).

Suppose now that (ii) holds. If $i, j \in I$ and V is open in X_j, then choosing $k \in I$ such that $i \leqq k$ and $j \leqq k$, we can infer that V is open in X_k, and hence that $V \cap X_i$ is open in X_i. Hence, by Theorem 1.2, (i) follows.

Remark 2.7. The condition that the identity mapping of X_i into X_j is open if $i \leqq j$ fails to hold in most of the applications.

Remark 2.8. If the identity mappings of all the spaces X_i into X are continuous, then by Theorem 1.4, the topology of X coincides with the finest topology on X for which the identity mappings of all the spaces X_i into X are continuous.

Theorem 2.9. *Suppose I is directed and $X_i \subset X_j$ if $i \leqq j$. Moreover, suppose that there exists a cofinal subset K of I such that X_k is Hausdorff for all $k \in K$. Then X is also Hausdorff.*

PROOF. If $x_1, x_2 \in X$, then by the assumptions there exists $k \in K$ such that $x_1, x_2 \in X_k$. Thus, if $x_1 \neq x_2$, there are disjoint open subsets V_1 and V_2 of X_k such that $x_1 \in V_1$ and $x_2 \in V_2$. Now, since V_1 and V_2 are also open in X, the proof is complete.

Theorem 2.10. *Let (x_α) be a net in X and $x \in X$. Then the following conditions are equivalent:*

(i) *$x \in \lim_\alpha x_\alpha$ in X;*

(ii) *for each $i \in I$ such that $x \in X_i$, there exists α_0 such that $\{x_\alpha\}_{\alpha \geqq \alpha_0} \subset X_i$ and $x \in \lim_{\alpha \geqq \alpha_0} x_\alpha$ in X_i.*

PROOF. Suppose first that we have (i). Let $i \in I$ such that $x \in X_i$, and let V be an open subset of X_i such that $x \in V$. Then, since V is also open in X, there exists α_0 such that $x_\alpha \in V$ for all $\alpha \geqq \alpha_0$. Hence, it is clear that $\{x_\alpha\}_{\alpha \geqq \alpha_0} \subset X_i$ and $x \in \lim_{\alpha \geqq \alpha_0} x_\alpha$ in X_i.

Suppose now that (ii) holds. Let V be an open subset of X such that $x \in V$. Then, by Theorem 1.2, there is a finite subset $\{i_k\}_{k=1}^n$ of I, and for each $k = 1, 2, \ldots, n$ an open subset V_k of X_{i_k} such that $x \in \bigcap_{k=1}^n V_k \subset V$. Hence, by (ii), for each $k = 1, 2, \ldots n$, there exists α_k such that $x_\alpha \in V_k$ for all $\alpha \geqq \alpha_k$. Choosing α_0 such that $\alpha_k \leqq \alpha_0$ for all $k = 1, 2, \ldots, n$, we have $x_\alpha \in V$ for all $\alpha \geqq \alpha_0$. This proves (i).

Theorem 2.11. *Suppose that I is directed, and $X_i \subset X_j$ if $i \leqq j$. If C is a compact subset of X, then there exists $i \in I$ such that $C \subset X_i$, and moreover, if $C \subset X_j$, then C is also compact in X_j.*

PROOF. Let C be a compact subset of X. Since $\{X_i\}_{i \in I}$ is an open cover of C, there exists a finite subset $\{i_k\}_{k=1}^n$ of I such that $C \subset \bigcup_{k=1}^n X_{i_k}$. Choosing $i \in I$ such that $i_k \leqq i$ for all $k = 1, 2, \ldots, n$, we have $C \subset X_i$. Finally, if $C \subset X_j$, then by Theorem 1.2, it is clear that C is also compact in X_j.

Corollary 2.12. *Suppose that I is directed, and moreover, suppose that $X_i \subset X_j$ and the identity mapping of X_i into X_j is continuous and open if $i \leqq j$. Then C is a compact subset of X if and only if C is a compact subset of some X_i.*

PROOF. This follows immediately from Theorems 2.6 and 2.11.

Remark 2.13. Some of the results of this section can slightly be generalized according to the first section.

3. The case of topological vector spaces

Notation 3.1. Let X be a vector space over $\mathbf{K}$ ($=\mathbf{R}$ or $\mathbf{C}$), and $(X_i)_{i\in I}$ be a family of subspaces of X such that $X=U_{i\in I}X_i$. Suppose that each X_i is equipped with a vector topology, and consider X to be equipped with the coarsest topology for which the identity mappings of all the spaces X_i into X are open.

Theorem 3.2. (i) *If $x, y\in X$ such that $x+y\in X_i$ implies $x, y\in X_i$, then the addition in X is continuous at the point (x, y).*

(ii) *If $0\neq\lambda\in\mathbf{K}$ and $x\in X$, then the scalar multiplication in X is continuous at the point (λ, x).*

Proof. Let $x, y\in X$ as in (i), and suppose that W is an open subset of X such that $x+y\in W$. By Theorem 1.2, we may suppose that W is an open subset of some X_i. Then, by the assumption, we also have $x, y\in X_i$. Hence, since the addition in X_i is continuous, we can infer that there are open subsets U and V of X_i such that $x\in U$, $y\in V$ and $U+V\subset W$. Now, since U and V are also open in X, the proof of (i) is complete.

The assertion (ii) can be proved quite similarly, namely $\lambda x\in X_i$ implies $x\in X_i$ if $\lambda\neq 0$, and the scalar multiplication in X_i is continuous.

Theorem 3.3. *Suppose that I is a directed set, and moreover, suppose that $X_i\subset X_j$ and the identity mapping of X_i into X_j is open if $i\leqq j$. Then X is also a topological vector space.*

Proof. The proof is similar to that of Theorem 3.2.

Example 3.4. The Euclidean space $\mathbf{R}^2$ with its usual topology is a topological vector space over $\mathbf{R}$ and $\mathbf{R}\times\{0\}$ is a closed subspace of $\mathbf{R}^2$. Let $\mathcal{T}$ be the coarsest topology on $\mathbf{R}^2$ for which the identity mappings of the spaces $\mathbf{R}\times\{0\}$ and $\mathbf{R}^2$ into $\mathbf{R}^2$ are open. Then $(\mathbf{R}^2, \mathcal{T})$ is not a topological vector space.

To prove this, we show that the translation

$$(x, y)\to(x, y)+(0, -1)$$

is not continuous at the point $(1, 1)$ for $\mathcal{T}$. For this, let (y_n) be a sequence in $\mathbf{R}\setminus\{1\}$ such that $\lim_n y_n=1$. Then, by Theorem 2.10, it is clear that $\lim_n(1, y_n)=(1, 1)$ in $(\mathbf{R}^2, \mathcal{T})$, but the sequence $((1, y_n)+(0, -1))=((1, y_n-1))$ fails to converge in $(\mathbf{R}^2, \mathcal{T})$.

Example 3.5. The space $l^2=l^2(\{1, 2, \dots\})$ [5] with its usual topology is a topological vector space over $\mathbf{C}$, and

$$l_c^2=\{(x_n)\in l^2 : x_n\neq 0 \text{ only for finitely many } n\}$$

is a subspace of l^2 such that l_c^2 and $l^2\setminus l_c^2$ are also dense in l^2. Let $\mathcal{T}$ be the coarsest topology on l^2 for which the identity mappings of the spaces l_c^2 and l^2 into l^2 are open. Then $(l^2, \mathcal{T})$ is not uniformizable.

To prove this, we show that $(l^2, \mathcal{T})$ is not completely regular. For this, suppose indirectly that $(l^2, \mathcal{T})$ is completely regular. Then, in particular, for

$0 \in l_c^2 \in \mathcal{T}$, there exists a continuous function $\varphi\colon (l^2, \mathcal{T}) \to [0, 1]$ such that $\varphi(0)=1$ and $\varphi(x)=0$ for all $x \in l^2 \setminus l_c^2$. Since $0 \in \varphi^{-1}\left(\left]\frac{1}{2}, 1\right]\right) \in \mathcal{T}$ and l_c^2 is considered as a topological subspace of l^2, by Theorem 1.2, there exists an open subset U of l^2 such that $0 \in U \cap l_c^2 \subset \varphi^{-1}\left(\left]\frac{1}{2}, 1\right]\right)$. Since $l^2 \setminus l_c^2$ is dense in l^2, we can choose an $x_0 \in U \setminus l_c^2$. Since $x_0 \in \varphi^{-1}\left(\left[0, \frac{1}{2}\right[\right) \in \mathcal{T}$ and $x_0 \notin l_c^2$, again by Theorem 1.2, there exists an open subset V of l^2 such that $x_0 \in V \subset \varphi^{-1}\left(\left[0, \frac{1}{2}\right[\right)$. Since $U \cap V$ is a nonvoid open subset of l^2 and l_c^2 is also dense in l^2, $(U \cap V) \cap l_c^2 \neq \emptyset$. Thus, we have $\varphi^{-1}\left(\left[0, \frac{1}{2}\right[\right) \cap \varphi^{-1}\left(\left]\frac{1}{2}, 1\right]\right) \neq \emptyset$, and this is a contradiction. Consequently, $(l^2, \mathcal{T})$ can not be completely regular.

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