Pure subgroups of non-abelian groups

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1. Introduction

Prüfer [8], [9] introduced the notion of *Servanzuntergruppe,* now more commonly called *pure subgroup,* into the theory of abelian groups, where it is now firmly established as a useful tool. A modern and comprehensive account of pure subgroups of abelian groups, together with some natural generalizations, can be found in the monograph [3] by FUCHS; see also GACSÁLYI [5]. We define here the natural extension of the concept to not necessarily abelian groups, and answer some of the questions that naturally present themselves. In fact the basic definitions apply to more general algebraic systems than groups, and some of the fundamental reults are capable of the corresponding generalization; these results are collected together in § 3, formulated for groups but so that they can be extended without difficulty.

Given a group G, we consider systems of equations of the form

$$w_i(g_1, g_2, \ldots, x_1, x_2, \ldots) = 1, \tag{1.1}$$

indexed by the elements i of some index set I, the w_i being words in the *coefficients* $g_j \in G$ and *variables* x_j. Each equation involves only a finite number of coefficients and variables, but if the index set I is infinite, then the total number of coefficients or of variables or of both may be infinite. We call (1.1) a system of

*) The authors started work on this paper in 1960 when the late A. KERTÉSZ visited the University of Manchester. It is published now, retaining the form in which it was written in 1961, to complete the record of his mathematical work. — A. KERTÉSZ died on April 3, 1974. For an obituary by K. Győri, see these Publicationes, Vol. 21, pp. 159—160.

equations *over* G, and we say that it has a *solution* $h_1, h_2, \ldots$ in a group H containing G as a subgroup if $h_1, h_2, \ldots$ are elements of H which, when substituted for the variables $x_1, x_2, \ldots$, turn (1.1) into a system of valid equalities:

$$w_i(g_1, g_2, \ldots, h_1, h_2, \ldots) = 1, \quad i \in I.$$

We now define the subgroup G of the group H to be *pure* in H if every system of equations over G in a finite number of variables which has a solution in H also has a solution in G. Note that we restrict the number of variables involved in the equations, but not the number of equations themselves — if the number of equations is restricted instead of, or in addition to, the number of variables, one arrives at different, fruitful and interesting notions that are, however, outside the scope of the present paper.

The restrictions on the number of variables can be varied, leading to a notion that depends on a cardinal $\mathfrak{n}$ and that we call $\mathfrak{n}$-*pure*. We defer the precise (and incidentally less intuitive) definition; what we have called *pure* will coincide with $\aleph_0$-*pure*. Most of this paper is concerned with pure subgroups and with $\mathfrak{n}$-pure subgroups for finite $\mathfrak{n}$.

In abelian groups there is no distinction between pure subgroups and $\mathfrak{n}$-pure subgroups for finite $\mathfrak{n}$. In fact the subgroup G of the abelian group H is pure in H if (and trivially only if) every *single* equation over G in a *single* variable has a solution in G if it has a solution in H: thus G is pure in H if for all $h \in H$ and integers n,

$$h^n \in G \quad \textit{implies} \quad g^n = h^n \quad \textit{for some} \quad g \in G.$$

This then could be made, and has been made (PRÜFER [8]), the definition of a pure subgroup of an abelian group; for a proof of the equivalence of this with our definition, see FUCHS [3]. Theorem 25.5. In non-abelian groups, as we shall see, the position is different.

2. Notation and definitions

The following notation is used throughout.

If S is a set, $|S|$ is its cardinal. If G is a group, $|G|$ is its order. If g is an element of a group, its order is $|g|$; we make the usual convention that $|g| = 0$ means that g generates an infinite cyclic group. If G is a subgroup of the group H, write $G \leqq H$, and $G \lhd H$ if it is a normal subgroup. The index of G in H is $|H : G|$. The subgroup of H generated by a subset $S \subseteq H$ is $\mathrm{gp}(S)$, and the subgroup generated by $G \leqq H$ and $S \subseteq H$ is $\mathrm{gp}(G, S)$. The group with generators $g_1, g_2, \ldots$ and defining relations $u_i(g_1, g_2, \ldots) = 1$, $i \in I$, is denoted by

$$\mathrm{gp}(g_1, g_2, \ldots;\ u_i(g_1, g_2, \ldots) = 1, \quad i \in I).$$

Conjugates and commutators are written

$$x^y = y^{-1}xy, \quad [x, y] = x^{-1}x^y = x^{-1}y^{-1}xy, \quad [x, y, z] = [[x, y], z].$$

The derived group of G is G'; the trivial group is denoted by E.

An idempotent endomorphism π of a group H will be called a *projection* of H (Baer [1] calls this a retraction, the image a retract). If π is a projection of H and if $H\pi=G$, then π restricted to G is the identity mapping of G. If, further, the kernel of π is N, then N is a normal complement of G in H, that is to say

$$N \lhd H, \quad GN = H, \quad G \cap N = E.$$

A normally complemented subgroup is also often called a semi-direct factor. To every normally complemented subgroup G of H and to every normal complement N of G in H there is a projection π of H such that $H\pi=G$ and N is the kernel of π. A normally complemented subgroup is normal if and only if it is a direct factor.

We denote by X a set of variables, and $w(G, X)$ will denote a word in *coefficients* $g \in G$ and *variables* $x \in X$, or, briefly, a word *over* G *in* X. A *system of equations* over G in X is a family

$$\{w_i(G, X)\}_{i \in I} \tag{2.1}$$

of words indexed by the elements of a set I; strictly speaking this is a system of left-hand sides of equations only; and in more general algebraic systems, where there is no unit element to provide a convenient universal right-hand side, one would instead consider families

$$\{(u_i(G, X),\ v_i(G, X))\}_{i \in I} \tag{2.2}$$

of pairs of words that are to be equated.

If θ is a mapping of the set X of variables into a group H containing G, and if $w(G, X)$ is a word over G in X, then

$$w(G, X\theta)$$

is obtained by substituting $x\theta$ for each $x \in X$ and then evaluating the resulting word as an element of H. The system of equations (2.1) is *soluble in* H if there is a mapping θ of X into H such that

$$\textit{for all} \quad i \in I, \quad w_i(G, X\theta) = 1.$$

[Correspondingly, the system (2.2) is soluble in H if there is a mapping θ of X into H such that *for all* $i \in I$, $u_i(G, X\theta)=v_i(G, X\theta)$.] We then call $X\theta$ a solution of the system of equations.

We now make the definition that is fundamental to this paper.

Definition 2.3. The subgroup G of the group H is $\mathfrak{n}$-*pure* in H, where $\mathfrak{n}$ is a cardinal number, if every system of equations over G in X with $|X| < \mathfrak{n}+1$ has a solution in G if it has a solution in H.

The condition $|X| < \mathfrak{n}+1$ concisely and conveniently expresses the restriction on the number of variables, namely that it is to be $\leqq \mathfrak{n}$ if $\mathfrak{n}$ is finite and $< \mathfrak{n}$ if $\mathfrak{n}$ is infinite; it allows one also to stipulate "$\leqq \aleph_\alpha$", namely as "$< \aleph_{\alpha+1}$". Thus "G is pure in H" means precisely the same as "G is $\aleph_0$-pure in H", and both terms can be used interchangeably.

Definition 2.4. The subgroup G of the group H is *absolutely pure* in H if every system of equations over G, irrespective of the cardinality of the set X of variables, has a solution in G if it has a solution in H.

3. General results

In this section we collect together some simple facts that flow from the definitions; most of them are not peculiar to group theory but require only universal algebraic concepts (exceptions are those which use the group-theoretic notion of a normal complement). Although group-theoretic language is used throughout, it will be obvious how the results can be generalized to other classes of algebraic systems.

Lemma 3.1. *Let G be $\mathfrak{n}$-pure in H and let $\mathfrak{m} \leqq \mathfrak{n}$; then G is $\mathfrak{m}$-pure in H. If G is absolutely pure in H, then G is $\mathfrak{n}$-pure in H for all $\mathfrak{n}$, and conversely.*

We omit the (obvious) proof. The question whether conversely an $\mathfrak{m}$-pure subgroup is also $\mathfrak{n}$-pure will be considered in the next section.

Lemma 3.2. *If $G \leqq K \leqq H$ and if G is $\mathfrak{n}$-pure (absolutely pure) in H then G is $\mathfrak{n}$-pure (absolutely pure) in K.*

Lemma 3.3. *If G is $\mathfrak{n}$-pure (absolutely pure) in K and if K is $\mathfrak{n}$-pure (absolutely pure) in H then G is $\mathfrak{n}$-pure (absolutely pure) in H; in other words, $\mathfrak{n}$-purity (absolute purity) is transitive.*

Again the (obvious) proofs are omitted. For countable $\mathfrak{n}$ we also have the following.

Lemma 3.4. *Let $\mathfrak{n} \leqq \aleph_0$ and let $H = \bigcup_{n=1}^{\infty} H_n$ be the direct limit of a chain*

$$G \leqq H_1 \leqq H_2 \leqq \ldots$$

of groups in each of which G is $\mathfrak{n}$-pure. Then G is $\mathfrak{n}$-pure in H.

Proof. We deal with the case that $\mathfrak{n} = \aleph_0$, that is that G is pure in each H_n, and we show that then G is pure in H. The case of finite $\mathfrak{n}$ is similar. Let a system of equations over G in X, where $|X| < \aleph_0$, have a solution $X\theta$ in H. As $X\theta$ is finite subset of H, there is an integer n and that $X\theta \leqq H_n$. Now G is pure in H_n, and so the given system of equations has a solution in G; thus the lemma follows.

The following criterion is implicit in Erdélyi [2].

Theorem 3.5. *The group G is $\mathfrak{n}$-pure in H if and only if, to every subgroup K of H that is generated by G and a set S of cardinal $|S| < \mathfrak{n}+1$, there is a projection π of K onto G.*

Proof. Let G have the property that to every subgroup $K = \mathrm{gp}(G, S)$ of H with $|S| < \mathfrak{n}+1$ there is a projection π of K onto G. Let $W = \{w_i(G, X)\}_{i \in I}$ be a system of equations over G in X with $|X| < \mathfrak{n}+1$, and assume it has a solutions $X\theta$ in H. Put $X\theta = S$ and $K = \mathrm{gp}(G, S)$, and let π be a projection of

K onto G — such a projection exists by hypothesis. Applying π to the equations

$$w_i(G, X\theta) = 1, \quad \text{for all} \quad i \in I,$$

and noting the homomorphism property of π together with the fact that π acts on G as the identity, we see that also

$$w_i(G, X\theta\pi) = 1, \quad \text{fot all} \quad i \in I.$$

Thus $X\theta\pi$ is a solution in G of the given system W, and it follows that every system of equations over G in fewer than $\mathfrak{n}+1$ variables has a solution in G if it has a solution in H: In other words, G is $\mathfrak{n}$-pure in H.

Conversely, assume G is $\mathfrak{n}$-pure in H, and let $K=\mathrm{gp}(G, S)$ with $|S|<\mathfrak{n}+1$. Let X be a set of variables of the same cardinal number as S, and let θ denote a one-to-one mapping of X onto S. Denote by W the set of all words $w(G, X)$ that satisfy

$$w(G, X\theta) = 1.$$

Then the system W has a solution in H, namely $X\theta$, and as $|X|=|S|<\mathfrak{n}+1$ and as G is, by hypothesis, $\mathfrak{n}$-pure in H, the system W also has a solution in G, say $X\eta$. We define a mapping π of K into G as follows: if $k \in K$ then k can be written as a word

$$k = u(G, S)$$

in elements of G and of S. Put

$$g = u(G, S\theta^{-1}\eta);$$

this is the element of G obtained from u by first substituting the variable $x=s\theta^{-1} \in X$ for the corresponding element $s \in S$ and then the element $x\eta \in G$ for the variable x. Now g depends only on the element $k \in K$, not on the particular word u chosen to represent it; for if another representation is

$$k = v(G, S)$$

then $w(G, S)=u(G, S)^{-1}v(G, S)=1$, and so $w(G, X)=w(G, S\theta^{-1}) \in W$. It follows that $w(G, X\eta)=1$, and therefore also

$$v(G, S\theta^{-1}\eta) = g.$$

Thus we may put $g=k\pi$, without ambiguity. One now verifies without difficulty that π is a projection of K onto G, and the theorem follows.

Corollary 3.6. *If $H=\mathrm{gp}(G, S)$ where $|S|<\mathfrak{n}+1$ and if G is $\mathfrak{n}$-pure in H, then there is a projection of H onto G [that is to say, G has a normal complement in H].*

Corollary 3.7. *The subgroup G of H is absolutely pure in H if, and only if, there is a projection of H onto G [that is to say, if, and only if, G is normally complemented in H].*

Corollary 3.8. *If G is $\mathfrak{n}$-pure in $H=\mathrm{gp}(G, S)$ where $|S|<\mathfrak{n}+1$, then G is absolutely pure in H.*

Corollary 3.9. *If $H=\mathrm{gp}(S)$ where $|S|<\mathfrak{n}+1$ then the $\mathfrak{n}$-pure subgroups of H are the absolutely pure subgroups [that is the normally complemented subgroups]; the pure subgroups of a finitely genrated group are absolutely pure.*

4. Comparison of $\mathfrak{n}$-purity for different $\mathfrak{n}$

For abelian groups there is no difference between $\mathfrak{n}$-purity for different finite values of $\mathfrak{n}$: If the subgroup G of the abelian group H is 1-pure in H, then it is pure in H (see FUCHS [3], Theorem 25.5); and then it is also $\mathfrak{n}$-pure in H for all $\mathfrak{n}$ between 1 and $\aleph_0$ (see Lemma 3.1). This is no longer so for non-abelian groups, as will be shown in this section; and to heighten the contrast to abelian groups, we choose our groups nilpotent of class 2. Before we define them, we prove two simple lemmas.

Lemma 4.1. *Let G be an abelian pure subgroup of the group H; then $G\cap H'=E$.*

PROOF. If $G\cap H'\neq E$, then there is a subgroup $K=\mathrm{gp}(G, k_1, \ldots, k_n)$ of H, finitely generated over G, such that $G\cap K'\neq E$. If $N\lhd K$ and $GN=K$, then $K/N\cong G/G\cap N$ is abelian, hence $K'\leqq N$, and $G\cap N\neq E$. Thus G is not a normally complemented subgroup of K, and by Corollary 3.6, G is not pure in K. By Lemma 3.2 then G is not pure in H, and the lemma follows.

Lemma 4.2. *Let H be a group such that $H'H'$ is elementary abelian, and let G be a subgroup of H such that $G\cap H'=E$. Then G is absolutely pure in H.*

PROOF. In H/H', every subgroup is a direct factor; hence there is a subgroup N of H, which we may take to contain H', such that

$$H/H' = GH'/H'\times N/H'.$$

Clearly N is normal in H, and $GH'\cap N=H'$, whence $G\cap N\leqq G\cap H'=E$; finally $GN=H$, and N is seen to be a normal complement of G in H. The lemma then follows from Corollary 3.7. — It could also have been derived from the more general Lemma 7.3.

Let now p be an odd prime, d a positive integer, and put

$$(4.3)\quad H = H_d = gp(h_1, h_2, \ldots, h_d;\quad h_i^p = [h_i, h_j, h_k] = 1,\quad i, j, k = 1, 2, \ldots, d).$$

These relations ensure that H is nilpotent of class 2 and has exponent p; and it is in fact the free d-generator group of the variety of second nilpotent groups of exponent p. The derived group H' coincides with the centre of H and is an elementary abelian group of order $p^{\binom{d}{2}}$, and H/H' is an elementary abelian group of order p^d.

Let $a\in H'$, and write it in the form

$$a = \prod_{i<j} [h_i, h_j]^{\alpha_{ij}}.$$

We assign to a the $d\times d$ skew matrix A over $GF(p)$ whose (i,j)-element is α_{ij} if $i<j$, 0 if $i=j$, and $-\alpha_{ij}$ if $i>j$. If b is another element of H' and if B is the skew matrix that corresponds to b, then the matrix that corresponds to ab is $A+B$. In this way an isomorphism is defined between the multiplicative group H' and the additive group of all skew $d\times d$ matrices over $GF(p)$.

The isomorphism depends on the particular set of generators chosen for H. If new generators $h_1', h_2', \ldots, h_d'$ are chosen for H such that

$$h_i' \equiv \prod_j h_j^{\pi_{ij}} \pmod{H'}, \quad i = 1, 2, \ldots, d.$$

so that the π_{ij} form a non-singular matrix P over $GF(p)$, and if A' denotes the matrix assigned to $a\in H'$ in terms of the new basis $h_1', h_2', \ldots, h_d'$, then

$$A = P^T A' P.$$

Hence A and A' have the same rank, and this rank is, therefore, an invariant of the element a; we shall call it simply the *rank of* a. As the rank of a skew matrix it is necessarily an even number.

Lemma 4.4. *If the rank of $a\in H'$ is r then there is a subgroup of H with r generators, but no subgroup with fewer than r generators, in whose derived group a is contained.*

Proof. Let $a\in K'$ where $K\leqq H$ is generated by δ elements $k_1, \ldots, k_\delta$ which we may assume independent modulo H', as H', being central, does not affect the derived group of K. We complete the basis of K to a basis $k_1, \ldots, k_\delta, k_{\delta+1}, \ldots, k_d$ of H. The skew matrix that corresponds to a has non-zero entries only in the first δ rows and columns, hence has rank at most δ, and it follows that $r\leqq\delta$: that is to say, no subgroup with fewer than r generators contains a in its derived group. On the other hand, we can choose a basis $h_1', \ldots, h_p'$ of H in terms of which the matrix corresponding to a becomes

$$A' = \begin{pmatrix} J & & & \\ & J & & \\ & & \ddots & \\ & & & J \\ & & & & 0 \end{pmatrix},$$

where $J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The number of terms J is $\frac{1}{2}r$, and in terms of this new basis of H,

$$a = [h_1', h_2'] \ldots [h_{r-1}', h_r'].$$

Hence $a\in K'$ where $K=gp(h_1', h_2', \ldots, h_r')$ is an r-generator subgroup of H; and the lemma follows.

Corollary 4.5. *If the number d of generators of $H=H_d$ (given by 4.3) is even, then the element*

$$g = [h_1, h_2][h_3, h_4] \ldots [h_{d-1}, h_d] \in H'$$

does not lie in the derived group of any proper subgroup of H.

The rank of g is d, and if K is a proper subgroup of H, then K can be generated by fewer than d elements modulo its centre; hence $g \notin K'$.

Corollary 4.6. *With H and g as in Corollary* 4.5, $G=\mathrm{gp}(g)$ *is* $(d-1)$*-pure in H but not d-pure.*

As an abelian subgroup that intersects the derived group non-trivially, G is not pure in H, nor then d-pure, as H has d generators (Lemma 4.1; Corollary 3.9); but as G lies in the Frattini subgroup of H, all groups generated by G and at most $d-1$ further elements are proper subgroups of H, and in such a subgroup G is absolutely pure (Corollary 4.5; Lemma 4.2). Hence G is $(d-1)$-pure in H.

Corollary 4.7. *With H as in Corollary* 4.5, *with* $d \geqq 4$, *and with g as in Corollary* 4.5, $G_1=\mathrm{gp}(h_1, g)$ *is* $(d-2)$*-pure in H but not* $(d-1)$*-pure.*

As an abelian subgroup that intersects the derived group non-trivially, G_1 is not pure in H, nor then $(d-1)$-pure, as $H=\mathrm{gp}(G_1, h_2, \ldots, h_d)$ has $d-1$ generators in addition to G_1 (Lemma 4.1; Corollary 3.8); but all groups K generated by G_1 and at most $d-2$ further elements are proper subgroups of H, and then $G_1 \cap K' = E$; hence G_1 is absolutely pure in all such K, and thus $(d-2)$-pure in H (Corollary 4.5; Lemma 4.2; Corollary 3.7; Theorem 3.5).

Thus we can distinguish between $\mathbf{n}$-purity and $(\mathbf{n}+1)$-purity for all finite values of $\mathbf{n}$, and we can do it within the variety of nilpotent groups of class 2 and odd prime exponent p. By an obvious modification of our construction we can even do it within a single group

$$H_\infty = \bigcup_d H_d.$$

Theorem 4.8. *Let p be an odd prime and put*

$$H_\infty = gp(h_1, h_2, \ldots;\ h_i^p = [h_i, h_j, h_k] = 1,\ i, j, k = 1, 2, \ldots).$$

Put further

$$g_t = [h_1, h_2][h_3, h_4] \ldots [h_{2t-1}, h_{2t}],$$

and

$$G_{2t} = \mathrm{gp}(g_t), \quad G_{2t+1} = \mathrm{gp}(h_1, g_{t+1}), \quad t = 1, 2, \ldots.$$

Then, for every $n=2, 3, \ldots$, the group G_n is $(n-1)$-pure in H_∞ but not n-pure.

We omit the proof. It may be remarked that a similar construction, with the same result, can be carried out in the variety of nilpotent groups of class 2 and exponent 4.

It will be noticed that the even-indexed subgroups G_{2t} are central and therefore normal in H_∞. One can further modify the example so as to produce also normal subgroups that are $(n-1)$-pure but not n-pure for odd n.

Theorem 4.9. *Let p be an odd prime and put*

$$H_\infty^* = \mathrm{gp}(h_0, h_1, h_2, \ldots;\ h_0^{p^2} = h_1^p = h_2^p = \ldots = [h_i, h_j, h_k] = 1,\quad i, j, k_z = 0, 1, 2, \ldots).$$

Put further

$$g_t^* = h_0^p [h_1, h_2][h_3, h_4] \ldots [h_{2t-1}, h_{2t}],$$

and

$$G^*_{2t+1} = \mathrm{gp}(g^*_t), \quad t = 1, 2, \ldots .$$

Then for every odd $n=3, 5, \ldots$, *the group* G^*_n *is central and* $(n-1)$*-pure in* H^*_∞ *but not* n*-pure.*

The proof is not difficult, and we omit it. It can be based on the fact — also easily proved — that *a subgroup* G *of prime order is normally complemented in a nilpotent group* H *if, and only if, it is not contained in the Frattini subgroup of* H.

The exponent of H^*_∞ is p^2; we do not know whether there are similar examples of exponent p.

5. Normal n-pure subgroups

A normal subgroup G of H is a direct factor if (and only if) it has a normal complement in H. Thus we may expect a normal n-pure subgroup of H to be in some sense "nearly" a direct factor of H. In fact we immediately deduce from Theorem 3.5 and its corollaries a criterion for our present case:

Lemma 5.1. *The normal subgroup* G *of* H *is* n*-pure in* H *if, and only if,* G *is a direct factor of every subgroup* $K=\mathrm{gp}(G, S)$ *of* H *with* $|S|<\mathfrak{n}+1$. *The normal subgroup* G *is absolutely pure in* H *if, and only if, it is a direct factor of* H. *The normal* n*-pure subgroups of a group with fewer than* $\mathfrak{n}+1$ *generators are its direct factors.*

We now look at the particular case $\mathfrak{n}=1$.

Lemma 5.2. *Let* G *be a normal* 1*-pure subgroup of* H, *and let* C *denote the centralizer of* G *in* H. *Then* $H=GC$. *Thus* H *is the generalized direct product of* G *and* C, *amalgamating the centre of* G.

PROOF. Let $h\in H$, and consider the system $\{w_g(G, x)\}_{g\in G}$ of equations over G, indexed by G itself, in the single variable x, where

$$w_g(G, x) = g^x g^{-h}$$

— noting that $g^{-h}=h^{-1}g^{-1}h$ is an element of G, as G is normal in H. This system has a solution in H, namely h; hence it has a solution, say k, in G. It follows that $k^{-1}h$ commutes with all $g\in G$, whence $k^{-1}h\in C$, and $h\in GC$ as required.

Corollary 5.3. *A normal* 1*-pure subgroup* G *of* H *with trivial centre is a direct factor of* H, *hence absolutely pure in* H.

Corollary 5.4. *An abelian normal* 1*-pure subgroup is central.*

The following criterion generalizes a theorem of PRÜFER [8] (see FUCHS [3], Theorem 25.1).

Theorem 5.5. *Let* G *be a normal subgroup of* H. *For* G *to be* 1*-pure in* H *it is necessary and sufficient that every coset of* G *in* H *contains an element that centralizes* G *and whose order equals the order of the coset modulo* G.

PROOF. Assume that G is 1-pure in H, and consider the coset Gs of G. Now $K=\mathrm{gp}(G, s)$ is a direct product, $K=G\times B$ (Lemma 5.1), hence we can write s in the form $s=g\times b$ with $g\in G$ and $b\in B$. Clearly $b\in Gs$ and b centralizes G. Also, if the order of Gs modulo G is q, then $s^q=g^q\times b^q\in G$, whence $b^q=1$; and conversely, if q is the order of b, then $(Gs)^q=(Gb)^q=G$. This proves the necessity. For the sufficiency, assume every coset Gs of G contains an element b, say, which centralizes G and whose order equals the order of Gs modulo G. Let $K=\mathrm{gp}(G, s)$; then also $K=\mathrm{gp}(G, b)$, with the b chosen as above. As b centralizes G, it generates a normal subgroup $B=\mathrm{gp}(b)$ of K. Moreover $b^q\in G$ implies $b^q=1$, so that $G\cap B=E$. Thus $K=G\times B$; and it now follows from Lemma 5.1 that G is 1-pure in H, as required.

This criterion is of limited usefulness, as its conditions are in general not easy to verify; it is included only as a formal generalization of Prüfer's theorem. The following theorem provides a more natural criterion.

Theorem 5.6. *Let G be a normal subgroup of H; denote by C the centralizer of G in H and by $Z\ (=C\cap G)$ the centre of G. Then G is $\mathfrak{n}$-pure in H if, and only if, $H=GC$ and Z is $\mathfrak{n}$-pure in C.*

PROOF. Assume that $H=GC$ and that Z is $\mathfrak{n}$-pure in C. Let $K=\mathrm{gp}(G, S)$ where $|S|<\mathfrak{n}+1$. By our assumption every $s\in S$ can be written in the form

$$s = gc, \quad g\in G, \quad c\in C.$$

We may then replace s by c, and thus assume, without loss of generality, that $S\subseteqq C$. Put $L=\mathrm{gp}(Z, S)$. Then $L\leqq C$ and as Z is assumed $\mathfrak{n}$-pure in C and $|S|<\mathfrak{n}+1$, there is a subgroup B of C such that

$$L = Z\times B$$

(Lemma 5.1); now

$$B\cap G = B\cap C\cap G = B\cap Z = E,$$

and B centralizes G and thus is normal in $\mathrm{gp}(G, B)=K$. It follows that $K=G\times B$, and by Lemma 5.1 again G is $\mathfrak{n}$-pure in H.

Conversely, let G be $\mathfrak{n}$-pure in H. Then G is 1-pure in H (Lemma 3.1), hence $H=GC$ (Lemma 5.2). Next let $S\subseteqq C$ and $|S|<\mathfrak{n}+1$. and consider $L=\mathrm{gp}(Z, S)$ and $K=\mathrm{gp}(G, S)$. By the assumption on G and by Lemma 5.1, $K=G\times B$ with a suitable subgroup B of H. But B clearly centralizes G and so is a subgroup of C. Hence also $L=Z\times B$, and an application of Lemma 5.1 again shows Z to be $\mathfrak{n}$-pure in C, as required.

Corollary 5.7. *With the notation of Theorem* 5.6, *G is a direct factor of H if, and only if, Z is a direct factor of C.*

The theorem reduces the investigation of normal $\mathfrak{n}$-pure subgroups to that of central $\mathfrak{n}$-pure subgroups. We remark that our examples of normal subgroups that are $(n-1)$-pure but not n-pure (see Theorems 4.8, 4.9) are in fact made with central subgroups.

Our results enable us to demonstrate some further points of divergence between the abelian and the non-abelian cases. Let $G_1\leqq G_2\leqq G_3\leqq\ldots$ be an ascending

chain of direct factors of the group H, and put $G=\bigcup_1^\infty G_i$. If H is abelian, then G is pure ($=\aleph_0$-pure) in H (see FUCHS [3], p. 77), but if H is non-abelian, then G need not even be 1-pure in H. To show this, we form the cartesian product H of an infinite sequence of strictly non-abelian groups $A_1, A_2, A_3, \ldots$ and consider the direct product G of the same family as a subgroup of H. Then $G=\bigcup G_n$ where

$$G_n = A_1 \times A_2 \times \ldots \times A_n$$

is a direct factor of H (a complementary factor being the cartesian product of $A_{n+1}, A_{n+2}, \ldots$). Now G is normal in H, and the centralizer C of G in H is the cartesian product of the centres $Z(A_1), Z(A_2), \ldots$ of the component groups. As $GC \neq H$ — no element of H whose components in all A_i are outside the centre can belong to GC — we can apply Lemma 5.2 to deduce that G is not even 1-pure in H.

If we take all the A_i isomorphic to the non-abelian group of order p^3 and exponent p, where p is an odd prime, or as one of the non-abelian groups of order 8, then H will be nilpotent of class 2 and of odd prime exponent or exponent 4, respectively: Note that by contrast in an abelian group of exponent p every subgroup is absolutely pure. Again, with the same choice of A_i, the direct product G is countable, but it is not contained in any countable 1-pure subgroup of H; we omit the proof, which is not difficult. This again contrasts with abelian groups, because every countable subgroup of an abelian group is contained in a countable pure subgroup; see FUCHS [3], p. 8.

6. The case of infinite $\mathfrak{n}$

The case $\mathfrak{n}=\aleph_0$, that is the case of normal pure subgroups, deserves closer study because of its importance for abelian groups. We shall also extend the results to $\mathfrak{n}>\aleph_0$. We first deal with central subgroups, in order then to apply Theorem 5.6.

Lemma 6.1. *Let Z be a subgroup of the centre of the group C. Then Z is pure in C if, and only if,* (i) $Z\cap C'=E$, *and* (ii) ZC'/C' *is pure in* C/C', *or, as we shall say, Z is pure in C modulo C'.*

Proof. Assume firstly that $Z\cap C'=E$ and that Z is pure in C modulo C'. If γ denotes the canonic epimorphism of C on to C/C', then $Z\gamma$ is pure in $C\gamma$. Let S be a finite subset of C, and let $K=\mathrm{gp}(Z,S)$. Then $Z\gamma$ is a direct factor of $K\gamma=\mathrm{gp}(Z\gamma, S\gamma)$; thus there is a subgroup $B\gamma$ of $K\gamma$ such that $K\gamma=Z\gamma\times B\gamma$. Here we can take B as a subgroup of K containing the kernel $K\cap C'$ of the restriction of γ to K. This ensures that $K=ZB$; for every $k\in K$ is of the form

$$k = zbc' \quad \text{with} \quad z\in Z, \quad b\in B, \quad c'\in C'\cap K;$$

and $bc'\in B$, too. Moreover, B is clearly normal in K. Finally

$$Z\cap B \leqq Z\cap K\cap C' \leqq Z\cap C' = E,$$

as $Z\gamma\cap B\gamma=E\gamma$. Hence $K=Z\times B$, showing Z to be pure in C. (This is a special case of Lemma 7.3.)

Conversely, assume Z to be pure in C. Then $Z\cap C'=E$ by Lemma 4.1. Next, if $(c\gamma)^q=z\gamma$ for some $c\in C$, $z\in Z$, and integer q, where again γ is the canonic epimorphism of C onto C/C', then $c^q=zc'$ with $c'\in C'$. We choose a finite set S so that $c\in S$ and that $c'\in(\mathrm{gp}(S))'$, and put $K=\mathrm{gp}(Z, S)$. Then Z is a direct factor of K, that is $K=Z\times B$ with a suitable subgroup B. Now $c'\in K'=B'$, and if $c=z_0b$ with $z_0\in Z$, $b\in B$, then $z_0^q=z$. Thus $(z_0\gamma)^q=z\gamma$. It follows — as $C\gamma$ is abelian — that $Z\gamma$ is pure in $C\gamma$, in other words, Z is pure in C modulo C', and the lemma follows.

The extension of this lemma to uncountable $\mathfrak{n}$ requires a more elaborate argument.

Lemma 6.2. *Let Z be a subgroup of the centre of the group C. Then Z is $\mathfrak{n}$-pure in C, where $\mathfrak{n}>\aleph_0$, if, and only if,* (i) $Z\cap C'=E$, *and* (ii) Z *is $\mathfrak{n}$-pure in C modulo C'.*

Proof. The sufficiency of the pair of conditions (i), (ii) is proved as before, with the set S now being only required to have cardinal $|S|<\mathfrak{n}+1$. To prove the necessity, we assume that Z is $\mathfrak{n}$-pure in C. Then $Z\cap C'=E$ by Lemma 4.1. It remains to prove that Z is $\mathfrak{n}$-pure in C modulo C'. Let T be a subset of $C\gamma$ with $|T|<\mathfrak{n}+1$. We may assume that T is infinite. Thus

$$\aleph_0 \leqq |T| = \mathfrak{m} < \mathfrak{n}.$$

Let S_0 be a subset of C such that $S_0\gamma=T$ and $|S_0|=\mathfrak{m}$, and put $K_0=\mathrm{gp}(Z, S_0)$. Let $K_0\cap C'=D_0$. Then $|D_0|\leqq\mathfrak{m}$, because $|K_0: Z|\leqq\mathfrak{m}$ and $Z\cap C'=E$. Each $d\in D_0$ can be written in the form

$$d = [c_1, c_2][c_3, c_4]\dots[c_{r-1}, c_r],$$

with finitely many $c\in C$. Thus there is a subset S_1 of C such that (i) $S_0\subseteqq S_1$, (ii) $D_0\leqq(\mathrm{gp}(S_n))'$, (iii) $|S_1|=\mathfrak{m}$. We put $K_1=\mathrm{gp}(Z, S_1)$, and then continue to define, inductively, $D_n=K_n\cap C'$; then $S_{n+1}\subseteqq C$ so that (i) $S_n\subseteqq S_{n+1}$, (ii) $D_n\leqq(\mathrm{gp}(S_{n+1}))'$, and (iii) $|S_{n+1}|=\mathfrak{m}$; finally $K_{n+1}=\mathrm{gp}(Z, S_{n+1})$. Observe that if $|S_n|=\mathfrak{m}$, then $|K_n: Z|\leqq\mathfrak{m}$, and then as $Z\cap C'=E$, also $|D_n|=|K_n\cap C'|\leqq\mathfrak{m}$; this makes it possible to choose S_{n+1} so that again $|S_{n+1}|=\mathfrak{m}$, and so the inductive definition proceeds. It is so arranged that

$$K_n\cap C' \leqq K'_{n+1}$$

for all n. If now we put $K=\bigcup_n K_n$, then

$$K\cap C' = K'; \tag{6.21}$$

for obviously $K'\leqq K\cap C'$; and if $k\in K\cap C'$, then there is an integer n such that $k\in K_n\cap C'$, and then $k\in K'_{n+1}\leqq K'$, so that also $K\cap C'\leqq K'$. Also $K=\mathrm{gp}(Z, \bigcup_n S_n)$, and

$$\Big|\bigcup_n S_n\Big| = \mathfrak{m}.$$

It follows — as Z is $\mathfrak{n}$-pure in C and $\mathfrak{m}<\mathfrak{n}$ — that Z is a direct factor of K, say $K=Z\times B$. From (6.21) and the fact that Z is abelian, we see that

$$K\cap C' = B'. \tag{6.22}$$

From this again we deduce that

(6.23) $$BC' \cap Z = E;$$

for if $z \in BC' \cap Z$, say $z = bc'$ where $b \in B$ and $c' \in C'$, then $c' = b^{-1}z \in K$, and so $c' \in B'$ by (6.22); but then also $z \in B$, and so $z = 1$. Now $K_0 = Z \times B_0$ where $B_0 = K_0 \cap B$; and

$$K_0\gamma = \mathrm{gp}(Z\gamma, S_0\gamma) = \mathrm{gp}(Z\gamma, T) = Z\gamma B_0\gamma.$$

By (6.23)

$$Z\gamma \cap B_0\gamma \leqq Z\gamma \cap B\gamma = E,$$

so we finally have

$$\mathrm{gp}(Z\gamma, T) = Z\gamma \times B_0\gamma.$$

As this is true for evrey subset T of C_γ with $|T| < \mathfrak{n}+1$, we see that $Z\gamma$ is $\mathfrak{n}$-pure in $C\gamma$, and the lemma follows.

As a corollary we now have the following criterion.

Theorem 6.3. *Let G be a normal subgroup of H; denote by C the centralizer of G in H and by $Z\,(=C \cap G)$ the centre of G. Then G is* **n***-pure in H, where* **n** *is an infinite cardinal, if, and only if,* (i) $H = GC$, (ii) $G \cap C' = E$, *and* (iii) Z *is* **n***-pure in C modulo C'.*

This follows at once from Lemmas 6.1, 6.2 and Theorem 5.6. The following corollary could also have been proved directly, and much more simply than Lemma 6.2.

Corollary 6.4. *With the same notation, G is a direct factor of H if, and only if, Z is a direct factor of C modulo C' (that is to say, ZC'/C' is a direct factor of C/C').*

This follows by choosing **n** so large that **n**-pure becomes absolutely pure and noting that a normal absolutely pure subgroup is a direct factor (Lemma 5.1).

Theorem 6.3 and Corollary 6.4 allow us to translate the splitting theorems for pure subgroups of abelian groups to analogous theorems for non-abelian groups. The following are examples.

Theorem 6.5. *Let G be a normal subgroup of H, let C be the centralizer of G in H, and Z the centre of G. If Z is the direct product of cyclic groups of (fixed) finite order n, then the following three propositions are equivalent.*

(i) *G is a direct factor of H.*

(ii) *G is pure in H.*

(iii) *$H = GC$ and $G \cap C^n C' = E$, where C^n is the group generated by the n-th powers in C.*

This follows from a theorem of SZELE [10]; see FUCHS [3], Theorem 24.1.

Theorem 6.6. *A pure normal subgroup whose centre has finite exponent is a direct factor.*

This follows from a theorem of KULIKOV [7]; see FUCHS [3], Theorem 24.5.

Theorem 6.7. *If G is a pure normal subgroup of H and if H/GH' is a direct product of cyclic groups, then G is a direct factor of H.*

PROOF. We note first that, with the notation of Lemma 6.1 and its proof,

$$H/GH' \cong (H/G)/(H/G)' \cong (C/Z)/(C/Z)' \cong C\gamma/Z\gamma.$$

By a theorem of KULIKOV [7] (see FUCHS [3], Theorem 25.2) then $Z\gamma$ is a direct factor of $C\gamma$, and the theorem now follows from Corollary 6.4. Similarly we have:

Theorem 6.8. *Let G be an $\mathfrak{n}$-pure normal subgroup of H, where $\mathfrak{n}$ is an infinite cardinal, and let H/GH' be a direct product of groups of orders less than $\mathfrak{n}$. Then G is a direct factor of H.*

PROOF. By Theorem 6.3, Z is $\mathfrak{n}$-pure in C modulo C', hence a direct factor of C modulo C' by Proposition G, p. 88, of Fuchs [3]; application of Corollary 6.4 completes the proof.

7. Homomorphisms and $\mathfrak{n}$-purity

The proofs of Lemmas 6.1, 6.2 indicate that preservation of $\mathfrak{n}$-purity under a homomorphism is not in general a simple matter. It is easy to show by examples that if G is an $\mathfrak{n}$-pure subgroup of H and if v is a homomorphism of H, then Gv need not be $\mathfrak{n}$-pure in Hv; in fact one can take H abelian, G as a direct factor (hence absolutely pure) in H, and yet have Gv not even 1-pure in Hv. Let H be the abelian group of order 8 generated by an element g of order 2 and an element h of order 4. Then $G=\mathrm{gp}(g)$ is a direct factor. If v is the projection of H onto the complementary direct factor $\mathrm{gp}(h)$ which is defined by

$$gv = h^2, \quad hv = h,$$

then gv has a square root in Hv but not in Gv: thus Gv is not 1-pure in Hv. Note that the kernel N of v has the property

$$N \cap G = E;$$

this corresponds to the situation met with in Lemmas 6.1, 6.2.

One is thus led to look for necessary and sufficient conditions for a homomorphism to preserve $\mathfrak{n}$-purity; but we have unable to find such conditions, and can present some partial results only.

Lemma 7.1. *Let G be an $\mathfrak{n}$-pure subgroup of the group H, and let v be a homomorphism of H with kernel N. Then Gv is $\mathfrak{n}$-pure in Hv if to every subgroup $K=\mathrm{gp}(G, S)$ with $|S|<\mathfrak{n}+1$ there is a projection π of K onto G whose kernel P satisfies the condition*

$$N \cap K = (N \cap G)(N \cap P). \tag{7.11}$$

PROOF. Under the assumptions, we first see that

$$\text{if } k, k' \in K \text{ and if } kv = k'v \text{ then } k\pi v = k'\pi v; \tag{7.12}$$

for then $k'=kn$ with $n \in N \cap K$, hence by (7.11) $n=gp$ with $g \in N \cap G$ and $p \in N \cap P$: thus

$$n\pi v = (gp)\pi v = gv = 1,$$

and (7.12) follows. If now we define π_1 by

$$kv\pi_1 = k\pi v,$$

then (7.12) ensures that π_1 is a mapping of Kv. It is evidently a homomorphism of Kv into Gv, and easily seen to be an epimorphism; in fact it is a projection, as (on K)

$$v\pi_1^2 = \pi v\pi_1, = \pi^2 v = \pi v = v\pi_1,$$

hence (on Kv) π_1 is idempotent.

Now if $L = \mathrm{gp}(Gv, T)$ is a subgroup of Hv generated by Gv and a subset $T \subseteqq H$ of cardinal $|T| < \mathfrak{n}+1$, then we can choose a subset $S \subseteqq H$ such that $Sv = T$ and $|S| = |T|$. Then $L = Kv$ where $K = \mathrm{gp}(G, S)$; and there is then, as we have shown, a projection π_1 of L onto Gv. By Theorem 3.5 then Gv is $\mathfrak{n}$-pure in Hv, and the lemma follows. The assumption that G is $\mathfrak{n}$-pure in H is seen to be superfluous, as this is ensured by the existence of the projections π.

Corollary 7.2. *If G is $\mathfrak{n}$-pure in H and if the normal subgroup N of H is contained in G, then G/N is $\mathfrak{n}$-pure in H/N.*

We now consider conditions that ensure conversely that $\mathfrak{n}$-purity of Gv in Hv implies $\mathfrak{n}$-purity of G in H. The following simple lemma is a natural generalization of the easier part of Lemmas 6.1, 6.2.

Lemma 7.3. *Let G be a subgroup of the group H and v a homomorphism of H such that Gv is $\mathfrak{n}$-pure in Hv. If v restricted to G is a monomorphism or equivalently, if*

$$N \cap G = E, \tag{7.31}$$

where N is the kernel of v, then G is $\mathfrak{n}$-pure in H.

Proof. Let $K = \mathrm{gp}(G, S)$ where $S \subseteqq H$ and $|S| < \mathfrak{n}+1$. Then there is a projection π_1, of $Kv = \mathrm{gp}(Gv, Sv)$ onto Gv. Now denote the restriction of v to G, which by assumption is a monomorphism, by v_1; then this has an inverse v_1^{-1}, and we can define a homomorphism π of K into G by

$$\pi = v\pi_1 v_1^{-1}.$$

This is in fact an epimorphism, because $v\pi_1$, is an epimorphism (on K to Gv) and v_1^{-1} is even an isomorphism (on Gv to G). Moreover π is idempotent, because

$$\pi^2 = v\pi_1 v_1^{-1} v\pi_1 v_1^{-1} = v\pi_1^2 v_1^{-1} = v\pi_1 v_1^{-1} = \pi.$$

Thus π is a projection of K onto G, and the lemma now follows by an application of Theorem 3.5.

A certain duality may be noted: If

$$N \cap G = E, \tag{7.31}$$

then $\mathfrak{n}$-purity of Gv in Hv entails $\mathfrak{n}$-purity of G in H, but not in general conversely; if

$$N \leqq G, \tag{7.32}$$

then $\mathfrak{n}$-purity of G in H entails $\mathfrak{n}$-purity of Gv in Hv, but again, as we shall now see, the converse is not generally true. For a simple counter-example we take H to be the quaternion group, with G a subgroup of order 4 and N the subgroup of order 2 (which is the derived group and the centre of H). Then G/N is a direct factor of H/N, but G is not even 1-pure in H — a generator of G is transformed into its inverse by an element of H outside G, but not by any element of G (alternatively, Corollary 5.4 may be applied).

Again we have no necessary and sufficient criteria, but only sufficient conditions, and examples to show that these conditions cannot be relaxed very far. The first result, a partial converse of Corollary 7.2 for abelian groups, is due to FUCHS [3].

Theorem 7.4. *If H is abelian, if N is a subgroup of the subgroup G of H, if N is $\mathfrak{n}$-pure in H and if G/N is $\mathfrak{n}$-pure in H/N, then G is $\mathfrak{n}$-pure in H.*

For a proof, the reader is referred to FUCHS [3], p. 88. If H is non-abelian, we add the assumptions that G and N are normal and N is also 2-pure in H: the first of these assumptions is trivially satisfied in abelian groups, and the second is then implied by the $\mathfrak{n}$-purity of N. The theorem of Fuchs is, therefore, a special case of the theorem which follows.

Theorem 7.5. *Let G be a normal subgroup of the group H and let v be a homomorphism of H with kernel N; further let Gv be $\mathfrak{n}$-pure in Hv. If* (i) $N \leqq G$, *and* (ii) *N is $\mathfrak{m}$-pure in H, where $\mathfrak{m} = \max(2, \mathfrak{n})$, then G is $\mathfrak{n}$-pure in H.*

PROOF. Let $K = \mathrm{gp}(G, S)$ where $S \subseteq H$ and $|S| < \mathfrak{n}+1$. Then — as Gv is normal and normally complemented in Kv — there is a direct decomposition

$$Kv = Gv \times Mv$$

of Kv. Here we can take M to be a normal subgroup of K containing the kernel N of v. Then

$$GM = K, \quad G \cap M = N.$$

Now every $s \in S$ can be written in the form $s = gs_1$ with $g \in G$ and $s_1 \in M$, and the set S_1 of second components s_1 then generates M modulo N, that is $M = \mathrm{gp}(N, S_1)$. As $|S_1| \leqq |S| < \mathfrak{n}+1 \leqq \mathfrak{m}+1$, and as N is a normal $\mathfrak{m}$-pure subgroup of H, there is a direct decomposition

$$M = N \times P.$$

It remains to show that P is the kernel of a projection of K onto G. Clearly $K = GP$ and $G \cap P = E$, and we only have to prove that P is normal in K or, equivalently, that G and P centralize each other. Let $g \in G$ and $p \in P$ be arbitrary, and consider

$$[g, p] = n, \tag{7.51}$$

say. Then — as $[g, p]$ must belong to the normal subgroups G and M that contain g and p, respectively, — we have $n \in N$. Now the set of "equations" over N

$$\{n^{-1}[x_1, x_2],\ [n', x_2]\}_{n' \in N}, \tag{7.52}$$

where n is fixed by (7.51) but n' ranges over all elements of N, has a solution θ in H given by $x_1\theta=g$, $x_2\theta=p$, as $p\in P$ clearly centralizes N. Now N is (at least) 2-pure in H, and (7.52) must then also have a solution, say $x_1\eta=n_1$, $x_2\eta=n_2$, in N. But then $[n_1, n_2]=1$ as $[n', n_2]=1$ for all $n'\in N$, and finally $n=1$ because $n^{-1}[n_1, n_2]=n^{-1}=1$. We see then from (7.51) that g and p commute, and as they were arbitrary elements of G and P, respectively, G and P centralize each other; and the theorem now follows by an application of Theorem 3.5.

To see that it is not sufficient to assume N to be 1-pure even when only 1-purity is to be deduced, one considers the non-abelian group H of exponent 3 and order 27, and takes N to be the centre of H and G an arbitrary subgroup of order 9. Then N is 1-pure in H because all groups generated by N and one further element are elementary abelian and so contain N as a direct factor; and G/N is absolutely pure in H/N, as this is also an elementary abelian group; but G is not even 1-pure in H, by Lemma 5.2, because G is its own centralizer in H.

If we drop the assumption that G is normal in H, then again the conclusion of Theorem 7.5 need not remain valid. We take H to be the group H_∞ of Theorem 4.8, G as the subgroup G_{2t+1} and N as $G_{2(t+1)}$ in the notation of that theorem. Then N is central in H, hence normal, and $N\trianglelefteqq G$. Moreover, G/N is absolutely pure in H/N, because it is a subgroup of prime order not contained in the Frattini subgroup of the nilpotent group H/N; also N is $(2t+1)$-pure in H but G is not $(2t+1)$-pure in H, by Theorem 4.8.

Hence, to deal with the case of non-normal G, we have to strengthen the assumptions on N. If we assume N to be absolutely pure in H, we obtain an almost obvious lemma.

Lemma 7.6. *Let G be a subgroup of H and let v be a homomorphism of H with kernel N. If* (i) *$N\trianglelefteqq G$, and* (ii) *N is absolutely pure in H, that is a direct factor of H, then G is $\mathfrak{n}$-pure in H if, and only if, Gv is $\mathfrak{n}$-pure in Hv.*

Proof. We may assume that v is the projection of H onto a complementary direct factor of N. Thus $H=N\times Hv$ and $G=N\times Gv$. If $K=\mathrm{gp}(G, S)$ with $|S|<\mathfrak{n}+1$, then $K=N\times Kv$, and $Kv=\mathrm{gp}(Gv, Sv)$; then, assuming Gv to be $\mathfrak{n}$-pure in Hv, there is a projection, say π_1, of Kv onto Gv. From this one obtains a projection π of K onto G by defining π to act as the identity on N and as π_1 on Kv:

$$k\pi = k(kv)^{-1}\times kv\pi_1 \quad \text{for all} \quad k\in K.$$

It follows that G is $\mathfrak{n}$-pure in H. The converse is a trivial consequence of Corollary 7.2.

More generally we can prove the following theorem.

Theorem 7.7. *Let G be a subgroup of H and let v be a homomorphism of H with kernel $N\trianglelefteqq G$. If* (i) *$G=\mathrm{gp}(N, R)$ with $|R|<\mathfrak{m}+1$, and* (ii) *N is $(\mathfrak{m}+\mathfrak{n})$-pure in H, then G is $\mathfrak{n}$-pure in H if, and only if, Gv is $\mathfrak{n}$-pure in Hv.*

Proof. Assume that Gv is $\mathfrak{n}$-pure in Hv. Let $K=\mathrm{gp}(G, S)$ where $|S|<\mathfrak{n}+1$. Then $K=\mathrm{gp}(N, R\cup S)$ and $|R\cup S|<\mathfrak{m}+\mathfrak{n}+1$; thus N, as a normal $(\mathfrak{m}+\mathfrak{n})$-pure subgroup of H and thus of K (Lemma 3.2) is absolutely pure in K

(Corollary 3.8). Then, as similarly Gv is absolutely pure in Kv, Lemma 7.6 shows G to be absolutely pure in K, and it follows that G is $\mathfrak{n}$-pure in H. The converse is again a trivial consequence of Corollary 7.2.

8. Cartesian products, direct products, and $\mathfrak{n}$-purity

If H is the cartesian product of a family $\{H_i\}_{i\in I}$ of groups, and if G is the cartesian product of a family $\{G_i\}_{i\in I}$ of subgroups, $G_i \leqq H_i$, then one would expect G to be $\mathfrak{n}$-pure in H if eads G_i is $\mathfrak{n}$-pure in H_i. This is indeed the case, and the converse is also true. We formulate the proof, which is not deep, so that it can be easily generalized to other algebraic systems.

Theorem 8.1. *Let H be the cartesian product of the family $\{H_i\}_{i\in I}$ of groups, let $G_i \leqq H$, and let G be the cartesian product of the family $\{G_i\}_{i\in I}$. Then G is $\mathfrak{n}$-pure in H if, and only if, each G_i is $\mathfrak{n}$-pure in H_i.*

Proof. It is convenient for the sake of the proof (and for the sake of generalizations not here presented) to characterize the cartesian product H by a family $\{\eta_i\}_{i\in I}$ of epimorphisms

$$\eta_i\colon H \to H_i$$

with the property that to every family $\{h_i\}_{i\in I}$ of elements $h_i \in H_i$ there is one and only one element $h\in H$ such that $h\eta_i = h_i$ for all $i\in I$. Similarly to every family $\{g_i\}_{i\in I}$ of $g_i \in G_i$ there is one and only one element $g\in G$ such that $g\eta_i = g_i$ for all $i\in I$.

Now assume that each G_i is $\mathfrak{n}$-pure in H_i. Let $K = \mathrm{gp}(G, S)$ where $S \subseteqq H$ and $|S| < \mathfrak{n}+1$. Then $K\eta_i = \mathrm{gp}(G\eta_i, S\eta_i)$, and as $|S\eta_i| \leqq |S|$, there is a projection, say π_i, of $K\eta_i$ onto $G\eta_i = G_i$. We define a mapping $\pi\colon K \to G$ by

$$\pi\eta_i = \eta_i\pi_i.$$

This is legitimate, as there is to given $k\in K$ precisely one $g\in G$ such that $g\eta_i = k\eta_i\pi_i$; and then $k\pi = g$. One easily verifies that π is a projection of K onto G. This shows, by Theorem 3.5, that G is $\mathfrak{n}$-pure in H.

Conversely, assume that G is $\mathfrak{n}$-pure in H. Let $j\in I$ be fixed, and let $K_j = \mathrm{gp}(G_j, S_j)$ where $S_j \subseteqq H_j$ and $|S_j| < \mathfrak{n}+1$. Let K be the cartesian product of the family defined by $K\eta_i = G_i$, $(i \neq j)$, $K\eta_j = K_j$. Then $K = \mathrm{gp}(G, S)$ where S consists of all $s\in H$ for which $s\eta_i = 1$ $(i\neq j)$, $s\eta_j \in S_j$. Hence $|S| = |S_j|$, and there is a projection π of K onto G. Then π_j defined by

$$\eta_j\pi_j = \pi\eta_j$$

is a well-defined mapping; for if $k, k'\in K$ and $k\eta_j = k'\eta_j$, then $(k^{-1}k')\eta_j = 1 \in G_j$; now $(k^{-1}k')\eta_i \in G_i$ also for all $i \neq j$, and so $k^{-1}k' \in G$ and $(k^{-1}k')\pi = k^{-1}k'$: hence $(k^{-1}k')\pi\eta_j = (k^{-1}k')\eta_j = 1$, and finally $k\pi\eta_j = k'\pi\eta_j$. It is easy to verify that π_j is an epimorphism of $K\eta_j$ onto G_j, and as

$$\eta_j\pi_j^2 = \pi\eta_j\pi_j = \pi^2\eta_j = \pi\eta_j = \eta_j\pi_j,$$

it is idempotent. Thus π_j is a projection of K_j onto G_j, and application of Theorem 3.5 now completes the proof of the theorem.

In the case of groups we also have the direct product H^* of the family $\{H_i\}_{i\in I}$ available, and one would expect also the direct product G^* of $\{G_i\}_{i\in I}$ to be $\mathfrak{n}$-pure in H if each G_i is $\mathfrak{n}$-pure in H_i, and conversely. This is in fact true, and can be proved by adapting the proof of Theorem 8.1.

Theorem 8.2. *Let H^* be the direct product of the family $\{H_i\}_{i\in I}$ of groups, let $G_i \leqq H$, and let G^* be the direct product of the family $\{G_i\}_{i\in I}$. Then G^* is $\mathfrak{n}$-pure in H^* if, and only if, each G_i is $\mathfrak{n}$-pure in H_i.*

PROOF. The direct product H^* consists of those elements $h^*\in H$, the cartesian product, whose *support*

$$\sigma(h^*) = \{i\in I \mid h^*\eta_i \neq 1\}$$

is finite. Now the proof proceeds as that of Theorem 8.1, noting that in the direct part again the projection π is well defined by

$$\pi\eta_i = \eta_i\pi_i,$$

because to a given $k\in K=\mathrm{gp}\,(G^*, S)$ there is precisely one $g^*\in G^*$ such that $g^*\eta_i=k\eta_i\pi_i$, as $\sigma(k)$ must be finite. In the converse part of the proof, we use again that an element $k^{-1}k'$ lies in G^* if $(k^{-1}k')\eta_i\in G_i$ for all i — again because $\sigma(k^{-1}k')$ is finite.

It should be noted that in general the direct product G^* will not be $\mathfrak{n}$-pure in the cartesian product H — an example was given at the end of § 5.

9. Groups in which certain subgroups are $\mathfrak{n}$-pure

In this and the following section we examine groups in which all subgroups of a given kind are $\mathfrak{n}$-pure, for some $\mathfrak{n}$. Thus we might ask for those groups in which all subgroups are absolutely pure: it is not difficult to see that such groups are simply the elementary abelian groups, that is the direct products of groups of prime orders; in fact, as we shall see, much less is sufficient for the same conclusion. In an abelian group it suffices to assume that all its cyclic subgroups are 1-pure in it: then it must be elementary, so that then all its subgroups are direct factors of it (see FUCHS, KERTÉSZ, SZELE [4], Theorem 4). In non-abelian groups the position is less simple. Our first goal is the following characterization of groups in which all cyclic subgroups are 1-pure.

Theorem 9.1. *The group H has the property (*) that all its cyclic subgroups are 1-pure in it if, and only if, it is the direct product of groups P with the following two properties:*

(i) *P has prime exponent, say p;*

(ii) *if $K\leqq P$ is generated by two elements but not by one, then $|K : K'|=p^2$.*

The proof requires several lemmas. We first show that a p-group has property (*) if, and only if, it has properties (i) and (ii); then that a direct product of such

p-groups also has property (*); and finally that a group H that has property (*) is a direct product of its Sylow subgroups.

Lemma 9.2. *The p-group P has the property (*) that every cyclic subgroup is 1-pure in it if, and only if,* (i) *P has exponent p, and* (ii) *if $K \leqq P$ is generated by two elements but not by one, then $|K: K'| = p^2$.*

PROOF. Let P have properties (i), (ii). If $G = \mathrm{gp}(g)$ is a cyclic subgroup, and if

$$K = \mathrm{gp}(G, h) = \mathrm{gp}(g, h) \neq G,$$

then $M = \mathrm{gp}(K', h)$ is clearly a normal complement of G in K; hence, by Theorem 3.5, G is 1-pure in P. Conversely, let P be a p-group with property (*). If $g \in P$ is an element of order p then g has no p-th root in $\mathrm{gp}(g)$ and therefore can have no p-th root in P: thus P contains no elements of order p^2, and must have exponent p. Next, let $K = \mathrm{gp}(g, h)$ be a non-cyclic two-generator subgroup of P; then $\mathrm{gp}(g)$ must have a normal complement, say M, in K. The p elements $h, gh, g^2h, \ldots, g^{p-1}h$ are mutually incongruent modulo M, hence one of them, say $h' = g^m h$, must belong to M. As $K = \mathrm{gp}(g, h')$ also, $\mathrm{gp}(h')$ has a normal complement, say N, in K. Both M and N contain K' and have index p in K; moreover $M \neq N$, as $h' \in M$ but $h' \notin N$. Thus $|K: M \cap N| = p^2$ and $|K: K'| \geqq p^2$; as K is generated by two elements of order p, also $|K: K'| \leqq p^2$, and the lemma follows.

Corollary 9.3. *If P is a group of exponent p in which all 2-generator subgroups are finite, then all cyclic subgroups of P are 1-pure in P. In particular, the cyclic subgroups of a locally finite group of prime exponent are 1-pure in it.*

It is known that there are groups of prime exponent which are not locally finite; but it appears to be unknown whether a group must be locally finite if all its 2-generator subgroups are finite, or even boundedly finite. We draw attention to the following "hyper-Burnside" problem.

PROBLEM 9.4. *Is there a positive integer d^* such that a group is locally finite if all its d^*-generator subgroups are finite? Is there a positive integer d_* such that a group is locally finite if all its d_*-generator subgroups have orders dividing a fixed integer n? Is there a bound $b = b(d, d_*, n)$ for the orders of all finite d-generator groups whose d_*-generator subgroups have orders dividing n?*

We advance no conjectures beyond the guess that these problems are difficult.

Lemma 9.5. *If $\{P_i\}_{i \in I}$ is a family of p-groups, for fixed p, each of which has the property (*) that its cyclic subgroups are 1-pure in it, then both the cartesian product P and the direct product P^* of the family have property (*).*

PROOF. Clearly P has exponent p, like all the P_i. Let $\{\eta_i\}_{i \in I}$ be the family of epimorphisms

$$\eta_i \colon P \to P_i.$$

If $K = \mathrm{gp}(g, h)$ is a non-cyclic subgroup of P, then either there is an $i \in I$ such that

$K\eta_i$ is non-cyclic, in which case

$$p^2 = |K\eta_i\colon (K\eta_i)'| \leqq |K\colon K'| \leqq p^2;$$

or else there are $i, j \in I$ and an integer m such that

$$g\eta_i \neq 1, \quad (g^m h)\eta_i = 1, \quad (g^m h)\eta_j \neq 1,$$

and then, if $\eta = \eta_i \times \eta_j$ denotes the epimorphism of P onto $P_i \times P_j$, we similarly have

$$p^2 = |K\eta\colon (K\eta)'| \leqq |K\colon K'| \leqq p^2.$$

Thus P has property (*), by Lemma 9.2. As property (*) is evidently inherited by subgroups, P^* also has it, and the lemma follows.

We are now ready to prove the sufficiency part of Theorem 9.1.

PROOF OF THEOREM 9.1. FIRST PART. Let H be the direct product of groups P with the properties (i), (ii) of the theorem. Those factors that have the same prime exponent p can be combined to a single group with the same properties, by Lemma 9.2, 9.5; thus we lose no generality if we assume that H is the direct product of its Sylow p-subgroups P, for varying p, and that these Sylow subgroups all have property (*). Then a cyclic subgroup of H is the direct product of its intersections with these Sylow subgroups P, and as each such intersection is cyclic and thus 1-pure in P, the given cyclic subgroup is 1-pure in H, by Theorem 8.2. Thus H has property (*).

To prove the converse, we require another lemma.

Lemma 9.6. *Let the group H have the property (*) that all its cyclic subgroups are 1-pure in it. Then H is periodic, and if g, h are two elements of H, then the order $|gh|$ of their product divides the product $|g||h|$ of their orders.*

PROOF. As property (*) is inherited by subgroups, and as the infinite cyclic group does not possess it, H must be periodic. Put $K = \mathrm{gp}(g, h)$. Then also $K = \mathrm{gp}(gh, h)$, and so the cyclic subgroup $\mathrm{gp}(gh)$ has a normal complement, say N, in K. As K/N is cyclic, $K' \leqq N$; and as $|K\colon N| = |gh|$, we see that $|gh|$ divides $|K\colon K'|$. Now $|K\colon K'|$ divides $|g||h|$, as g and h generate K. Thus $|gh|$ divides $|g||h|$, and the lemma follows.

We can now prove the necessity part of Theorem 9.1.

PROOF OF THEOREM 9.1. SECOND PART. We assume that H has property (*). By Lemma 9.6 then H is periodic, and the elements whose orders are powers of a fixed prime p form a subgroup P, say, of H. This is clearly characteristic in H, and the Sylow p-subgroup of H. Different such Sylow subgroups have trivial intersection; their product is H, as H is periodic, and the product is clearly direct. Each P inherits property (*) from H, and by Lemma 9.2 has properties (i), (ii). This completes the proof of the theorem.

Corollary 9.7. *If $\{H_i\}_{i \in I}$ is a family of groups each of which has the property (*) that its cyclic subgroups are 1-pure in it, then the direct product of the family also has property (*).*

The same is not true for the cartesian product, as it is in general not even periodic.

Theorem 9.8. *If every cyclic subgroup of a group H is 2-pure in H, then H is elementary abelian, that is to say, a direct product of groups of prime order. Thus every subgroup of H is then a direct factor.*

PROOF. Theorem 9.1 shows that it suffices to show that H is abelian. If g, h are two arbitrary elements of H, the cyclic subgroup generated by their commutator $[g, h]$ must be 2-pure in $K=\mathrm{gp}(g, h)$, and so $[g, h]$ must also be the commutator of two elements in the cyclic group $\mathrm{gp}([g, h])$; it follows that $[g, h]=1$, and as g, h were arbitrary, H is abelian, and the theorem follows.

Theorem 9.9. *If every two-generator subgroup of a group H is 1-pure in H, then H is elementary abelian, and thus every subgroup is a direct factor of H.*

PROOF. Again it suffices, by Theorem 9.1, to prove that H is abelian; and, still by Theorem 9.1, we need only show that the elements of prime order p commute. Let then $g, h \in H$ and $|g|=|h|=p$, and put $G=\mathrm{gp}(g, [g, h])$ and $K=\mathrm{gp}(G, h)=$ $=\mathrm{gp}(g, h)$. As a 2-generator group, G is 1-pure in H, hence, by Theorem 3.5, there is a projection π of K onto G. Now

$$[g, h]=[g, h]\pi=[g\pi, h\pi]=[g, h\pi]\in G', \tag{9.91}$$

as $h\pi \in G$. Hence $|G: G'|$ divides p, and Lemma 9.2 (with G here taking the place of K there) shows that G must be cyclic. Thus $G'=E$, and then from (9.91) we deduce $[g, h]=1$; and the theorem follows.

Thus we see that though the non-abelian groups in which all cyclic subgroups are 1-pure form a wider class than the corresponding abelian groups, strengthening the assumption to 2-purity of cyclic subgroups instead of 1-purity, or to 1-purity of 2-generator subgroups instead of cyclic subgroups, leaves us with only the same groups as in the abelian case, and with the narrowest class of groups that can occur in this context.

10. Groups whose normal subgroups are n-pure

A different way of generalizing the theorem of FUCHS, KERTÉSZ, and SZELE ([4], Theorem 4) referred to and extended in the preceding section restricts the assumption of purity to normal subgroups. If all normal subgroups are absolutely pure, and thus direct factors, a simple description of the groups is available.

Theorem 10.1 (WIEGOLD). *The normal subgroups of the group H are absolutely pure in H if, and only if, H is a direct product of simple groups.*

For the proof we refer the reader to WIEGOLD [11]. We shall again find that the conditions of the theorem can be relaxed. The main criterion we establisch here is as follows.

Theorem 10.2. *Let H be a group with centre Z. Then all normal subgroups of H are* **n***-pure in H if, and only if,* (i) *Z is* **n***-pure in H;* (ii) *Z is elementary abelian; and* (iii) *H/Z is a direct product of non-abelian simple groups.*

It is to be understood that Z or H/Z may be trivial. Throughout the proof

we shall denote by ξ the canonic epimorphism of H onto H/Z. We require two lemmas, of which the first makes no reference to **n**-purity.

Lemma 10.3. *Let H be a group such that $H\xi$ is a direct product of non-abelian simple groups. If G is a normal subgroup of H, then the centralizer C of G in H is the unique subgroup with the properties* (i) $Z \leqq C$, *and* (ii) $H\xi = G\xi \times C\xi$.

PROOF. As a normal subgroup of the direct product $H\xi$ of non-abelian simple groups, $G\xi$ is the direct product of some of the simple direct factors of $H\xi$, and the remaining factors combine to form the unique complementary direct factor of $G\xi$ in $H\xi$; the inverse image under ξ of this complementary direct factor is then the unique subgroup C_0, say, such that $Z \leqq C_0$ and $H\xi = G\xi \times C_0\xi$. Clearly $C \leqq C_0$, and it remains to prove that $C_0 \leqq C$, that is that all elements of C_0 centralize G. Now $[G, C_0] \leqq Z$, and it follows that if $g, g' \in G$ and $c \in C_0$, then

$$[gg', c] = [g, c][g', c].$$

Thus for fixed $c \in C_0$ the mapping γ defined by

$$g\gamma = [g, c]$$

maps G homomorphically into Z. The kernel of γ clearly contains G', as $G\gamma$ is abelian, and also $G \cap Z$. But $G'(G \cap Z) = G$, because

$$G'\xi = (G\xi)' = G\xi,$$

this being a direct product of non-abelian simple groups. Hence the kernel of γ is G, and $[g, c] = 1$ for all $g \in G$ and $c \in C_0$. This shows that C_0 centralizes G, and the lemma follows.

Corollary 10.4. *Under the assumptions and with the notation of the lemma,*

$$H = GC.$$

Lemma 10.5. *If all normal subgroups that contain the centre Z of the group H are* 1*-pure in H, then the centre of $H\xi$ is trivial or, differently put, Z is the hypercentre of H.*

PROOF. If h is an element of the second centre of H, that is if $h\xi$ is central in $H\xi$, then $K = \mathrm{gp}(Z, h)$ is an abelian normal subgroup of H; by Corollary 5.4 then $K \leqq Z$, and $h \in Z$; and the lemma follows.

PROOF OF THEOREM 10.2. First assume that H has properties (i)—(iii) of the theorem. We remark that then every subgroup Z_0 of Z is $\mathfrak{n}$-pure in H; for Z_0 is absolutely pure in Z and Z is $\mathfrak{n}$-pure in H, so that Lemmas 3.1, 3.2 are applicable. Now let G be a normal subgroup of H, and let C be its centralizer. Then $H = GC$, by Corollary 10.4. Moreover, if $G \cap C = Z_0$ is the centre of G, then Z_0 is, as we have just remarked, $\mathfrak{n}$-pure in H and thus also in C. It now follows from Theorem 5.6 that G is $\mathfrak{n}$-pure in H, proving the sufficiency of the stated conditions.

Conversely assume that every normal subgroup of H is $\mathfrak{n}$-pure in H. Then in particular Z is $\mathfrak{n}$-pure in H; also every subgroup of Z must be at least 1-pure in Z, and so Z must be elementary abelian; thus (i) and (ii) are established. Now

let $G\xi$ be an arbitrary normal subgroup of $H\xi$; we may take G as a normal subgroup of H containing Z. Then G is π-pure in H by hypothesis, and $G\xi$ is π-pure in $H\xi$ by Corollary 7.2. Moreover the centre of $G\xi$ is trivial, as by Lemma 5.2 it is contained in the centre of $H\xi$, and this latter is trivial by Lemma 10.5. We apply Corollary 5.3 and see that $G\xi$ is a direct factor of $H\xi$. As this is true for every normal subgroup $G\xi$ of $H\xi$, Wiegold's Theorem 10.1 shows $H\xi$ to be a direct product of simple groups. Finally, as by Lemma 10.5 the centre of $H\xi$ is trivial, these simple groups must be non-abelian. Thus (iii) also follows, completing the proof of the theorem.

If $\pi=\aleph_0$, we can say more.

Theorem 10.6. *The normal subgroups of the group H are pure in H if, and only if, H is a direct product of simple groups; thus the normal subgroups are then absolutely pure in H.*

PROOF. If H is a direct product of simple groups, then all its normal subgroups are direct factors, hence absolutely pure, hence pure in H; see also Theorem 10.1. Conversely, assume that all normal subgroups of H are pure in H. From condition (iii) of Theorem 10.2 we see that $(H'\xi)=(H_\xi)'=H\xi$, hence $H=H'Z$. By Lemma 6.1, with H here for C there, $H'\cap Z=E$. It follows that $H=H'\times Z$. Now $H'\cong H\xi$ is a direct product of (non-abelian) simple groups, and also Z is a direct product of (abelian) simple groups, both by Theorem 10.2. Thus H is a direct product of simple groups, and the theorem follows.

We do not know whether in this situation purity can be distinguished from π-purity for finite π. It is conceivable that H must necessarily be a direct product of simple groups even if the normal subgroups are only assumed to be 1-pure in H. This would be the case if the conditions of Theorem 10.2 imply that Z is a direct factor of H, and this is so if and only if $H'\cap Z=E$; see the proof of Theorem 10.6. The answer depends on the solution to the following problem.

PROBLEM 10.7. *Does there exist a group H whose centre Z is 1-pure in H, non-trivial elementary abelian, and contained in the derived group H', and such that H/Z is a non-trivial simple group?*

11. Groups that are π-pure as subgroups

The divisible abelian groups have the property that they are 1-pure, and indeed absolutely pure, in every abelian group that contains them. For non-abelian groups there are two possible analogues of this: the groups that are π-pure in every group that contains them as subgroups, and the groups that are π-pure in every group that contains them as normal subgroups. These conditions depend on π, and become more restrictive as π is made to increase. In fact it turns out that the first of these possibilities leads to no interesting groups even for $\pi=1$.

Theorem 11.1. *The only group G that is 1-pure in every group H that contains it as a subgroup is the trivial group, $G=E$.*

PROOF. Let G be 1-pure in every group that contains it. We first form the group

$$H_1 = \mathrm{gp}(G, a; a^2 = 1, \quad [a^{-1}ga, g'] = 1 \quad \textit{for all} \quad g, g' \in G).$$

This is the so-called wreath product of G and the cyclic group of order 2 generated by a. As H_1 is generated by G and one further element, there must be a projection π_1 of H_1 onto G. Put $a\pi_1 = g_1$. Then, on applying π_1 to $[a^{-1}ga, g']$, we obtain

$$[g_1^{-1}gg_1, g'] = 1 \quad \text{for all} \quad g, g' \in G.$$

As g, g' range independently over G, so do $g_1^{-1}gg_1$ and g'; hence G must be abelian. Next form the group

$$H_2 = \mathrm{gp}(G, b; b^2 = 1, \quad (bg)^2 = 1 \quad \textit{for all} \quad g \in G).$$

This is the splitting extension of G by the involutory automorphism that inverts all its elements. Again there must be a projection, π_2 say, of H_2 onto G. Put $b\pi_2 = g_2$ and apply π_2 to bg: then

$$(g_2 g)^2 = 1 \quad \text{for all } g \in G,$$

and as $g_2 g$ ranges with g over all elements of G, we see that G has exponent 2. Finally G must be divisible in order to be 1-pure even in the abelian groups only that contain it; and the only divisible group of finite exponent is the trivial group. This proves the theorem. The theorem sharpens Theorem 2 of Baer [1], which says that only the trivial group is absolutely pure in all groups that contain it.

If we assume G only to be $\mathfrak{n}$-pure in all groups that contain G as a normal subgroup, we get more interesting results.

Theorem 11.2. *The group G is 1-pure in every group H that contains G as a normal subgroup if, and only if,* (i) *G has no outer automorphisms, and* (ii) *the centre Z of G is divisible.*

PROOF. First assume that G has properties (i) and (ii). Let G be a normal subgroup of H, and let $K = \mathrm{gp}(G, s)$ where $s \in H$. Then G is normal in K, and thus s induces an automorphism of G. By (i) this is an inner automorphism: hence there is an element $g_0 \in G$ such that for all $g \in G$

$$s^{-1}gs = g_0^{-1}gg_0.$$

Thus $a = g_0^{-1}s$ centralizes G; and also $K = \mathrm{gp}(G, a)$. Next let the order of a modulo G be m. Then $a^m = z \in Z$. Let $z_0 \in Z$ be chosen so that $z_0^m = z$. Then $b = z_0^{-1}a$ also, like a, centralizes G; and also $K = \mathrm{gp}(G, b)$. Now if $B = \mathrm{gp}(b)$, then $K = GB$ and B is normal in K; moreover $G \cap B = E$, as the order of B is m, and this is also the order of b modulo G. Thus $K = G \times B$, and it follows that G is 1-pure in H. Conversely, assume G to be 1-pure in every group H that contains G as a normal subgroup. Let α be an automorphism of G and form the splitting extension of G by a cyclic group whose generator induces this automorphism:

$$H_1 = \mathrm{gp}(G, a; \; a^{-1}ga = g^{\alpha} \quad \textit{for all} \quad g \in G).$$

Now G is normal in H_1, and H_1 is generated by G and one further element a. There must then be a projection π_1 of H_1 onto G, and transformation by $a\pi_1 \in G$ induces the same automorphism α of G; hence the automorphism is inner, and G has property (i). Next let $z \in Z$ and let n be a positive integer. Adjoin an n-th

root centrally to z by forming the group

$$H_2 = \mathrm{gp}(G, b; b^n = z, \quad [b, g] = 1 \quad \textit{for all} \quad g \in G).$$

This can be done by forming the generalized direct product of G with a cyclic group $B = \mathrm{gp}(b)$ of order mn, where m is the order of z, amalgamating z with b^n. Again G is normal in H_2, and H_2 is generated by G and one further element; thus there is a projection π_2, say, of H_2 onto G. Now $b\pi_2$ is an n-th root of z in Z; thus, as z and n were arbitrary, Z must be divisible, and G has property (ii). This completes the proof of the theorem.

All values of $\mathbf{n} \geqq 2$ give one and the same result, as is seen from the following theorem.

Theorem 11.3. *The group G is 2-pure in every group H that contains G as a normal subgroup if, and only if, G has no outer automorphisms and has trivial centre. Then G is absolutely pure, that is a direct factor, in every H in which it is normal.*

PROOF. Groups without outer automorphisms and with trivial centre, sometimes called "complete groups", are well known to be direct factors of every group that contains them as normal subgroups (HÖLDER [6], Lehrsatz I, p. 325): If G has the property and is a normal subgroup of H, then every element of H induces an inner automorphism of G, hence is congruent to an element of G modulo the centralizer C of G. It follows that $H = GC$; and $G \cap C = E$, as G has trivial centre; finally C, as the centralizer of a normal subgroup, is itself normal in H, and it follows that $H = G \times C$. Conversely assume that G is 2-pure in every group H in which G is normal. From Theorem 10.2 we know that G has no outer automorphisms, and has divisible centre. If z is an element of the centre of G, we form the group

$$H = \mathrm{gp}(G, a, b; \ [a, b] = z, \ [a, g] = [b, g] = 1 \quad \textit{for all} \quad g \in G).$$

This group can be described as the generalized direct product of G and a nilpotent group of class 2, say

$$A = \mathrm{gp}(a, b; \ [a, b]^m = [a, b, a] = [a, b, b] = 1),$$

where m is the order of z, amalgamating $z \in G$ with $[a, b] \in A$. As G is normal in H and H is generated by G and two further elements, there is a projection of H onto G. Put $b\pi = g_2$. Then, as $[a, g_2] = 1$, we get

$$z = z\pi = [a, b]\pi = [a\pi, b\pi] = [a\pi, g_2] = [a\pi, g_2\pi] = [a, g_2]\pi = 1.$$

As z was an arbitrary element of the centre of G, the centre is trivial, and the theorem follows.

The question now arises whether Theorems 10.2 and 10.3 describe actually different classes of groups, that is to say, whether there are groups without outer automorphism and with divisible but non-trivial centre. We conclude this paper with the construction of an example of such a group.

12. Construction of an example

Let R denote the additive group of rational numbers. We begin by forming the group M of triplets

$$m = (r, s, t), \quad r, s, t \in R.$$

with multiplication defined by

$$(r, s, t)(r', s', t') = (r+r', \ s+s', \ t+t'-r's). \tag{12.1}$$

The unit element is $(0, 0, 0)$, and the inverse of $m=(r, s, t)$ is

$$m^{-1} = (-r, \ -s, \ -t-rs). \tag{12.2}$$

M is nilpotent of class 2; it is known also as the "free second nilpotent square" of R. The derived group M' coincides with the centre Z and consists of the triplets

$$m' = (0, 0, t).$$

The mapping of M to M that maps each (r, s, t) on $(-r, -s, t)$ is easily seen to be an involutory automorphism of M. Let F denote the splitting extension of M by this automorphism:

$$F = \mathrm{gp}\big(M, a; \ a^2 = 1, \quad (r, s, t)^a = (-r, -s, t) \quad \textit{for all} \quad (r, s, t) \in M\big).$$

Let N denote the subgroup of M generated by $Z\ (=M')$ and all elements

$$\left.\begin{aligned} b_i &= (2^{-i}, 0, 0), \\ c_i &= (0, 3^{-i}, 0), \\ d_i &= (5^{-i}, 5^{-i}, 0). \end{aligned}\right\} \quad i = 0, 1, 2, \ldots$$

Then N will consist of those triplets $n=(r, s, t)$ whose components are of the form

$$\left.\begin{aligned} r &= u \cdot 2^{-u'} + w \cdot 5^{-w'}, \\ s &= v \cdot 3^{-v'} + w \cdot 5^{-w'}, \\ t &\in R \text{ arbitrary}, \end{aligned}\right\} \tag{12.3}$$

with integers u, u', v, v', w, w'. Finally we examine the subgroup

$$G = \mathrm{gp}\,(N, a)$$

of F.

Theorem 12.4. *The group G has no outer automorphism, and the centre Z of G is non-trivial and divisible.*

It will follow from this that all normal subgroups of G are characteristic; we begin by exhibiting several characteristic subgroups of G.

Lemma 12.5. *Among the characteristic subgroups of* G *are* $N, B, C, D, B_0, C_0, D_0, Z_0$, *where*

$$\begin{aligned} B &= \mathrm{gp}\,((r, 0, t),\ r = u \cdot 2^{-u'},\ t \in R), \\ C &= \mathrm{gp}\,((0, s, t),\ s = v \cdot 3^{-v'},\ t \in R), \\ D &= \mathrm{gp}\,((q, q, t),\ q = w \cdot 5^{-w'},\ t \in R), \\ B_0 &= \mathrm{gp}\,((u, 0, t),\ t \in R), \\ C_0 &= \mathrm{gp}\,((0, v, t),\ t \in R), \\ D_0 &= \mathrm{gp}\,((w, w, t),\ t \in R), \\ Z_0 &= \mathrm{gp}\,((0, 0, p)), \end{aligned}$$

with u, u', v, v', w, w', p *ranging over the integers.*

PROOF. As $|G: N| = 2$, and as N is nilpotent while G is not, N is the maximal nilpotent normal subgroup of G, hence characteristic in G. Next, B is characteristic in N, and hence in G, because it consists of all those elements that are 2^i-th powers for all i; similarly C consists of the 3^i-th and D of the 5^i-th powers. Finally

$$B_0 = B \cap CD, \quad C_0 = C \cap BD, \quad D_0 = D \cap BC, \quad Z_0 = [B_0, C_0],$$

which shows that these groups are also characteristic in G.

PROOF OF THEOREM 12.4. The centre $M' = Z$ of M is also the centre of N, of F, and of G, because a clearly commutes with all triplets $(0, 0, t)$. It remains to show that all automorphisms of G are inner. Let α be an automorphism of G. Then α maps $b_0 = (1, 0, 0)$, which is a generator of B_0 modulo Z, onto a generator of B_0 modulo Z, that is onto an element of the form $(\pm 1, 0, k)$. We may assume without loss of generality, that the first component is $+1$; for otherwise we replace α by its product α' with the inner automorphism induced by a; and α and α' are both inner or both outer. Thus we now have

$$b_0\alpha = (1, 0, k).$$

Similarly $c_0 = (0, 1, 0)$ is mapped by α on an element $(0, \pm 1, l)$; however, here we now must have the positive sign; for also $d_0 = (1, 1, 0) = b_0 c_0$ must be mapped on

$$d_0\alpha = b_0\alpha c_0\alpha = (1, \pm 1, k + l),$$

and this has to lie in D_0 again: hence

$$c_0\alpha = (0, 1, l).$$

Now $z_0 = (0, 0, 1) = [b_0, c_0]$, and so

$$z_0\alpha = [b_0\alpha, c_0\alpha] = [(1, 0, k),\ (0, 1, l)] = (0, 0, 1) = z_0.$$

Next, as $b_i = (2^{-i}, 0, 0)$ is the unique 2^i-th root of b_0, it must be mapped on

$$b_i\alpha = (2^{-i}, 0, 2^{-i}k),$$

which is the unique 2^i-th root of $b_0\alpha$. Similarly

$$c_i\alpha = (0, 3^{-i}, 0)\alpha = (0, 3^{-i}, 3^{-i}l),$$

$$d_i\alpha = (5^{-i}, 5^{-i}, 0)\alpha = (5^{-i}, 5^{-i}, 5^{-i}k+5^{-i}l),$$

and finally, if $z=(0, 0, t)\in Z$, then $z\alpha=z$. Now the b_i, c_i, d_i, and z between them generate N, so that the effect of α on $n=(r, s, t)$ is now determined. One readily verifies that it is

$$(r, s, t)\alpha = (r, s, kr+ls+t). \tag{12.6}$$

We gather further information by considering the effect of α on a. As $a\notin N$, the image must also be outside N, that is to say, it must be of the form

$$a\alpha = a(x, y, z), \tag{12.7}$$

where $(x, y, z)\in N$. As $a^2=1$, we must have $(a\alpha)^2=1$, and this gives

$$1 = (a\alpha)^2 = a(x, y, z)a(x, y, z) = a^2(-x, -y, z)(x, y, z) =$$

$$= (0, 0, 2z+xy) = (0, 0, 0).$$

It follows that

$$z = -\frac{1}{2}xy. \tag{12.8}$$

Next we apply α to the equation

$$a(r, s, t) = (-r, -s, t)a,$$

giving

$$a\alpha(r, s, t)\alpha = (-r, -s, t)\alpha a\alpha,$$

or, using (12.6) and (12.7),

$$a(x, y, z)(r, s, kr+ls+t) = (-r, -s, -kr-ls+t)a(x, y, z) =$$

$$= a(r, s, -kr-ls+t)(x, y, z).$$

This leads to

$$(x+r, y+s, z+kr+ls+t-ry) = (r+x, s+y, -kr - ls+t+z-xs),$$

and this to

$$(2k-y)r+(2l+x)s = 0.$$

This must be true for all $(r, s, t)\in N$, and therefore implies

$$x = -2l, \quad y = 2k.$$

On substituting this and (12.8), (12.7), we obtain

$$a\alpha = a(-2l, 2k, 2kl). \tag{12.9}$$

We now put $g=(-l, k, 0)$, and compute the effect of the inner automorphism, γ say, induced by g. We note that by (12.2),

$$g^{-1} = (\chi, -k, kl).$$

Now

$$a\gamma = g^{-1}ag = (l, -k, kl)a(-l, k, 0) = a(-l, k, kl)(-l, k, 0) =$$
$$= a(-2l, 2k, 2kl) = a\alpha,$$

by (12.9). Next, with $(r, s, t) \in N$, we have

$$(r, s, t)\gamma = (l, -k, kl)(r, s, t)(-l, k, 0) =$$
$$= (l+r, -k+s, kl+t+kr)(-l, k, 0) =$$
$$= (r, s, kl+t+kr+l(-k+s)) = (r, s, kr+ls+t) =$$
$$= (r, s, t)\alpha,$$

by (12.6). Thus the effect of γ equals that of α on a and on the elements of N, hence on all elements of G: that is, α equals the inner automorphism γ. This completes the proof of the theorem. The group G here constructed is 1-pure in every group in which it is contained as a normal subgroup; but it is not 2-pure in the generalized direct product H, say, of G and an isomorphic copy of M, amalgamating the centre Z, though G is normal in H.

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