

On some questions concerning the differential geometry of curves in n -dimensional euclidean spaces

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1. Introduction

It is well-known that the classical Frenet-theory of curves in euclidean 3-space applies only to those curves which have non-vanishing curvature at each of their points. In this way, for example, the trivial case of a line segment is not covered by the usual treatment. Noticing this incompleteness K. NOMIZU set about to extend the classical theory to a larger class of curves, to the so-called Frenet-curves [1]. A rigorous treatment of Frenet-curves including such special ones as plane curves, spherical curves and helices was elaborated by YUNG-CHOW WONG and HON-FEI LAI [2] for the 3-dimensional case.

In the present paper some general methods will be developed in order to be able to extend a considerable part of the results in [2] for curves in higher dimensional euclidean spaces. Moreover some important results of R. BISHOP [3] will be also proved here for $n > 3$.

Our methods are based mainly on a simple formula which relates two different families of moving frames being adapted to the same curve.

2. On moving frames of curves in $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the n -dimensional euclidean space and let us consider such a curve in $\mathbb{R}^n$ which can be given in the following arc-length representation:

$$\Gamma: \underline{x} = \underline{x}(s), \quad s \in L = [0, \lambda],$$

where λ is the total length of the curve Γ , and the vectorfunction $\underline{x}(s)$ is supposed to be of class C^∞ on L . According to this, we will mean by a curve in this paper always an oriented C^∞ [and regular] curve having a finite length λ where, of course, by a C^∞ -function on a closed interval L , we mean a function which may be extended to a C^∞ -function on an open interval containing L . Γ is called regular, if $\underline{x}'(s) \neq \underline{0}$. Regularity will also be supposed.

Let $E(s), s \in L$ be a matrix-function of class C^∞ . It will be called a *moving frame* if

$$E(s) \in SO(n) \quad \text{holds for each } s \in L,$$

or in other words if the vectors

$$e_1(s), e_2(s), \dots, e_n(s),$$

which are the consecutive columns of the matrix $E(s)$, form a positively oriented orthonormal basis of $\mathbf{R}^n$ for each $s \in L$.

We will say that a moving frame $E(s)$ is *adapted* to a curve $\Gamma: \underline{x} = \underline{x}(s), s \in L$, if

$$\underline{x}'(s) = e_1(s)$$

holds for each $s \in L$.

Let $E(s)$ and $\tilde{E}(s), s \in L$ be two different moving frames. They are called *congruent* if there exists an orthogonal matrix $R \in SO(n)$ such that $\tilde{E}(s) = RE(s)$ holds on the whole interval L . It is easy to see that if $E(s)$ and $\tilde{E}(s), s \in L$, are congruent moving frames being adapted to the curves $\Gamma: \underline{x} = \underline{x}(s)$ and $\tilde{\Gamma}: \underline{\tilde{x}} = \underline{\tilde{x}}(s), s \in L$, respectively, then there exists an orientation preserving isometry of $\mathbf{R}^n$ which carries the curve Γ into the curve $\tilde{\Gamma}$, that is $\underline{\tilde{x}}(s) = R\underline{x}(s) + \underline{a}$ holds for each $s \in L$, where $\underline{a} \in \mathbf{R}^n$.

Let $E(s), s \in L$ be an arbitrary moving frame. The usual derivational formulae for the frame-vectors can be conveniently expressed in the following matrix-equation:

$$(1) \quad E'(s) = -E(s)C(s), \quad s \in L$$

where $C(s)$ is the so-called *Cartan-matrix*.

It is uniquely defined by the equation:

$$C(s) = -E^*(s)E'(s), \quad s \in L$$

where asterisk denotes transposition.

It is evident that a Cartan-matrix is always skew-symmetric. This follows immediately from the fact that the matrix $(E^*(s)E(s))'$ is identically zero and thus

$$C^*(s) = -(E'(s))^*E(s) = E^*(s)E'(s) = -C(s)$$

holds for each $s \in L$.

Notice also that congruent moving frames must have the same Cartan-matrix. In fact, let $\tilde{E}(s) = RE(s), s \in L$ and $R \in SO(n)$, then

$$\tilde{C}(s) = -\tilde{E}^*(s)\tilde{E}'(s) = -(E^*(s)R^*)(RE'(s)) = -E^*(s)E'(s) = C(s)$$

holds on L since R^*R is the unit matrix of $SO(n)$.

Theorem 1. *Let $C(s), s \in L$ be an arbitrary skew-symmetric matrix-function of class C^∞ . Then there always exists a moving frame $E(s), s \in L$ whose Cartan-matrix is $C(s)$, and all the moving frames having the same Cartan-matrix $C(s)$ are congruent.*

PROOF. The matrix-equation (1) can be considered as a linear system of differential equations for the unknown entries of $E(s), s \in L$. Let the initial condition be chosen so that, at a fixed value of parameter s_0 $E(s_0) \in SO(n)$ holds. If $E = \{e_j^i | i, j = 1, 2, \dots, n\} \in m_n(\mathbf{R})$, where $m_n(\mathbf{R})$ denotes the set of all matrices of degree n with coefficients in $\mathbf{R}$, then $E \cdot C(s)$ is a continuous function on the closed (n^2+1) -dimensional square domain given by $s \in L$ and $|e_j^i| \leq 1$ for

$i, j=1, 2, \dots, n$; consequently the existence of a solution satisfying the given initial condition is assured ([4], pp. 85—86).

On the other hand for the solution

$$(E(s)E^*(s))' = (-E(s)C(s))E^*(s) + E(s)(C(s)E^*(s)) = 0$$

holds on the whole interval L , and hence $E(s) \in SO(n)$ also holds for each $s \in L$ showing that $E(s)$ is a moving frame.

According to the usual proof of the uniqueness let $E(s)$ and $\tilde{E}(s)$, $s \in L$ be two solutions of the equation (1) for which $e_j(s_0) = \tilde{e}_j(s_0)$ holds for $j=1, 2, \dots, n$. It is easy to see that the derivative of the scalar-function

$$f(s) = \sum_{j=1}^n \langle e_j(s), \tilde{e}_j(s) \rangle$$

is identically zero.

So, since $f(s) = f(s_0) = n$ and $|\langle e_j(s), \tilde{e}_j(s) \rangle| \leq 1$ we have that $E(s) = \tilde{E}(s)$ holds for each $s \in L$.

At last, following of the uniqueness, all the moving frames which are solutions of the equation (1) must be congruent.

Corollary. *A curve in $\mathbf{R}^n$ can be given uniquely up to an orientation preserving isometry by a prescribed skew-symmetric matrix-function of class C^∞ .*

Theorem 2. *Let $E(s)$ and $\tilde{E}(s)$, $s \in L$ be two different moving frames and denote by $C(s)$ and $\tilde{C}(s)$ their Cartan-matrices, respectively. Then the following matrix-equation holds on the whole interval L :*

$$(2) \quad \tilde{C}(s) = A'(s)A^*(s) + A(s)C(s)A^*(s),$$

where the matrix-function $A(s)$ expressing the unique C^∞ transformation between the given moving frames is defined by

$$(3) \quad A(s) = \tilde{E}^*(s)E(s), \quad s \in L.$$

PROOF. It is enough to give a short verification of (2). From (3) we get that $\tilde{E}^*(s) = A(s)E^*(s)$ and $\tilde{E}(s) = E(s)A^*(s)$ hold for each $s \in L$.

On the other hand we will use the identity $(A(s)A^*(s))' = 0$, $s \in L$, and also the very definitions of the Cartan-matrices $C(s)$ and $\tilde{C}(s)$. Thus

$$\begin{aligned} \tilde{C}(s) &= -\tilde{E}^*(s)\tilde{E}'(s) = -A(s)E^*(s)(E'(s)A^*(s) + E(s)A^{*'}(s)) = \\ &= A(s)C(s)A^*(s) + A'(s)A^*(s) \end{aligned}$$

identically holds on L .

Remarks. Let $E(s)$, $s \in L$ be an arbitrary moving frame which is adapted to a curve Γ . It is easy to see that a matrix-function $\tilde{E}(s)$, $s \in L$ given in the form $\tilde{E}(s) = E(s)A^*(s)$, $s \in L$ will be also a moving frame of the same curve Γ if and only if the following conditions are satisfied for the C^∞ matrix-function $A(s)$, $s \in L$:

- (i) $A(s) \in SO(n)$;
- (ii) for entries in the first row and first column of $A(s)$ $a_{11}(s) = 1$ and $a_{1j}(s) = a_{j1}(s) = 0$ hold for $i, j \neq 1$ and for each $s \in L$.

We will call here a C^∞ matrix-function with the properties (i) and (ii) simply a *frame-transformator*.

Now some special types of moving frames will be defined for C^∞ and regular curves lying in $\mathbf{R}^n$:

(a) A moving frame $E(s)$, $s \in L$ will be called a *Frenet-frame* if for the frame vectors the following derivational formulae (the so-called Frenet-equations) hold for each $s \in L$:

$$\begin{aligned} e'_1(s) &= k_1(s)e_2(s), \\ e'_i(s) &= -k_{i-1}(s)e_{i-1}(s) + k_i(s)e_{i+1}(s) \\ &\quad \text{for } i = 2, 3, \dots, n-1 \text{ and} \\ e'_n(s) &= -k_{n-1}(s)e_{n-1}(s), \end{aligned}$$

where the suitable coefficients $k_i(s)$, $i = 1, 2, \dots, n-1$, are called the *pseudo-curvatures* belonging to $E(s)$.

The definition shows that the entries in the corresponding Cartan-matrix $C(s)$, $s \in L$ are the following:

$$\begin{aligned} c_{ij}(s) &= k_i(s) \quad \text{for } i = 1, 2, \dots, n-1 \text{ and } j = i+1; \\ c_{ij}(s) &= 0 \quad \text{for } i = 1, 2, \dots, n-2 \text{ and } j > i+1; \end{aligned}$$

and using the skew-symmetry

$$c_{ij}(s) = -c_{ji}(s) \quad \text{for } j \leq i.$$

(b) A moving frame $E(s)$, $s \in L$ will be called here a *Bishop-frame* if for the frame vectors the following derivational formulae (the so-called Bishop-equations) hold for each $s \in L$:

$$\begin{aligned} e'_1(s) &= \sum_{i=1}^{n-1} b_i(s)e_{i+1}(s) \quad \text{and} \\ e'_{i+1}(s) &= -b_i(s)e_1(s) \quad \text{for } i = 1, 2, \dots, n-1, \end{aligned}$$

where the suitable coefficients $b_i(s)$, $i = 1, 2, \dots, n-1$, will be called the *Bishop-coefficients* belonging to $E(s)$.

The definition shows that the entries in the corresponding Cartan-matrix $C(s)$, $s \in L$ are the following

$$\begin{aligned} c_{ij}(s) &= b_{j-1}(s) \quad \text{for } i = 1 \text{ and } j > 1; \\ c_{ij}(s) &= 0 \quad \text{for } i > 1 \text{ and } j > i; \end{aligned}$$

and using the skew-symmetry

$$c_{ij}(s) = -c_{ji}(s) \quad \text{for } j \leq i.$$

Notice that in the Cartan-matrices of both a Frenet-frame and a Bishop-frame $\binom{n-1}{2}$ out of the $\binom{n}{2}$ independent entries are identically zero.

3. On Frenet-curves

Following Nomizu, a curve will be called a *Frenet-curve* if there exists a Frenet-frame which is adapted to it. It should be noted that a Frenet-curve may have more than one Frenet-frames adapted to it. In other words, there may exist many different systems of pseudo-curvatures which determine the same Frenet-curve in $\mathbf{R}^n$.

The problem of finding a necessary and sufficient condition for a curve to be a Frenet-curve has been studied by several authors including K. NOMIZU [1] and A. WINTNER [8] but the problem in its entire generality has not been solved yet. In any case an example of Nomizu shows that there are C^∞ and regular curves in $\mathbf{R}^n$ which do not admit Frenet-frames.

The following simple but rather strong sufficient condition is well-known from the classical treatment: Let $\Gamma: x=x(s)$, $s \in L$ be a C^∞ and regular curve in $\mathbf{R}^n$. If the consecutive derivative vectors $x^{(i)}(s)$, for $i=1, 2, \dots, n-1$ are linearly independent at every $s \in L$ then Γ is a Frenet-curve.

The method for obtaining a Frenet-frame of such a curve Γ has been concisely presented by H. GLUCK [5] and [6], as follows: The Gram—Schmidt orthonormalization process applied to the vectors

$$x'(s), x''(s), \dots, x^{(n-1)}(s)$$

gives the unit vectors

$$e_1(s), e_2(s), \dots, e_{n-1}(s)$$

uniquely at every $s \in L$ including the fact that

$$\langle x^{(i)}(s), e_i(s) \rangle > 0 \quad \text{for } i = 1, 2, \dots, n-1.$$

The choice of the last unit vector $e_n(s)$, $s \in L$ is already independent of the derivative $x^{(n)}(s)$, it is defined uniquely by the fixed orientation of $\mathbf{R}^n$. The pseudo-curvatures $k_i(s)$, $i=1, 2, \dots, n-1$, $s \in L$ belonging to the above given Frenet-frame will be positive for $i=1, 2, \dots, n-2$ and $\text{sign } k_{n-1}(s) = \text{sign } \langle x^{(n)}(s), e_n(s) \rangle$ holds on L . Moreover the absolute values of the pseudo-curvatures, called simply curvatures in this case, have concrete geometrical meaning. Namely, they are the velocities of the so-called osculating subspaces as it was pointed out first by E. EGERVÁRY [7].

In fact, the p -dimensional osculating subspaces ($p=1, 2, \dots, n-1$) are spanned now just by the vectors $e_1(s), e_2(s), \dots, e_p(s)$, $s \in L$ instead of the derivative vectors

$$x'(s), x''(s), \dots, x^{(p)}(s), \quad s \in L.$$

Thus, as the method of H. Gluck shows, it is enough to consider the turning of the following unit p -vector:

$$n_p(s) = e_1(s) \wedge e_2(s) \wedge \dots \wedge e_p(s), \quad s \in L$$

where $p=1, 2, \dots, n-1$.

It can be noticed that the p -th curvature of Γ at a given parameter $s \in L$ is nothing else than the norm of the derivative p -vector $n'_p(s)$, where the norm in the $\binom{n}{p}$ -dimensional euclidean vector space $\Lambda^p(\mathbf{R}^n)$ is defined by the inner product induced by that of $\mathbf{R}^n$. Let us differentiate $n_p(s)$ with respect to the arc-length s .

Using the Frenet-equations and also some basic properties of the exterior product, we get the above mentioned result:

$$\|\underline{n}'_p(s)\| = |k_p(s)| \quad \text{holds for } p = 1, 2, \dots, n-1 \quad \text{and } s \in L.$$

It is convenient to say that a curve $\Gamma: \underline{x} = \underline{x}(s), s \in L$ has regularity of order p ($1 \leq p \leq n$) if the consecutive derivative vectors $\underline{x}'(s), \underline{x}''(s), \dots, \underline{x}^{(p)}(s)$ are linearly independent at every $s \in L$. Now, it is clear that the classical Frenet-theory of curves covers only those Frenet-curves which have regularity of order $(n-1)$.

For a further discussion of Frenet-curves we will prove here the following important theorem of R. BISHOP [3] for the general n -dimensional case:

Theorem 3. *Let $\Gamma: \underline{x} = \underline{x}(s), s \in L$ be an arbitrary C^∞ and regular curve in $\mathbf{R}^n$. Then there always exists a Bishop-frame which is adapted to it.*

On account of this theorem we may call here all C^∞ and regular curves Bishop-curves, as well.

PROOF. Let s_0 be an arbitrarily fixed value in the parameter interval L and denote $\underline{e}_1^*, \underline{e}_2^*, \dots, \underline{e}_n^*$ a positively oriented orthonormal basis of $\mathbf{R}^n$ where $\underline{e}_1^* = \underline{x}'(s_0)$ holds. As there always exists a closed interval $L(s_0) \subseteq L$ containing s_0 where the vectors

$$\underline{x}'(s), \underline{e}_2^*, \dots, \underline{e}_n^*, \quad s \in L(s_0)$$

remain linearly independent, the Gram—Schmidt orthonormalization process can be applied to these vectors at every $s \in L(s_0)$. So we have a local moving frame $E(s), s \in L(s_0)$ being adapted to the corresponding arc of the curve Γ .

Assume now that there is a frame transformer $A(s), s \in L(s_0)$ which carries the above chosen local moving frame $E(s)$ into a local Bishop-frame $\tilde{E}(s), s \in L(s_0)$. Then on account of Theorem 2 the unknown entries in the matrix of the frame transformer $A(s), s \in L(s_0)$ have to satisfy the following system of differential equations:

$$a'_{ij}(s) = - \sum_{l=1}^n a_{il}(s) c_{lj}(s) \quad \text{for } i, j = 2, 3, \dots, n \quad \text{and } s \in L(s_0).$$

We can write these equations in the more convenient matrix-form:

$$A'_0(s) = -A_0(s)C_0(s), \quad s \in L(s_0)$$

where $A_0(s)$ and $C_0(s)$ denote the matrices obtained at every $s \in L(s_0)$ by omitting the first rows and columns of the matrices $A(s)$ and $C(s)$, respectively. So, as it was shown in the proof of the Theorem 1, there exists a unique solution $A_0(s), s \in L(s_0)$ satisfying a given initial condition $A_0(\bar{s}) \in SO(n-1)$, where $\bar{s} \in L(s_0)$, and

$$A_0(s) \in SO(n-1) \quad \text{holds on the whole interval } L(s_0).$$

The uniqueness of the local Bishop-frame $\tilde{E}(s), s \in L(s_0)$ belonging to the above chosen initial condition

$$\tilde{E}(\bar{s}) = E(\bar{s})A^*(\bar{S}), \quad \bar{s} \in L(s_0)$$

is merely a consequence of the fact that the frame-transformator $A(s)$ has been uniquely determined on the interval $L(s_0)$.

Only the global existence of a Bishop-frame adapted to the whole curve Γ remained to be shown. It is easy to see that there exists for L a finite system of open covering intervals on each of which the existence of a local Bishop-frame is assured. These local Bishop-frames can be patched together, and due to the above mentioned uniqueness they link together smoothly.

Remark. Let $E(s)$, $s \in L$ be a Bishop-frame adapted to a given curve Γ . The corresponding Bishop-coefficients will be denoted by $b_i(s)$ for $i=1, 2, \dots, n-1$. Let further $A(s)$, $s \in L$ be an arbitrary frame-transformator.

Then, on account of Theorem 2. the C^∞ entries in the transformed Cartan-matrix $\tilde{C}(s)$, $s \in L$ are the following:

$$\tilde{c}_{ij}(s) = \sum_{l=2}^n b_{l-1}(s) a_{jl}(s) \quad \text{for } i=1 \quad \text{and } j > i$$

$$\tilde{c}_{ij}(s) = \sum_{l=2}^n a'_{il}(s) a_{jl}(s) \quad \text{for } i=2, 3, \dots, n-1 \quad \text{and } j > i$$

and using the skew-symmetry

$$\tilde{c}_{ij}(s) = -\tilde{c}_{ji}(s) \quad \text{for } j \geq i.$$

Corollary 1. *It is easy to see that $\tilde{C}(s)$, $s \in L$ will belong also to a Bishop-frame of the given curve Γ if and only if the frame-transformator $A(s)$ is constant on the parameter interval L .*

In fact, from $A'(s)=0$ we get that $\tilde{c}_{ij}(s)=0$ holds for $i, j \geq 2$. Conversely, if $\tilde{C}(s)$ belongs to a Bishop-frame of Γ then $A'(s)A^*(s)=0$ has to hold for each $s \in L$ implying that $A(s)$ is constant on the parameter interval L . This is in a complete accordance with the fact that if once the frame-vectors $\underline{e}_2^*, \underline{e}_3^*, \dots, \underline{e}_n^*$ orthogonal to the tangent vector $\underline{x}'(s_0)$ are chosen at an initial parameter $s_0 \in L$ then the Bishop-frame of Γ is already unique.

Corollary 2. *K. Nomizu's crucial problem of finding a necessary and sufficient condition for a C^∞ and regular curve to be a Frenet-curve can be answered now, as follows:*

The considered curve Γ is a Frenet-curve if and only if there exists a suitable frame-transformator $A(s)$, $s \in L$ for which

$$\sum_{l=2}^n b_{l-1}(s) a_{jl}(s) = 0 \quad \text{for } j > 2$$

and

$$\sum_{l=2}^n a'_{il}(s) a_{jl}(s) = 0 \quad \text{for } i=2, 3, \dots, n-2 \quad j > i+1$$

hold at every $s \in L$.

Notice that the existence of a solution for $A(s)$ having $\binom{n-1}{2}$ unknown independent C^∞ entries does not depend on the actual choice of the Bishop-coefficients $b_i(s)$, $i=1, 2, \dots, n-1$, characterizing the given curve Γ .

In the most interesting special case, where $n=3$, the above condition expressing which entries have to vanish in the Cartan-matrix of the transformed moving frame reduces to the following single equation:

$$b_1(s) \sin \varphi(s) = b_2 \cos \varphi(s) \quad \text{at every } s \in L,$$

where $\varphi(s)$, $s \in L$ is the only C^∞ scalar-function to be determined in the frame-transformator

$$A(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi(s) & \sin \varphi(s) \\ 0 & -\sin \varphi(s) & \cos \varphi(s) \end{pmatrix}, \quad s \in L.$$

Let us consider now only Frenet-curves in $\mathbf{R}^n$. The following theorems will show how far the pseudo-curvatures are determined by a given Frenet-curve.

Theorem 4. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a Frenet-curve in $\mathbf{R}^n$ and p ($1 \leq p \leq n$) a fixed natural number. Let further $s_0 \in L$ be an arbitrarily chosen value of parameter

Then the following two conditions are equivalent:

(i) The consecutive derivative vectors

$$\underline{x}'(s_0), \underline{x}''(s_0), \dots, \underline{x}^{(i)}(s_0) \quad \text{are}$$

linearly independent for $i \leq p$, and linearly dependent for $i > p$,

(ii) $k_i(s_0) \neq 0$ for $i < p$ and

$$k_i(s_0) = 0 \quad \text{for } i = p$$

hold at the given parameter s_0 , where $k_i(s)$ for $i=1, 2, \dots, n-1$ and for $s \in L$ denote the pseudo-curvatures belonging to an arbitrary Frenet-frame adapted to the curve Γ .

PROOF. Let $E(s)$, $s \in L$ be one of the Frenet-frames adapted to the given curve Γ . Then the higher derivatives of $\underline{x}(s)$ can be obtained at each $s \in L$ as linear combinations of the frame vectors $\underline{e}_1(s)$, $\underline{e}_2(s)$, ..., $\underline{e}_n(s)$. Applying the Frenet-equations we can write:

$$\underline{x}'(s) = \underline{e}_1(s),$$

$$\underline{x}''(s) = k_1(s) \underline{e}_2(s),$$

...

$$\underline{x}^{(r)}(s) = \sum_{j=1}^n \lambda_{jr}(s) \underline{e}_j(s),$$

where the coefficient $\lambda_{jr}(s)$ is equal to zero for each $j > r$; moreover, it can be verified that $\lambda_{jr}(s) = \prod_{l=1}^{r-1} k_l(s)$ holds for $j=r$ and $2 \leq r \leq n$. On account of basic properties of the exterior multiplication we can get the following expression:

$$\underline{x}'(s) \wedge \underline{x}''(s) \wedge \dots \wedge \underline{x}^{(r)}(s) = \left\{ \prod_{l=1}^{r-1} k_l^{-1}(s) \right\} \underline{e}_1(s) \wedge \underline{e}_2(s) \wedge \dots \wedge \underline{e}_r(s)$$

for each $s \in L$ and $2 \leq r \leq n$.

Using the fact that the vectors $\underline{x}'(s), \underline{x}''(s), \dots, \underline{x}^{(r)}(s)$ are linearly independent in $\mathbf{R}^n$ if and only if the corresponding r -vector

$$\underline{x}'(s) \wedge \underline{x}''(s) \wedge \dots \wedge \underline{x}^{(r)}(s)$$

is not zero, the proof of our theorem can easily be completed, since

$$\prod_{i=1}^{p-1} k_i(s_0) \neq 0 \quad \text{and} \quad \prod_{i=1}^p k_i(s_0) = 0$$

has to hold at $s_0 \in L$.

Remark. Let $E(s), s \in L$ be a Frenet-frame adapted to a given curve Γ and $k_i(s)$ for $i=1, 2, \dots, n-1$ and for $s \in L$ the corresponding pseudo-curvatures. Then the parameter interval L can be decomposed in the following form:

$$L = \bigcup_{i=1}^n L_i$$

where

$$L_1 = \{s: k_1(s) = 0\},$$

$$L_i = \left\{s: \prod_{i=1}^{i-1} k_i(s) \neq 0 \quad \text{and} \quad k_i(s) = 0\right\} \quad \text{for} \quad 2 \leq i \leq n-1$$

and

$$L_n = \left\{s: \prod_{i=1}^{n-1} k_i(s) \neq 0\right\}.$$

Theorem 5. Let $E(s)$ and $\tilde{E}(s)$ be two different Frenet-frames adapted to a given Frenet-curve $\Gamma: \underline{x} = \underline{x}(s), s \in L$. If $\underline{x}'(s), \underline{x}''(s), \dots, \underline{x}^{(p)}(s)$ are linearly independent vectors at a given parameter $s_0 \in L$ then $\tilde{e}_i(s_0) = \varepsilon_i e_i(s_0)$ holds for $i=1, 2, \dots, p$ and $s_0 \in L$, where ε_i is either $+1$ or -1 .

PROOF. Let $A(s), s \in L$ be the frame-transformator carrying $E(s)$ into $\tilde{E}(s)$. Then, on account of Theorem 2., the following conditions are satisfied for each $s \in L$ and $1 \leq i < j \leq n$:

(4)

$$\sum_{i=1}^n a'_{ii}(s) a_{ji}(s) + \sum_{i=1}^{n-1} \{a_{i,i}(s) a_{j,i+1}(s) - a_{i,i+1}(s) a_{ji}(s)\} \cdot k_i(s) = \begin{cases} \tilde{k}_i(s) & \text{for } j = i+1 \\ 0 & \text{for } j > i+1, \end{cases}$$

where $k_i(s)$ and $\tilde{k}_i(s)$, for $i=1, 2, \dots, n-1$ and for $s \in L$ denote the pseudo-curvatures belonging to $E(s)$ and $\tilde{E}(s)$, respectively.

Let $I \subseteq L$ denote an open neighbourhood of s_0 where the vectors $\underline{x}'(s), \underline{x}''(s), \dots, \underline{x}^{(p)}(s), s \in I$ are still linearly independent, and so we have that $k_i(s) \neq 0$ holds for each $s \in I$ and $i=1, 2, \dots, p-1$.

Now, we start proving our theorem step by step. First it is trivial that $\tilde{e}_1(s) = \varepsilon_1 e_1(s)$ holds on I , where $\varepsilon_1 = 1$. From (4) $\tilde{k}_1(s) = a_{11}(s) a_{22}(s) k_1(s)$, $s \in L$ can be obtained since for $i, j \neq 1$ $a_{ii}(s)$ and $a_{ij}(s)$ are identically zero on I . Let us compare now the Frenet-equations valid for $\tilde{e}'_i(s)$ and $e'_i(s)$. Using also the condition that $k_i(s) \neq 0$ for $s \in I$, we get that $a_{22} \tilde{e}_2(s) = e_2(s)$ holds for each $s \in I$.

Thus, we already know that $\underline{e}_2(s_0) = \varepsilon_2 \underline{e}_1(s_0)$ holds at $s_0 \in L$. Continuing this procedure analogously we will arrive at the last step. Then from (4)

$$\tilde{k}_{p-1}(s) = a_{p-1,p-1}(s)a_{pp}(s)k_{p-1}(s), \quad s \in I$$

can be obtained since we have already that $a_{i,p-1}(s)$ and $a_{p-1,j}(s)$ are identically zero on I for $i, j \neq p-1$. Let us compare now the Frenet-equations valid for $\tilde{e}_{p-1}(s)$ and $\underline{e}_{p-1}(s)$. Using also the condition that $k_{p-1}(s) \neq 0$ for $s \in I$, we get that $a_{pp}(s)\tilde{e}_p(s) = \underline{e}_p(s)$ holds for each $s \in I$. Thus, we know that $\tilde{e}_p(s_0) = \varepsilon_p \underline{e}_p(s_0)$ holds at $s_0 \in L$, as well.

Remark 1. Notice that the r -dimensional osculating subspaces of the given curve Γ surely exist at the parameter $s_0 \in L$ for $r = 1, 2, \dots, p$. So, as it was shown earlier, their velocities also exist and are given by the absolute values of the corresponding pseudo-curvatures $k_r(s_0)$ for $r = 1, 2, \dots, p$. It is reasonable to ask whether these values do not depend on the choice of the Frenet-frame which is actually adapted to the curve Γ . In fact, we can see from the proof of the above theorem that for the different pseudo-curvatures $k_r(s)$ and $\tilde{k}_r(s)$ the following conditions hold at the given parameter $s_0 \in L$: $|\tilde{k}_r(s_0)| = |k_r(s_0)| > 0$ for $r = 1, 2, \dots, p-1$, and due to Theorem 4. $|\tilde{k}_p(s_0)| = |k_p(s_0)|$.

Finally, in order to cover also the case $p = n$, it seems convenient to accept for the n -th pseudo-curvature function $k_n(s)$ the following definition: $k_n(s) = 0$ for each $s \in L$.

Remark 2. As the proof of the Theorem 5. shows the matrix of the frame-transformator $A(s)$, $s \in L$ must have the following form at $s_0 \in L$:

$$A(s_0) = \begin{pmatrix} \varepsilon_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \varepsilon_p & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{p+1,p+1} & \dots & a_{p+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & a_{n,p+1} & \dots & a_{nn} \end{pmatrix}$$

In particular, let $p = n-1$.
Then

$$A(s_0) = \begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{pmatrix} \quad \text{holds, where}$$

$$\varepsilon_1 = 1 \quad \text{and} \quad \varepsilon_n = \prod_{l=1}^{n-1} \varepsilon_l.$$

Consequently there are only 2^{n-2} possibilities for the choice of different Frenet-frames adapted to a curve Γ having regularity of order $(n-1)$.

4. Characterization of curves lying in p -planes

Definition 1. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a C^∞ and regular curve in $\mathbf{R}^n$. It is said that Γ lies in a p -plane ($1 \leq p < n$) if there exists a p -dimensional linear submanifold of $\mathbf{R}^n$ containing Γ .

Definition 2. We say that Γ lies uniformly in a p -plane if it lies in a p -plane but it has no subarcs lying in an r -plane, where $r < p$.

Theorem 6. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a Frenet-curve. If there exists a Frenet-frame $E(s)$, $s \in L$ adapted to the curve Γ where $k_p(s) = 0$ identically holds on the whole interval L ($1 \leq p < n$), then Γ lies in a p -plane.

PROOF. Let $s_0 \in L$ be an arbitrarily chosen parameter value. It is enough to show that $(\underline{x}(s) - \underline{x}(s_0)) \wedge \underline{n}_p(s_0) = 0$ identically holds for each $s \in L$, where $\underline{n}_p(s)$ denotes the p -vector formed by the vectors $\underline{e}_j(s)$, for $j = 1, 2, \dots, p$, of the considered Frenet-frame $E(s)$, $s \in L$. First, on account of the Frenet-formulas and some basic properties of the exterior multiplication, it is easy to see that $\frac{d}{ds}(\underline{n}_p(s)) = 0$ holds for each $s \in L$. Thus we can get the following identity:

$$\underline{e}_1(s) \wedge \underline{n}_p(s_0) = 0, \quad s \in L.$$

An integration of this last equation already shows the desired result.

Remark. It should be noticed that the condition $k_p(s) = 0$ for each $s \in L$, is only a sufficient but not a necessary condition for a Frenet-curve to lie in a p -plane.

Theorem 7. If a Frenet-curve Γ lies in a p -plane then for each system of its pseudo-curvatures

$$\prod_{i=1}^p k_i(s) = 0 \quad \text{holds for every } s \in L.$$

PROOF. Since Γ lies in a p -plane, there exists a suitable p -vector $\underline{n}_p$, so that $(\underline{x}(s) - \underline{x}(s_0)) \wedge \underline{n}_p = 0$ identically holds on L . Thus for the derivative vectors $\underline{x}^{(r)}(s)$ $\underline{x}^{(r)}(s) \wedge \underline{n}_p = 0$ also holds for each natural number r and $s \in L$. It is easy to see that the consecutive derivative vectors $\underline{x}'(s)$, $\underline{x}''(s)$, ..., $\underline{x}^{(r)}(s)$ cannot be linearly independent if $r > p$ and so $\underline{x}'(s) \wedge \underline{x}''(s) \wedge \dots \wedge \underline{x}^{(p+1)}(s) = 0$ identically holds on L implying that

$$\prod_{i=1}^p k_i(s) = 0$$

also has to hold for each $s \in L$.

Remark: If a Frenet-curve Γ lies in a p -plane then the previously introduced decomposition of the parameter interval L is the following $L = \bigcup_{i=1}^p L_i$, in other words $L_i = 0$ for $i = p+1, \dots, n$.

Definition: A curve $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ will be called strictly p -regular curve if for the consecutive derivative-vectors $\underline{x}'(s)$, $\underline{x}''(s)$, ..., $\underline{x}^{(r)}(s)$, $s \in L$ the following conditions hold:

- (i) they are linearly independent for $r \leq p$,
- (ii) they are linearly dependent for $r > p$.

Theorem 8. *Every strictly p -regular curve Γ is a Frenet-curve lying uniformly in a p -plane.*

PROOF. First we show that Γ is a Frenet-curve. Applying the Gram—Schmidt orthonormalization process for the linearly independent derivative vectors $\dot{x}(s)$, $\ddot{x}(s)$, ..., $x^{(p)}(s)$, we get for each $s \in L$ the first p unit vectors of a Frenetframe $E(s)$. It is easy to see on account of the condition (ii) that

$$\frac{d}{ds} (\varepsilon_1(s) \wedge \varepsilon_2(s) \wedge \dots \wedge \varepsilon_p(s)) = 0$$

has to hold for each $s \in L$. So, from the first p Frenet-equations we get for the pseudo-curvatures $k_i(s)$ the following results:

$$\prod_{i=1}^{p-1} k_i(s) \neq 0 \quad \text{and} \quad k_p(s) = 0$$

hold on the whole interval L .

Notice that the unit vectors

$$\varepsilon_{p+1}(s), \varepsilon_{p+2}(s), \dots, \varepsilon_n(s), \quad s \in L,$$

of a suitable Frenet-frame $E(s)$ can be chosen from that constant $(n-p)$ -dimensional subspace of $\mathbf{R}^n$ which is orthogonal to the p -dimensional subspace spanned by the linearly independent vectors $\dot{x}(s)$, $\ddot{x}(s)$, ..., $x^{(p)}(s)$. This choice, however, has much freedom and so the corresponding pseudo-curvatures

$$k_{p+1}(s), k_{p+2}(s), \dots, k_{n-1}(s)$$

may be arbitrary C^∞ scalar-functions on L .

Let now $\tilde{E}(s)$, $s \in L$ denote a Frenet-frame adapted to the given curve Γ and $\tilde{k}_i(s)$, for $i=1, 2, \dots, n-1$ and for $s \in L$ be the corresponding pseudo-curvatures. Then by Theorem 4 $\tilde{k}_p(s)=0$ holds on L and therefore we have that Γ lies in a p -plane. Moreover, again by Theorem 4

$$\prod_{i=1}^{p-1} k_i(s) \neq 0, \quad s \in L$$

holds and therefore on account of Theorem 7 we have that Γ has no subarcs lying in any lower dimensional r -plane.

Remark. For the parameter interval L of a strictly p -regular curve we have:

$$L = L_p = \left\{ s: \prod_{i=1}^{p-1} k_i(s) \neq 0, k_p(s) = 0 \right\}.$$

Lemma. *Let $\Gamma: x=x(s)$, $s \in L$ be an arbitrary Frenet-curve in $\mathbf{R}^n$. Then it has a dense subset which is the union of a countable number of strictly i -regular curves, where $i=1, 2, \dots, n$.*

PROOF. Let us consider the above mentioned decomposition of the parameter interval L

$$L = \bigcup_{i=1}^n L_i.$$

Then, on account of a simple topological lemma applied by [2] we can state that

$$L = \bigcup_{i=1}^n \bar{L}_i^0$$

also holds, where L_i^0 denotes the interior of L_i . Assuming that $\mathbf{R}$ has the usual topology, the open set L_i^0 can be given as union of a countable family of disjoint open intervals. It is evident that to each of such an open subinterval a strictly i -regular subarc of the given curve Γ will belong since

$$\prod_{i=1}^{i-1} k_i(s) \neq 0 \quad \text{and} \quad k_i(s) = 0$$

hold there identically.

Theorem 9. Let $\Gamma: x=\underline{x}(s), s \in L$ be a Frenet-curve lying uniformly in a p -plane. Then for the p -th pseudo-curvature function of any Frenet-frame adapted to the given curve Γ $k_p(s)=0$ holds on the whole interval L .

PROOF. Since Γ lies in a p -plane $L = \bigcup_{i=1}^p L_i$ holds. On the other hand $L_i^0=0$ has to hold for $i=1, 2, \dots, p-1$ because Γ lies uniformly in a p -plane. Thus, for the parameter interval L is valid $L = \bar{L}_p^0$. Consider now a parameter value $s_0 \in L$ for which $s_0 \notin L_p^0$ holds. Then, due to the continuity of the function $k_p(s)$ $k_p(s_0)=0$ holds, as well, and our theorem is proved.

Notice that a necessary and sufficient condition for a Frenet-curve to lie in a p -plane has not been found yet. The following theorem gives such a condition generally for C^∞ and regular curves in terms of the Bishop-coefficients.

Theorem 10. Let $\Gamma: x=\underline{x}(s), s \in L$ be an arbitrary C^∞ and regular curve in $\mathbf{R}^n$. It lies in a p -plane if and only if there exists a Bishop-frame $E(s), s \in L$ adapted to Γ so that the corresponding Bishop-coefficients satisfy the following conditions: $b_p(s)=b_{p+1}(s)=\dots=b_{n-1}(s)=0$ for each $s \in L$.

PROOF. First suppose that Γ lies in a p -plane, i.e. the vectorfunction $\underline{x}(s)$ satisfies the following equation for each $s \in L$:

$$(\underline{x}(s) - \underline{x}(s_0)) \wedge \underline{u}_1 \wedge \underline{u}_2 \wedge \dots \wedge \underline{u}_p = 0$$

where $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p$ are mutually orthogonal unit vectors spanning the p -plane which goes through a fixed point $\underline{x}(s_0), s_0 \in L$. Let us complete now this system of vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_p$ to a positively oriented orthonormal basis $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ of $\mathbf{R}^n$ and denote by $R \in SO(n)$ the matrix of that orthogonal transformation which carries the above chosen basis into the canonical basis of $\mathbf{R}^n$. Then, it is easy to see that the curve $\tilde{\Gamma}$ given by

$$\tilde{\Gamma}: \underline{x} = R(\underline{x}(s) - \underline{x}(s_0))$$

will lie in the p -dimensional linear subspace of $\mathbf{R}^n$ spanned by the first p vectors of its canonical basis and so it can be identified with a curve $(\tilde{\Gamma})_p$ lying in the space $\mathbf{R}^p$.

Consequently on account of Theorem 3 we can adapt to $\tilde{\Gamma}$ a Bishop-frame $\tilde{E}(s)$, $s \in L$ so that only the first p frame-vectors

$$\tilde{e}_1(s), \tilde{e}_2(s), \dots, \tilde{e}_p(s)$$

may change along the curve $\tilde{\Gamma}$. Let then

$$E(s) = R^* \tilde{E}(s), \quad s \in L$$

be the corresponding Bishop-frame adapted to the curve Γ . Since $E(s)$ and $\tilde{E}(s)$ are congruent moving frames, they have the same Cartan-matrix so that

$$b_p(s) = b_{p+1}(s) = \dots = b_{n-1}(s) = 0$$

hold for each $s \in L$.

Conversely, let us suppose that the given curve Γ has a Bishop-frame $E(s)$, $s \in L$ for the Cartan-matrix of which the considered conditions hold. Then, on account of the Bishop-equations the p -vector

$$\underline{n}_p(s) = e_1(s) \wedge e_2(s) \wedge \dots \wedge e_p(s), \quad s \in L$$

is constant on the parameter interval L . Thus, it is easy to see that

$$(\underline{x}(s) - \underline{x}(s_0)) \wedge \underline{n}_p(s_0) = 0$$

identically holds on L implying that Γ lies in a p -plane.

Remark 1. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a Frenet-curve in $\mathbf{R}^n$ lying in a p -plane. As it was shown in the above proof, there always exists an orientation preserving isometry carrying the given curve Γ into a curve $\tilde{\Gamma}$ which can be identified with a C^∞ and regular curve $(\tilde{\Gamma})_p$ lying in $\mathbf{R}^p$. The curve $(\tilde{\Gamma})_p$, however, may not be a Frenet-curve in $\mathbf{R}^p$. This is the reason of the fact that the proof of the Theorem 10 cannot be modified so as to yield an analogous theorem for Frenet-curves where the conditions would be given in terms of the pseudo-curvatures.

Remark 2. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in T$, be a Frenet-curve in $\mathbf{R}^n$ and denote by $E(s)$, $s \in L$ a Frenet-frame which is adapted to it. The given curve Γ lies in a p -plane if and only if there exists a frame-transformator $A(s)$, $s \in L$ so that in the Cartan-matrix $\tilde{C}(s)$, $s \in L$ belonging to the transformed moving frame we have the following entries:

$$\tilde{c}_{1j}(s) = 0 \quad \text{for } j = p+1, p+2, \dots, n \quad \text{and for } s \in L,$$

$$\tilde{c}_{ij}(s) = 0 \quad \text{for } i, j = 2, 3, \dots, n \quad \text{and for } s \in L.$$

Notice that this condition given now for a Frenet-curve is nothing else then the condition used in Theorem 10.

5. On Curves lying on p -spheres

Definition. Let $\Gamma: \underline{x} = \underline{x}(s), s \in L$ be a C^∞ and regular curve in $\mathbb{R}^n$. It is said that Γ lies on a p -sphere having center-vector $\underline{c}$ and radius r if Γ lies in a $(p+1)$ -plane which contains this p -sphere and

$$\langle \underline{x}(s) - \underline{c}, \underline{x}(s) - \underline{c} \rangle = r^2$$

holds on the parameter interval L . ($1 \leq p < n$).

The following theorem gives a convenient characterization of the spherical curves in terms of the Bishop-coefficients:

Theorem 11. Let $\Gamma: \underline{x} = \underline{x}(s), s \in L$ be a C^∞ and regular curve in $\mathbb{R}^n$. It lies on a p -sphere (of radius r) if and only if there exists a Bishop-frame $E(s), s \in L$ adapted to Γ so that the corresponding Bishop-coefficients satisfy the following conditions:

$$b_p(s) = \frac{1}{r} \quad \text{and} \quad b_{p+1}(s) = b_{p+2}(s) = \dots = b_{n-1}(s) = 0$$

hold for each $s \in L$, where $r > 0$ is the radius of the p -sphere.

PROOF. First suppose that Γ lies on a p -sphere having center-vector $\underline{c}$ and radius r . Then by Theorem 10 there exists a Bishop-frame

$$E(s) = \{\underline{e}_1(s), \underline{e}_2(s), \dots, \underline{e}_n(s)\}, \quad s \in L$$

so that $\underline{e}_i(s)$ is constant for $i > p+1$ and

$$(\underline{x}(s) - \underline{x}(s_0)) \wedge \underline{e}_1(s) \wedge \underline{e}_2(s) \wedge \dots \wedge \underline{e}_{p+1}(s) = 0$$

identically holds on L .

Moreover by Theorem 3 the initial condition

$$\underline{e}_{p+1}(s_0) = \frac{1}{r} (\underline{c} - \underline{x}(s_0)), \quad s_0 \in L$$

can be satisfied at the choice of this Bishop-frame $E(s)$. Using now the fact that the vectors $\underline{c} - \underline{x}(s)$ and $\underline{e}_1(s)$ are everywhere perpendicular to each other, we can write:

$$\underline{c} - \underline{x}(s) = A_1(s)\underline{e}_2(s) + A_2(s)\underline{e}_3(s) + \dots + A_p(s)\underline{e}_{p+1}(s)$$

holds for each $s \in L$, where the coefficient-functions

$$A_i(s) = \langle \underline{c} - \underline{x}(s), \underline{e}_{i+1}(s) \rangle \quad \text{for } i = 1, 2, \dots, p$$

are constant on L since

$$A'_i(s) = \langle -\underline{e}_1(s), \underline{e}_{i+1}(s) \rangle + \langle \underline{c} - \underline{x}(s), -b_i(s)\underline{e}_1(s) \rangle = 0$$

holds for each $s \in L$. Thus

$$\underline{c} - \underline{x}(s) = r \underline{e}_{p+1}(s)$$

identically holds on the whole interval L .

A derivation with respect to the arc-length s already shows that $b_p(s)=1/r$ holds for each $s \in L$, as well.

Conversely, let us suppose that the given curve Γ has a Bishop-frame $E(s)$, $s \in L$ for the Cartan-matrix of which the considered conditions hold. Then the vector-function

$$\underline{c}(s) = \underline{x}(s) + r\underline{e}_{p+1}(s)$$

is constant on L since its derivative is identically zero. So, using simply $\underline{c}$ instead of $\underline{c}(s)$, it is evident that

$$\langle \underline{x}(s) - \underline{c}, \underline{x}(s) - \underline{c} \rangle = r^2$$

holds for each $s \in L$, and consequently the curve Γ lies on a p -sphere having center-vector $\underline{c}$ and radius r .

Remark 1. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a C^∞ and regular curve in $\mathbf{R}^n$ and denote by $b_1(s), b_2(s), \dots, b_{n-1}(s)$ the Bishop-coefficients belonging to a Bishop-frame $E(s)$, $s \in L$ adapted to Γ . Then, according to R. BISHOP [3], a curve defined in $\mathbf{R}^{n-1}$ by the vector-function $\underline{b}(s)$, $s \in L$ whose coordinates are just the Bishop-coefficients $b_i(s)$, $i=1, 2, \dots, n-1$ and $s \in L$, is called a *normal development* of the given curve.

So, the above theorem, involving also Theorem 10 where $1/r=0$, can be given in the following form:

A C^∞ and regular curve Γ lies on a p -sphere of radius r ($1 \leq p < n$) if and only if it has in $\mathbf{R}^{n-1}$ a normal development $\underline{b}(s)$, $s \in L$ which lies in the $(p-1)$ -plane spanned by the first $(p-1)$ element of the canonical basis of $\mathbf{R}^{n-1}$ and going through the point $P(0, 0, \dots, 0, 1/r, 0, \dots, 0)$ where only the p -th coordinate differs from zero for $r < \infty$.

On the other hand, if $\underline{\tilde{b}}(s)$, $s \in L$ is an arbitrary other normal development belonging to the given curve Γ , then on account of Theorem 2 and Theorem 3 we have that

$$\underline{\tilde{b}}(s) = A_0 \underline{b}(s)$$

holds for each $s \in L$, where $A_0 \in SO(n-1)$.

Thus the curve $\underline{\tilde{b}}(s)$, $s \in L$ lies also in a $(p-1)$ -plane which has the distance $1/r$ from the origin of the vector-space $\mathbf{R}^{n-1}$.

So we have obtained the characterization of the spherical curves given in [3]. A proof of the corresponding theorem for the usual 3-dimensional case can be found there.

Remark 2. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a Frenet-curve in $\mathbf{R}^n$ and denote by $E(s)$, $s \in L$ a Frenet-frame which is adapted to it. The given curve Γ lies on a p -sphere of radius r if and only if there exists a frame transformer $A(s)$, $s \in L$ so that in the Cartan-matrix $\tilde{C}(s)$, $s \in L$ belonging to the transformed moving frame we have the following entries:

$$\tilde{c}_{1j}(s) = \begin{cases} \frac{1}{r} & \text{for } j = p+1, \\ 0 & \text{for } j = p+2, \dots, n, \end{cases} \quad s \in L;$$

$$\tilde{c}_{ij}(s) = 0 \quad \text{for } i, j = 2, 3, \dots, n, \quad s \in L.$$

Notice that this condition given now for a Frenet-curve is nothing else then the condition used in Theorem 11.

The above formulae are closely related to the ones given in [2] for the usual case where $n=3$ and $p=2$. In fact, let $k_1(s)$ and $k_2(s)$, $s \in L$ be pseudo-curvatures belonging to a Frenet-curve Γ in $\mathbb{R}^3$. Then, on account of Theorem 2, the matrix $\tilde{C}(s)$, $s \in L$ has the following independent entries:

$$\tilde{c}_{12}(s) = k_1(s) \cos \varphi(s)$$

$$\tilde{c}_{13}(s) = -k_1(s) \sin \varphi(s)$$

$$\tilde{c}_{23}(s) = k_2(s) + \varphi'(s),$$

where $\varphi(s)$, $s \in L$ is a C^∞ scalar-function to be determined in the frame-transformator $A(s)$, $s \in L$. Thus, the Frenet-curve Γ lies on a sphere of radius r if and only if there exists a C^∞ scalar-function $\varphi(s)$, $s \in L$ so that

$$-k_1(s) \sin \varphi(s) = \frac{1}{r} \quad \text{and} \quad k_2(s) + \varphi'(s) = 0 \quad \text{hold on } L.$$

Finally, we can easily obtain the following consequence by eliminating $\varphi(s)$ from the above two equations: If the Frenet-curve Γ lies on a sphere having radius $r < \infty$ then for the pseudo-curvatures $k_1(s)$ and $k_2(s)$, $s \in L$ we have that

$$(k_1'(s))^2 = (k_1(s)k_2(s))^2((rk_1(s))^2 - 1)$$

identically holds on the parameter interval L .

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