

Note on maximal asymptotic nonbases of order h

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In [3] NATHANSON introduced the notion of a maximal nonbase sequence of order h . In his paper a sequence $\mathcal{A}$ is called an asymptotic nonbase sequence of order h if it has the following two properties:

- (1) $\mathcal{A}$ is not an asymptotic base sequence of order h ;
- (2) if " a " is a nonnegative integer and $a \notin \mathcal{A}$, then $\mathcal{A} \cup \{a\}$ is an asymptotic base sequence of order h .

In [3] NATHANSON constructs maximal asymptotic nonbase sequences which can be represented as a union of residue classes; moreover he poses the problem whether there exist asymptotic maximal nonbase sequences of another type. An affirmative answer to this problem was given by ERDŐS and NATHANSON in [2]. In [6] one finds asymptotic maximal nonbase sequences $\mathcal{A}$ of 2nd order for which $\liminf \frac{A(x)}{x} = 0$ (where $A(x)$ denotes as usual the number of those elements of the sequence which are not greater than x); in [4] and [7] one even finds a sequence for which the order of $A(x)$ is $O(x^{\frac{1}{2}})$. In the case $h > 2$, however, no maximal nonbase sequence of order h and of density O is known.

In the present paper we show that there exist asymptotic nonbases of order h such that the order of their density function is $O(x^{\frac{1}{h}})$. More exactly, we give a procedure by the aid of which we depart from nonbase sequences of order h whose density functions have the order $O(x^{\frac{1}{h}})$; from these sequences we construct sequences whose density functions have the same order but which will be maximal nonbase sequences of order h . The abovementioned procedure has the same main features as the procedure applied in [6] and in [7].

The basic idea of the proof stems from the following analysis of the definition of an asymptotic maximal nonbase sequence of order h .

What does the first condition mean for $\mathcal{A}$? This condition requires the existence of infinitely many natural numbers M_i such that $M_i \notin h\mathcal{A}$ where

$$h\mathcal{A} = \left\{ x \mid x = \sum_{i=1}^h a_{i_i} \text{ and } a_{i_i} \in \mathcal{A} \right\}.$$

The second condition means that in case $a \notin \mathcal{A}$ (" a " is a natural number) the sufficiently large M_i 's belong to $h\{\mathcal{A} \cup \{a\}\}$, that is: those numbers which are not

represented by $h\mathcal{A}$, are represented by $h\{\mathcal{A} \cup \{a\}\}$ provided that they are sufficiently large. This can be carried through in such a way that either

$$\begin{aligned}
 & \{M_i - a\} \in (h-1)\mathcal{A} \quad \text{or} \\
 & \{M_i - 2a\} \in (h-2)\mathcal{A} \quad \text{or} \\
 (3) \quad & \vdots \\
 & \{M_i - (h-1)a\} \in \mathcal{A} \quad \text{or} \\
 & M_i = la \quad (l = 1, 2, \dots, h).
 \end{aligned}$$

Not that the latter relation can be satisfied by only finitely many M_i 's.

Assume now $a \notin \mathcal{A}$ and choose " a " from the interval $[0; b]$ then there exists a number m which depends on b and on $\mathcal{A}$ in such a way that in case $M_i > m$ we have $M_i \in h\mathcal{A} \cup \{a\}$ and then at least one of the relations (3) is satisfied. It is more convenient for us to construct $\mathcal{A}$ in such a way that in case of given $b \cong a \cong 0$ and $M_i > m$ the relation

$$\{M_i - a\} \in (h-1)\mathcal{A}$$

is satisfied.

We shall construct sequences $\mathcal{A}^*$; $\{m_i\}$; $\{b_i\}$; $\{M_i\}$ for which the following conditions hold:

- (4) With the exception of the integers of $\{M_i\}$ the sequence $h\mathcal{A}^*$ contains each nonnegative integer.
- (5) If " a " is a nonnegative integer with $a \notin \mathcal{A}^*$, and if $0 \cong a \cong b$ and $m_i < M_i \leq M_j$, then

$$\{M_j - a\} \in (h-1)\mathcal{A}^*.$$

- (6) $b_i < m_i < M_i$ and $\{b_i\}$; $\{m_i\}$; $\{M_i\}$ are strictly monotonically increasing ($i = 1, 2, \dots, n, \dots$).

Condition (4) requires that $\mathcal{A}^*$ is not an asymptotic base of order h . From (5) it follows that $a \notin \mathcal{A}^*$ implies for sufficiently large M_i 's the relation $\{M_i - a\} \in (h-1)\mathcal{A}^*$, that is, $\{\mathcal{A}^* \cup \{a\}\}$ is an asymptotic basis of order h . By (6) the relations of the orders of the sequences $\{b_i\}$; $\{m_i\}$; $\{M_i\}$ are restricted. This restriction will be sharpened later on by other conditions.

Theorem. *There exists a sequence $\mathcal{A}^*$ which is an asymptotic maximal nonbase sequence whose density function has an order of $O(x^{\frac{1}{h}})$ ($h \geq 3$).*

Remark. This result cannot be improved in the following sense: an essentially less dense sequence cannot be an asymptotic maximal nonbase of order h .

PROOF. Let A_0 be a basis sequence of order h . Moreover let $0 \in A_0$ and $A(x) = O(x^{\frac{1}{h}})$. A proof of the existence of such a sequence can be found in [1] and [5]. From A_0 we form A_1 by a suitable choice of b_1, m_1, M_1 . In general, A_k is formed from A_{k-1} by a suitable choice of b_k, m_k, M_k . If b_k, m_k, M_k and A_{k-1} are

already determined, then define

$$(7) \quad \begin{cases} A_k = A_{k-1}^I \cup A_{k-1}^{II} \cup A_{k-1}^{III} \cup A_{k-1}^{IV} \\ A_{k-1}^I = \{x | x \leq m_k \text{ and } x \in A_{k-1}\} \\ A_{k-1}^{II} = \{x | x = M_k + a_i + 1 \text{ and } a_i \in A_{k-1}\} \\ A_{k-1}^{III} = \{h(M_k + 1) + M_i\} \cup \{sM_k + M_i + s\} \cup \{(s+1)M_k + r\} \end{cases}$$

where $i=1, 2, \dots, k-1$; $s=1, 2, \dots, h$ and $r=0, 1, \dots, h-1$. A_{k-1}^{IV} denotes a set of natural numbers which satisfies the following conditions:

$\alpha)$ if $a \notin A_{k-1}$ and $0 \leq a \leq b_k$, then $\{M_k - a\}$ is an element of

$$(h-1)\{A_{k-1}^I \cup A_{k-1}^{II} \cup A_{k-1}^{III} \cup A_{k-1}\},$$

$\beta)$ if $d \in A_{k-1}^{IV}$, then $\frac{1}{3h^2} M_k < d < M_k$,

$\gamma)$ the number of elements of A_{k-1}^{IV} is $|A_{k-1}^{IV}| \leq hb_k$,

$\delta)$ $A_{k-1}^I \cap A_{k-1}^{IV} = \emptyset$ (for certain k 's it can also occur that A_{k-1}^{IV} is the empty set).

Knowing the sequences A_k , the sequence $\mathcal{A}^*$ can be defined in the following manner:

$$(8) \quad x \in \mathcal{A}^* \Leftrightarrow x \leq M_k \text{ and } x \in A_k.$$

If the sequences A_k satisfy the following conditions

(9) With the exception of $M_1, M_2, \dots, M_k$ the set hA_k contains all nonnegative integers;

(10) there exists a constant c such that $A_k(x) \leq cA_0(x)$ and c does not depend on k ,

then the existence of the sequence $\mathcal{A}^*$ mentioned in the theorem is proved.

Indeed, $\mathcal{A}^*(x) = O(x^{\frac{1}{h}})$ holds because of (8) and (10), since in the case $x \leq M_k$ we have $|\mathcal{A}^*(x)| = A_k(x)$ and $A_k(x) \leq cA_0(x)$ but $A_0(x) = O(x^{\frac{1}{h}})$. Because of (8) and (9) the set $h\mathcal{A}^*$ contains all nonnegative integers with the exception of the elements of $\{M_k\}$. On the other hand, if x is a nonnegative integer with $x \leq b_k$ and $x \notin \mathcal{A}^*$, then we have $\{M_l - x\} \in (h-1)\mathcal{A}^*$ for $l \geq k$ because of (8) and the property (α) of the sets A_{l-1}^{IV} .

It is therefore sufficient to verify (9) and (10), which will be done by induction on k . First we give the numbers b_k, m_k, M_k and assume the existence of A_{k-1}^{IV} . After having seen in this way that (9) and (10) hold, we show that A_{k-1}^{IV} can be constructed for every $k \geq 1$ according to the corresponding conditions.

By c_0 we denote the least number for which $A_0(x) = c_0 x^{\frac{1}{h}}$ holds (provided that $x \geq 1$). Let $b_1 = h^{2h+1}$ and $m_1 = a_i$ but $a_i \geq h^h [c_0 b_1 h^2]^h$ where a_i denotes an arbitrarily large, but fixed element of A_0 . By M_1 we denote that natural number which is not representable as a sum of h members of elements of A_0 which are not greater than a_i , but every natural number less than M_1 is representable as a sum of h members of elements of A_0 which are not greater than a_i . In other words: by M_1 we mean the least positive integer which does not belong to hA_0 . M_1 is clearly greater than m_1 , since A_0 was a basis of order h with $0, 1 \in A_0$.

First we verify (9). The integers of the interval $[0; M_1 - 1]$ are contained in hA_0^I . The integers of the interval $[hM_1 + 1; \infty]$ are contained in hA_0^{II} . Namely, we have $M_1 + 1 \in A_0^{II}$ since $0 \in A_0$ and so $h(M_1 + 1) \in hA_0^{II}$. On the other hand, let $y > h(M_1 + 1)$ and $0 < y' = y - h(M_1 + 1)$. Then $y' \in hA_0$, that is,

$$y' = a_{i_1} + a_{i_2} + \dots + a_{i_h} \quad \text{and} \quad a_{i_j} \in A_0 \quad (j = 1, 2, \dots, h)$$

but

$$M_1 + a_{i_j} + 1 \in A_0^{II} \quad \text{and therefore} \quad y \in hA_0^{II} \quad \text{since}$$

$$y = M_1 + a_{i_1} + 1 + M_1 + a_{i_2} + 1 + \dots + M_1 + a_{i_h} + 1.$$

The integers of the interval $[M_1 + 1; hM_1 + h]$ can be written in the form $sM_1 + r$ where s and r are nonnegative integers with $0 < s \leq h$ and $0 \leq r < M_1$. We distinguish now two cases according to whether $r - s \geq 0$ or $r - s < 0$. If $r - s \geq 0$ then $r - s \in hA_0$, that is, $r - s = a_{r_1} + a_{r_2} + \dots + a_{r_h}$. In this case we have

$$sM_1 + r = M_1 + a_{r_1} + 1 + M_1 + a_{r_2} + 1 + \dots + M_1 + a_{r_s} + 1 + a_{r_{s+1}} + \dots + a_{r_h}.$$

If $r - s < 0$, then $r < s \leq h$, that is $r \leq h - 1$, $sM_1 \in A_0^{III}$ and $1 \in A_0^I$; thus we have

$$sM_1 + r = \underset{1}{sM_1} + \underset{2}{1} + \underset{r}{1} + \dots + 1,$$

that is,

$$sM_1 + r \in h \{A_0^I \cup A_0^{II} \cup A_0^{III}\}.$$

We ask now whether (10) is satisfied or not. If $x \leq \frac{1}{3h^2} M_1$, then $A_1(x) = A_0^I(x) = A_0(x)$. If $\frac{1}{3h^2} M_1 < x < m_1$, then $A_0(x) = A_1(x) = A_0(x) + hb_1$, since for A_0^{IV} condition (y) is satisfied. If $m_1 \leq x \leq M_1$, then $A_1(x) = A_0(m_1) + A_0^{IV}(x) \leq A_0(x) + hb_1$. If $1 + M_1 \leq x$, then $|A_0^{III}(x)| \leq h^2$ and

$$(11) \quad A_1(x) = A_0(m_1) + A_0^I(M_1) + A_0(x - (M_1 + 1)) \leq 3A_0(x) + h^2 = 4A_0(x).$$

Taking into consideration that $A_0^{IV}(M_1) \leq hb_1$, $b_1 = h^3$ and $M_1 > h^h(c_0 b_1 h^2)^h$, it follows that $A_1(x) \leq 2A_0(x)$ provided that $x \leq M_1$. On the other hand, (11) yields

$$\lim_{x \rightarrow \infty} \frac{A_0(x)}{A_1(x)} = 1.$$

At this place we remark that because of (8) it is appropriate to assign also to m_0 a value; by definition let this value be equal 0. By ε we denote an arbitrarily small but fixed positive number; moreover let c_i be numbers satisfying

$$(12) \quad \prod_{i=1}^{\infty} c_i = 1 + \varepsilon \quad \text{and} \quad c_i > 1.$$

Let now for 1 natural numbers which are smaller than k the values of A_i, b_i, m_i, M_i be given. Then $b_k = b_{k-1} + 1$ and we choose the m_k 's in such a manner that in case $x > m_k$ the relation $A_{k-1}(x)/A_{k-2}(x) < c_{k-1}$ is satisfied; furthermore let m_k be greater than $h^h(M_{k-1} h^2 b_k)^h$ and let m_k be equal to some

a_i which belongs to A_{k-1} . By M_k we denote that least positive integer which is, on the one hand, greater than M_{k-1} , and which, on the other hand, does not belong to hA_{k-1}^I .

First we verify that (9) holds true. With exception of the numbers $M_1, M_2, \dots, M_{k-1}$ the set hA_{k-1}^I contains the integers of the interval $[0; M_{k-1}]$. This follows from the definition of A_{k-1}^I , from the fact that (9) is satisfied for A_{k-1} and from the choice of M_k . The integers of the interval $[h(M_k+1); \alpha]$ are contained in $h\{A_{k-1}^{II} \cup A_{k-1}^{III}\}$. In fact, let $y > h(M_k+1)$ and $0 < y' = y - h(M_k+1)$; if y' is equal to some of the numbers $M_1, M_2, \dots, M_{k-1}$, then $y \in A_{k-1}^{III}$ and if y' is distinct from each of these numbers, then we have by assumption that $y' \in hA_{k-1}$, that is, $y' = a_{i_1} + a_{i_2} + \dots + a_{i_h}$ where $a_{i_j} \in A_{k-1}$ ($j = 1, 2, \dots, h$). It follows that $M_k + a_{i_j} + 1 \in A_{k-1}^{II}$ and so $y \in hA_{k-1}^{III}$ since

$$y = M_k + a_{i_1} + 1 + M_k + a_{i_2} + 1 + \dots + M_k + a_{i_h} + 1.$$

The integers of the interval $[M_k+1; (h+1)M_k]$ can be written in the form $sM_k + r$ where s and r are nonnegative integers with $0 < s \leq h$ and $0 \leq r < M_k$. We distinguish now two cases according to whether $r-s \geq 0$ or $r-s < 0$.

If $r-s \geq 0$ and $r-s \in hA_{k-1}$, that is,

$$r-s = a_{r_1} + a_{r_2} + \dots + a_{r_s} + a_{r_{s+1}} + \dots + a_{r_h},$$

then

$$sM_k + r = M_k + a_{r_1} + 1 + \dots + M_k + a_{r_s} + s + a_{r_{s+1}} + \dots + a_{r_h},$$

that is, $sM_k + r \in h\{A_{k-1}^{II} \cup A_{k-1}^I\}$. If $r-s \notin hA_{k-1}$, then $r-s = M_i$ and $1 \leq i \leq k-1$ but $sM_k + M_i + s \in A_{k-1}^{III}$.

If $r-s < 0$, then $r < s \leq h$, that is, $r \leq h-1$, $sM_k \in A_{k-1}^{III}$, $0; 1 \in A_{k-1}^I$ and $M_k + 1 \in A_{k-1}^{II}$ therefore

$$sM_k + r = \underset{1}{sM_k} + \underset{2}{1} + \underset{r}{1} + \dots + 1,$$

that is,

$$sM_k + r \in h\{A_{k-1}^I \cup A_{k-1}^{II} \cup A_{k-1}^{III}\}.$$

We investigate whether (10) is satisfied for A_k . If $x = \frac{1}{3h^2} M_k$, then $A_{k-1}(x) = A_k(x)$. If $(3h^2)^{-1} M_k \leq x \leq m_k$, then

$$A_{k-1}(x) \leq A_k(x) \leq A_{k-1}(x) + hb_k \quad \text{since} \quad |A_{k-1}^{IV}| \leq hb_k.$$

If $m_k \leq x \leq M_k$, then

$$A_k(x) = A_{k-1}(m_k) + A_{k-1}^{IV}(x) \leq A_{k-1}(x) + hb_k = A_{k-1}(x) + h(h+k-1).$$

If $1 + M_1 \leq x$, then

$$\begin{aligned} (13) \quad A_k(x) &= A_{k-1}(m_k) + A_{k-1}^{III}(x) + A_{k-1}^{IV}(M_k) + A_{k-1}(x - (M_k + 1)) \leq \\ &\leq A_{k-1}(m_k) + k - 1 + h(k-1) + h^2 + hb_k + A_{k-1}(x - (M_k + 1)) \leq \\ &\leq 4A_{k-1}(x). \end{aligned}$$

Here we used that the number of the elements of the set A_{k-1}^{III} is not greater than $(k-1+h(k-1)+h^2)$ and that $A_{k-1}^{\text{IV}}(M_k) \leq hb_k = h(h^3+k-1) = h^4+h(k-1)$; besides we took into consideration that $A_{k-1}(M_k) \leq A_{k-1}(x)$ provided that $x \geq M_k+1$, and then $M_k^{\frac{1}{h}} \leq A_{k-1}(M_k)$. Moreover we used that $M_k^{\frac{1}{h}}$ is essentially greater than $h^2+h(k-1)$ and $(h-1)k-1+h^2$ since $M_k > h^h(M_{k-1}h^2b_k)^h$ which implies $M_k^{\frac{1}{h}} > h(M_{k-1}h^2b_k)$ and $M_{k-1} > h^h(M_{k-2}h^2b_{k-1})^h$. On the other hand (13) immediately yields that

$$\lim_{x \rightarrow \infty} \frac{A_{k-1}(x)}{A_k(x)} = 1.$$

We show now that there exists a constant c which does not depend on k and for which $A_k(x) \leq cA_0(x)$. We have seen above that $A_k(x) \leq 4A_{k-1}(x)$ follows from $x \geq M_k+1$; because of the choice of m_k we have then $\frac{A_{k-1}(x)}{A_{k-2}(x)} < c_{k-1}$, that is, $A_{k-1}(x) < c_{k-1}A_{k-2}(x)$. Using $x \geq m_k \geq m_{k-1} \geq \dots \geq m_1$ we obtain that $A_k(x) \leq 4A_{k-1}(x) < 4c_{k-1}A_{k-2}(x) < \dots < 4c_{k-2}c_{k-3} \dots c_1A_0(x)$; because of the choice of the numbers $c_1, c_2, \dots, c_{k-1}$ we have here $c_1c_2 \dots c_{k-2} < \prod_{i=1}^{\infty} c_i = 1+\varepsilon$ and therefore $A_k(x) \leq 4(1+\varepsilon)A_0(x)$ where the constant $1+\varepsilon$ does not depend on k . If $x < m_k$ and if $m_i \leq x < M_{i+1}$ for some nonnegative integer, then

$$\begin{aligned} A_k(x) &= A_{i+1}(x) \leq 4A_i(x) < 4c_ic_{i-1} \dots c_1A_0(x) \leq \\ &\leq 4(1+\varepsilon)A_0(x), \end{aligned}$$

where, of course, $i < k$.

It remains now only to show the existence of the sets A_{k-1}^{IV} . First we deal with the case $h=3$. Let a be an integer such that $0 \leq a \leq b_k$ and $a \notin A_{k-1}$ (if such an a does not exist, then A_{k-1} is the empty set). We show now that there exist integers D_1 and D_2 such that the following conditions are satisfied:

$$(14) \quad \frac{1}{3}M_k < D_1; \quad D_2 < \frac{2}{3}M_k$$

and

$$(15) \quad M_k \notin 3A_{k-1}^{\text{I}} \cup \{D_1\} \cup \{D_2\}$$

$$(16) \quad M_k - a = D_1 + D_2.$$

It is obvious that D_1 and D_2 have to be determined only if $M_k - a \notin 2A_{k-1}^{\text{I}}$. In particular (15) requires that neither $M_k - D_i$ nor $M_k - 2D_i$ belong to $2A_{k-1}^{\text{I}}$ or A_{k-1}^{I} . We look now which numbers of the interval $\left(\frac{1}{3}M_k; \frac{2}{3}M_k\right)$ are at our disposal. In $\left(\frac{1}{3}M_k; \frac{2}{3}M_k\right)$ there are at least $\left[\frac{1}{3}M_k\right] - |2A_{k-1}^{\text{I}}| - 1$ integers y for which $M_k - y \notin 2A_{k-1}^{\text{I}}$. We denote the set of these integers y by $\{y\}$. We cancel now those elements of $\{y\}$ for which $M_k - 2y \in 2A_{k-1}^{\text{I}}$. The remaining elements will be denoted by $\{z\}$. For the number of the elements of $\{z\}$ we obviously have

the relation

$$|\{z\}| \cong \frac{1}{3} M_k - 2A_{k-1}^I - 1 - A_{k-1}^I = \frac{1}{3} M_k - 4c^2 M_k - 1 = \frac{3}{4} \frac{1}{3} M_k.$$

Here we took into consideration that $|A_{k-1}^I| \leq cM_k^{\frac{1}{3}}$ and therefore $|2A_{k-1}^I| = (cM_k^{\frac{1}{3}})^2 = c^2 M_k^{\frac{2}{3}}$. Among the elements of $\{z\}$ one can choose z_1 and z_2 such that $M_k - a = z_1 + z_2$. Namely, if the elements of $\{z\}$ are taken away from $M_1 - a$, then we get $\left[\frac{3}{4} \frac{1}{3} M_k\right]$ integers which belong to the interval $\left(\frac{1}{3} M_k - a; \frac{2}{3} M_k - a\right)$. Observing now that " a " is essentially smaller than M_k , that is, $a \leq b_k$ and $M_k \cong \cong 3^3(M_{k-1} 3^2 b_k)^3$ and therefore $a \leq b_k < \frac{1}{9} \frac{1}{4} \frac{1}{3} M_k$, it follows that there exists a z_1 such that $z_2 = M_k - a - z_1 \in \{z\}$. By definition of $\{z\}$, z_1 and z_2 correspond D_1 and D_2 , since the conditions (14), (15) and (16) are satisfied. If $D_i \in A_{k-1}^I$, then it will not be considered as an element of A_{k-1}^{IV} ; D_i will be considered as an element of A_{k-1}^I only if $D_i \notin A_{k-1}^I$ ($i=1, 2$). The described procedure can be repeated with an arbitrary a which satisfies $0 \leq a \leq b_k$ and $a \notin A_{k-1}^I$; note that then in (15) the set A_{k-1}^I has to be replaced by the set $\{A_{k-1}^I \cup \{D_1\} \cup \{D_2\} \cup \dots \cup \{D_t\}\}$ where the numbers $D_1, D_2, \dots, D_t$ have been constructed in the preceding step: the corresponding inequalities then preserve their validity. Looking at the conditions for A_{k-1}^{IV} we can see that (α) is satisfied because of (15) and (16), (β) is satisfied because of (14) and that instead of (γ) even $|A_{k-1}^{IV}| < 2b_k$ is satisfied. Concerning (δ) it suffices to remark that in case $D_j \in A_{k-1}^I$ this element is not taken to the elements of A_{k-1}^{IV} .

If $h > 3$, then the existence of A_{k-1}^{IV} will be proved in the following manner. For arbitrary integers $0 \leq a \leq b_k$ with $a \notin A_{k-1}^I$ we determine in the first step an integer F which has the following properties:

$$(17) \quad \frac{M_k}{h^2} - S = F = \frac{M_k}{h^2} + S \quad (i = 1, 2, \dots, h-3).$$

For reasons of convenience we choose S to be $M_k^{\frac{h-1}{h} + \frac{1}{2h}}$.

$$(18) \quad M_k \notin h\{A_{k-1} \cup \{F\}\}.$$

This means at the same time that $M_k - a - (h-3)F = K \in A_{k-1}^I$ and $a \in 2A_{k-1}^I$ cannot be satisfied simultaneously; therefore we shall require that

$$(19) \quad M_k - a - (h-3)F = K \notin A_{k-1}^I.$$

During the second step we determine to the already existing K numbers D_1 and D_2 such that the following relations are satisfied:

$$(20) \quad \frac{1}{3} M_k < D_i < \frac{2}{3} M_k \quad \text{and} \quad i = 1, 2$$

$$(21) \quad K = D_1 + D_2$$

$$(22) \quad M_k \notin h\left\{A_{k-1}^I \cup \{F\} \bigcup_{i=1}^2 \{D_i\}\right\}.$$

Simply spoken it is here our aim to represent $M_1 - a$ as a sum of $h-1$ integers, which fulfil the preceding conditions; the conditions on the order ensure that the density of A_k will not essentially change. For a given "a" the abovementioned procedure has clearly to be applied only if $M_1 - a \notin (h-1)A_{k-1}^I$; from the numbers F_j, D_i , of course, only those are taken to be elements of the set A_{k-1}^{IV} which are not elements of A_{k-1}^I . Our procedure for the determination of the numbers F and D_i will have the following property: if the numbers $\{F_j\}$ and $\{D_i\}$ have already been determined for some a , then the procedure can automatically be applied for an arbitrary new a satisfying the conditions; to this end we have in (18), (19) and (22) to write $A_{k-1}^I \cup \{F_j\} \cup \{D_i\}$ instead of A_{k-1}^I where $\{F_j\}$ and $\{D_i\}$ denotes all until now constructed and in A_{k-1}^{IV} collected numbers. Note that if A_{k-1}^{IV} is constructed in the above described manner, then it satisfies the conditions (α) , (β) , (γ) and (δ) . Indeed, (α) is satisfied since

$$M_k - a \in (h-1)\{A_{k-1}^I \cup A_{k-1}^{IV}\} \subset (h-1)\{A_{k-1}^I \cup A_{k-1}^{II} \cup A_{k-1}^{III} \cup A_{k-1}^{IV}\}$$

holds because of (19) and (21); moreover we have $M_k \notin hA_k$ because of (7), (18) and (22). (β) holds because of (17) and (20). In detail, we have to show that $F > \frac{1}{3h^2} M_k$ if

$$\frac{M_k}{h^2} - S < F \quad \text{and} \quad S = M_k^{\frac{h-1}{h} + \frac{1}{2h}}.$$

It is obviously sufficient to show that

$$\frac{M_k}{h^2} - M_k^{\frac{h-1}{h} + \frac{1}{2h}} \cong \frac{M_k}{3h^2},$$

that is,

$$\frac{2}{3} \cong h^2 M_k^{\frac{h-1}{h} + \frac{1}{2h} - 1} = h^2 M_k^{-\frac{1}{2h}}.$$

But $h^2 x^{-\frac{1}{2h}}$ is for positive values of x a monotonically decreasing function and therefore it is sufficient to prove the inequality for M_1 . Observing that

$$M_1 \cong h^h [c_0 b_1 h^2]^h \quad \text{and} \quad b_1 = h^3 + 1, \quad h > 3$$

we get

$$h^2 M_1^{-\frac{1}{2h}} \leq h^2 (h c_0 b_1 h^2)^{-\frac{1}{2}} = h^2 (h^3 c_0 h^3)^{-\frac{1}{2}} = \frac{1}{\sqrt{c_0 h}} \leq \frac{2}{3}$$

since $c_0 \geq 1$ and $h^h \geq 4$.

From the fact that A_{k-1}^{IV} can have maximally $3b_k$ elements, it follows that even more is true than (γ) . Condition (δ) is satisfied since the numbers F_j and D_i with $F_j, D_i \in A_{k-1}^I$ are not taken to be elements of A_{k-1}^{IV} .

We concentrate now on the question which integers of the interval $[h^{-2}M_k - S; h^{-2}M_k + S]$ play the role of F . By $\{y\}$ we denote the set of those integers for which

$$h^{-2}M_k - S \leq y \leq h^{-2}M_k + S \quad \text{and} \quad M_1 - sy \notin (h-1)A_{k-1}^I \quad (s = 1, 2, \dots, h)$$

is satisfied. The number of the y 's having the preceding property is clearly greater or equal than

$$\begin{aligned} 2[S] - 1 - h|(h-1)A'_{k-1}| &\cong 2[S] - h(A_{k-1}(M_k))^{h-1} - 1 \cong \\ &\cong 2[S] - h|A_0(M_k)|^{h-1} \cdot 5^h \cong 2[S] - 1 - hM_k^{\frac{h-1}{h}} (5c_0)^h \cong \\ &\cong 2M_k^{\frac{h-1}{h} + \frac{1}{2h}} - hM_k^{\frac{h-1}{h}} (5 \cdot c_0)^{h-2} > M_k^{\frac{h-1}{h} + \frac{1}{2h}}. \end{aligned}$$

From $\{y\}$ we choose now an y such that

$$(23) \quad M_k - a - (h-3)y = K \notin A_{k-1}^1.$$

We denote this element by y_0 ; this will be our F . This y_0 exists since among the elements of the set $\{y\}$ there is at most a number of $|A_{k-1}^1| \leq 5c_0 M_k^{\frac{1}{h}}$ which does not satisfy (23), whereas the number of the element of y is considerably greater, namely greater than $M_k^{\frac{h-1}{h} + \frac{1}{2h}}$. Knowing F and K the numbers D_1 and D_2 can be constructed similarly as in the case $h=3$.

At this place we also pose the question, how many integers V exist which satisfy (20) and for which

$$(24) \quad M_k \notin hA_{k-1}^1 \cup \{F\} \cup \{V\}.$$

This latter condition is necessary because we want to see directly from the determination of D_1, D_2 that (22) holds true. (24) can be written also as

$$(25) \quad M_k - i \cdot V \notin (h-1)\{A_{k-1}^1 \cup \{F\}\} \quad \text{and} \quad M_k \neq 2V \quad (i = 1, 2).$$

Here it is obviously sufficient to investigate the case $i=1, 2$ since $\frac{1}{3}M_k < V < \frac{2}{3}M_k$ and $3V > M_k$. At most $2|(h-1)A_{k-1}^1(M_k)| + 1$ numbers of the interval $\left[\frac{1}{3}M_k; \frac{2}{3}M_k\right]$ does not satisfy condition (25); on the other hand, we have $A_{k-1}(x) \leq 4(1+\varepsilon)A_0(x) \leq 4(1+\varepsilon)c_0x^{\frac{1}{h}}$, and so

$$A_{k-1}^1(M_k) \leq 5c_0M_k^{\frac{1}{h}},$$

that is,

$$|(h-1)A_{k-1}^1(M_k)| \leq (5c_0)^{h-1}M_k^{\frac{h-1}{h}}.$$

But in the interval $\left[\frac{1}{3}M_k; \frac{2}{3}M_k\right]$ there are at least $\left[\frac{1}{3}M_k\right] - 2$ integers. Hence we have

$$\left[\frac{1}{3}M_k\right] - 2 - 2|(h-1)A_{k-1}^1(M_k)| - 1 \geq \left[\frac{1}{3}M_k\right] - 3 - 2(5c_0)^{h-1}M_k^{\frac{h-1}{h}} > \frac{3}{4}\frac{1}{3}M_k$$

possibilities for the choice of the numbers V ; subtracting now each of these numbers from K , we obtain at least one among the numbers $K-V$, which likewise satisfies the conditions for the V 's; we denote this by V_1 and by V_2 we denote the number subtracted from K , that is, $V_1 = K - V_2$. The existence of V_1 is seen from the following fact: if we subtract numbers of type " V " from K , then we get numbers belonging to the interval

$$\left(\frac{1}{3} M_k - a - (h-3)F; \frac{2}{3} M_k - a - (h-3)F \right);$$

now the pigeon hole principle can be applied since $a + (h-3)F$ is essentially smaller than $\frac{1}{4} \frac{1}{3} M_k$ which ensures the existence of V_1 . It suffices now to remark that

V_1 and V_2 satisfy the conditions for D_1 and D_2 because of (25) and $\frac{1}{3} M_k < V_1$; $V_2 < \frac{2}{3} M_k$. If we write now instead of A_{k-1}^1 always the set $\{A'_{k-1} \cup \{F_j\} \cup \{D_i\}\}$ (where $\{F_j\}$ and $\{D_i\}$ denotes the numbers F and D_1, D_2 which have been constructed in the preceding steps), then it was already mentioned that the construction of D_1, D_2 and F is always possible, provided that $0 \leq a \leq b_k$. This completes the proof.

Remarks

1. Actually we have proved more than the assertion of the theorem, namely: from an arbitrary basis with order h whose density function has an order of $O(x^{\frac{1}{h}})$ we formed a maximal nonbase sequence of order h whose density function has likewise the order $O(x^{\frac{1}{h}})$. Given A_0 and B_0 in case of distinct basis sequences and choosing a suitably large m_1 , one sees that the corresponding maximal nonbases $\mathcal{A}^*$; B^* will also be distinct from each other.

2. If we would choose to A_0 a basis sequence of order h whose density function has an order of $O(x^K)$ (where $\frac{1}{h} \leq K \leq \frac{1}{h-1}$), then a slight modification of the construction would yield that there exists a maximal nonbase sequence of order h whose density function has an order of $O(x^K)$.

Open problem: Does there exist for an arbitrary K with $h^{-1} < K < 1$ an asymptotic maximal nonbase sequence of order h whose density function has an order of $O(x^K)$?

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References

- [1] J. W. S. CASSELS, Über Basen der natürlichen Zahlenreihe, *Abh. Math. Sem. Univ. Hamburg* **21** (1957), 247—257.
- [2] P. ERDŐS and M. B. NATHANSON, Maximal asymptotic nonbases, *Proc. Amer. Math. Soc.* **48** (1975), 57—60.
- [3] M. B. NATHANSON, Minimal bases and maximal nonbases in additive number theory, *J. Number Theory* **6** (1974), 324—333.
- [4] M. B. NATHANSON, s -maximal nonbases of density zero *J. London Math. Soc. (2)* **15** (1977), 29—34.
- [5] A. Stöhr, Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe, *J. Reine Angew. Math.*, **194** (1955), 40—65.
- [6] S. TURJÁNYI, On maximal asymptotic nonbases of zero density, *J. Number Theory*, **9** (1977), 271—275.
- [7] S. TURJÁNYI, Note on maximal asymptotic nonbases of zero density, *Publ. Math. (Debrecen)* **26** (1979), 229—235.

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