

Topogenous g -mappings

By KÁLMÁN MATOLCSY (Debrecen)

0. Introduction

This paper presents a generalization of the notion of an E -mapping, which was introduced by M. HACQUE in [3] and [4].

Considering a set E , an E -mapping is a mapping $\mathfrak{z}$ of the power set 2^E into the set of all systems of subsets of E with the following properties for $A, B, X, Y \subset E$:

(E0) $\mathfrak{z}(A) \neq \emptyset$, and $Y \supset X \in \mathfrak{z}(A)$ implies $Y \in \mathfrak{z}(A)$.

(E1) $\mathfrak{z}(\emptyset) = 2^E$.

(E2) $X \in \mathfrak{z}(A)$ implies $A \subset X$.

(E3) $A \subset B$ implies $\mathfrak{z}(A) \supset \mathfrak{z}(B)$.

In [2] S. GACSÁLYI proved that the correspondence

$$(0.1) \quad < \rightarrow \mathfrak{z}_< : \mathfrak{z}_<(A) = \{X : A < X\}$$

defined between semi-topogeneous orders on E and E -mappings is one-to-one, and its inverse can be produced as follows:

$$(0.2) \quad \mathfrak{z} \rightarrow <_{\mathfrak{z}} : A <_{\mathfrak{z}} B \text{ iff } B \in \mathfrak{z}(A).$$

These concepts can be improved by a simple idea. Let g be a (single-valued) mapping of a set E onto a dense subset of the syntopogenous space $[E', \mathcal{S}']$; we frequently meet this situation in theory of extensions of syntopogenous spaces. If $<' \in \mathcal{S}'$, and as a value of an arbitrary $A' \subset E'$ we prescribe the system of those subsets X of E , which include the inverse image $g^{-1}(X')$ of a set $X' \subset E'$ such that $A' <' X'$, we get a mapping of 2^E into the set of all systems of subsets of E with properties similar to (E0)—(E3).

Starting out from the observation mentioned above, in chapter 1 of the paper the notion of a g -mapping will be defined for a fixed mapping $g: E \rightarrow E'$. In part 2 a correspondence will be produced between semi-topogeneous orders on E' and g -mappings like in (0.1), but in general this will be not one-to-one. The main result of this paragraph is a generalization of Gacsályi's perception, namely the corre-

spondence in question is an injection (a surjection), iff the mapping g is a surjection (an injection), consequently it is one-to-one, iff g is of this kind, too.

The special families of topogeneous g -mappings (so called *syntopogenous g -families*) and their application in the theory of extensions of syntopogenous spaces will be found in another notice of the author [5].

1. The notion of a g -mapping

Let us consider two sets E, E' and a single valued mapping $g: E \rightarrow E'$ (E, E' and g will be fixed throughout the whole paper). By a g -mapping we shall mean a mapping $\mathfrak{z}$ defined on the power set $2^{E'}$ into the set of all systems of subsets of E , which satisfies the following conditions for $X, Y \subset E$ and $A', B' \subset E'$:

(M0) $\mathfrak{z}(A') \neq \emptyset$, and $Y \supset X \in \mathfrak{z}(A')$ implies $Y \in \mathfrak{z}(A')$.

(M1) $\emptyset \in \mathfrak{z}(A')$ iff $A' = \emptyset$.

(M2) $X \in \mathfrak{z}(A')$ implies $g^{-1}(A') \subset X$.

(M3) $A' \subset B'$ implies $\mathfrak{z}(A') \supset \mathfrak{z}(B')$.

(1.1) **Proposition.** If $E = E'$ and g is the identity of E , then the g -mappings are identical with the E -mappings.

PROOF. Under the conditions of the proposition the equivalences (M0) $\Leftrightarrow$ (E0), (M2) $\Leftrightarrow$ (E2), (M3) $\Leftrightarrow$ (E3) and the implication (M0) $\cap$ (M1) $\Rightarrow$ (E1) are obvious. For the verification of (E1) $\cap$ (E2) $\Rightarrow$ (M1) let us observe that by (E2) $\emptyset \in \mathfrak{z}(A)$ implies $A = g^{-1}(A) \subset \emptyset$, thus $A = \emptyset$. ■

(1.2) *Remark.* In (M1) we had to postulate that $\emptyset \in \mathfrak{z}(A')$ implies $A' = \emptyset$, because in general for a set $A' \subset E' - g(E)$ the assumption $\emptyset \in \mathfrak{z}(A')$ and (M2) does not imply $A' = \emptyset$. Leaving this out of consideration we should have got a generalization of the notion of an E -mapping infitted for the applications in extension theory.

A g -mapping $\mathfrak{z}$ will be said to be *topogenous*, if for any $X, Y \subset E$ and $A', B' \subset E'$

(MQ') $X \in \mathfrak{z}(A'), Y \in \mathfrak{z}(B')$ implies $X \cap Y \in \mathfrak{z}(A' \cap B')$

and

(MQ'') $X \in \mathfrak{z}(A'), Y \in \mathfrak{z}(B')$ implies $X \cup Y \in \mathfrak{z}(A' \cup B')$

(cf. ch. 3 of [1]). The topogenous g -mappings have a simple lattice theoretical characterization, in which, as in general topology it is usual, by a (filter) ideal in E we mean a (proper filter) ideal of the lattice of all subsets of E .

(1.3) **Lemma.** A g -mapping $\mathfrak{z}$ is topogenous iff

(1.3.1) $\mathfrak{z}(A')$ is a filter in E for any $\emptyset \neq A' \subset E'$,

and

(1.3.2) the system $\{A' \subset E': X \in \mathfrak{z}(A')\}$ is an ideal in E' for every $X \subset E$.

PROOF. Let $\mathfrak{z}$ be topogenous. By (M1) $\emptyset \notin \mathfrak{z}(A')$, and because of (M0) $\mathfrak{z}(A')$ is increasing in E . If $X, Y \in \mathfrak{z}(A')$, then $X \cap Y \in \mathfrak{z}(A' \cap A') = \mathfrak{z}(A')$. Similarly, by (M3) the system in question is decreasing in E' for every $X \subset E$. If $X \in \mathfrak{z}(A')$ and $X \in \mathfrak{z}(B')$, then $X = X \cup X \in \mathfrak{z}(A' \cup B')$.

Conversely, if (1.3.1) is satisfied by $\mathfrak{z}$, then $X \in \mathfrak{z}(A')$, $Y \in \mathfrak{z}(B')$ together with (M3) imply $X, Y \in \mathfrak{z}(A' \cap B')$, and from this $X \cap Y \in \mathfrak{z}(A' \cap B')$ follows. If (1.3.2) holds, then because of (M0) $X \in \mathfrak{z}(A')$, $Y \in \mathfrak{z}(B')$ gives $X \cup Y \in \mathfrak{z}(A' \cap \mathfrak{z}(B'))$, thus from our condition $X \cup Y \in \mathfrak{z}(A' \cup B')$. ■

Let $\mathfrak{z}_1, \mathfrak{z}_2$ be two g -mappings. We shall say that $\mathfrak{z}_1$ is *coarser* than $\mathfrak{z}_2$, or equivalently $\mathfrak{z}_2$ is *finer* than $\mathfrak{z}_1$, iff for every $A' \subset E'$ the inequality $\mathfrak{z}_1(A') \subset \mathfrak{z}_2(A')$ holds. This will be denoted by $\mathfrak{z}_1 \subset \mathfrak{z}_2$.

(1.4) *Remarks.* Let $\mathfrak{z}$ be a g -mapping. If there exists a topogenous g -mapping finer than $\mathfrak{z}$, then $\mathfrak{z}(A')$ is centred for any $A' \neq \emptyset$. Using the simple notation $\mathfrak{z}(x')$ instead of $\mathfrak{z}(\{x'\})$ for points x' of E' , this condition is satisfied by $\mathfrak{z}$ iff $\mathfrak{z}(x')$ is centred for every $x' \in E'$. In this case there exists a topogenous g -mapping $\mathfrak{z}^q$, which is the coarsest of all topogenous g -mappings finer than $\mathfrak{z}$. $\mathfrak{z}^q$ can be defined as follows: (1.4.1) $X \in \mathfrak{z}^q(A')$ iff there are natural numbers m and n such

that $A' = \bigcup_{i=1}^m A'_i$, $X = \bigcap_{j=1}^n X_j$ and $X_j \in \mathfrak{z}(A'_i)$ for any $1 \leq i \leq m$, $1 \leq j \leq n$.

(Cf. [1], (3.7).) These facts have no importance from the point of view of our further studies, therefore their proof will be omitted.

A g -mapping $\mathfrak{z}$ will be called *perfect*, if for an arbitrary set I of indices

$$(MP) \quad X_i \in \mathfrak{z}(A'_i) \quad (i \in I) \quad \text{implies} \quad \bigcup_{i \in I} X_i \in \mathfrak{z}\left(\bigcup_{i \in I} A'_i\right)$$

(cf. ch. 4 of [1]). We shall prove that the structure of a perfect g -mapping is uniquely determined by its values taken on the subsets of E' having one element.

(1.5) **Theorem.** A g -mapping $\mathfrak{z}$ is perfect iff

$$(1.5.1) \quad \mathfrak{z}(A') = \bigcap \{ \mathfrak{z}(x') : x' \in A' \}$$

holds for any $A' \neq \emptyset$. If $\mathfrak{z}$ is an arbitrary g -mapping, then there exists a perfect g -mapping $\mathfrak{z}^p$, which is the coarsest of all perfect g -mappings finer than $\mathfrak{z}$. For $A' \neq \emptyset$, $\mathfrak{z}^p(A')$ can be defined by

$$(1.5.2) \quad \mathfrak{z}^p(A') = \bigcap \{ \mathfrak{z}(x') : x' \in A' \}$$

and $\mathfrak{z}^p(\emptyset) = 2^E$.

PROOF. For an arbitrary g -mapping $\mathfrak{z}$ the inequality

$$\mathfrak{z}(A') \subset \bigcap \{ \mathfrak{z}(x') : x' \in A' \}$$

is obvious for $\emptyset \neq A' \subset E'$ (see (M3)). Let $\mathfrak{z}$ be perfect, and $X \in \mathfrak{z}(x')$ for any $x' \in A'$. In this case because of $A' = \bigcup_{x' \in A'} \{x'\}$ we have $X \in \mathfrak{z}(A')$. Conversely, suppose

(1.5.1) and $X_i \in \mathfrak{z}(A'_i) \quad (i \in I)$, $X = \bigcup_{i \in I} X_i$, $A' = \bigcup_{i \in I} A'_i$. Then $x' \in A'$ implies $x' \in A'_i$

for some $i \in I$, form this $X \supset X_i \in \mathfrak{z}(x')$. Consequently $X \in \mathfrak{z}(x')$ for every $x' \in A'$, thus by (1.5.1) $X \in \mathfrak{z}(A')$.

It is easy to prove that $\mathfrak{z}^p$ is a g -mapping for any g -mapping $\mathfrak{z}$, and it is perfect, since $\mathfrak{z}^p(x') = \mathfrak{z}(x')$ for any $x' \in E'$, hence $\mathfrak{z}^p$ satisfies (1.5.1). Finally if $\mathfrak{z}_1$ is a perfect g -mapping finer than $\mathfrak{z}$, then, for any $x' \in E'$, $\mathfrak{z}(x') \subset \mathfrak{z}_1(x')$, consequently by (1.5.1) $\mathfrak{z}^p(A') \subset \mathfrak{z}_1(A')$ for every $\emptyset \neq A' \subset E'$. ■

(1.6) **Theorem.** *A mapping $\mathfrak{z}$ of $2^{E'}$ into the set of all systems of subsets of E is a perfect topogenous g -mapping iff for every $x' \in E'$ there exists a filter $\mathfrak{f}(x')$ in E such that*

$$(1.6.1) \quad x \in E, X \in \mathfrak{f}(g(x)) \text{ implies } x \in X,$$

further

$$(1.6.2) \quad \mathfrak{z}(\emptyset) = 2^E \text{ and } \mathfrak{z}(A') = \bigcap \{\mathfrak{f}(x') : x' \in A'\} \text{ for any } \emptyset \neq A' \subset E'.$$

PROOF. If $\mathfrak{z}$ is a perfect topogenous g -mapping, then $\mathfrak{f}(x') = \mathfrak{z}(x')$ satisfies these conditions for any $x' \in E'$ (see (1.3) and (1.5)). Conversely, let $\mathfrak{f}(x')$ be a filter in E for $x' \in E'$ with conditions (1.6.1)–(1.6.2). $\emptyset \neq \mathfrak{z}(A')$ for any $A' \subset E'$, since $E \in \mathfrak{z}(A')$. Every $\mathfrak{f}(x')$ is increasing in E , therefore $Y \supset X \in \mathfrak{z}(A')$ implies $Y \in \mathfrak{z}(A')$. The fulfilment of (M1) follows from $\emptyset \notin \mathfrak{f}(x')$ ($x' \in E'$), and from (1.6.2). $X \in \mathfrak{z}(A')$, $x \in g^{-1}(A')$ implies $X \in \mathfrak{f}(g(x))$, thus $x \in X$. Finally if $A' \subset B'$, then

$$\bigcap \{\mathfrak{f}(x') : x' \in B'\} \subset \bigcap \{\mathfrak{f}(x') : x' \in A'\}$$

is trivial. We got that $\mathfrak{z}$ is a g -mapping. $\mathfrak{z}(x') = \mathfrak{f}(x')$ and (1.5.1) give that $\mathfrak{z}$ is perfect, consequently it satisfies axiom (MQ'), too. Let $X \in \mathfrak{z}(A')$ and $Y \in \mathfrak{z}(B')$. If $x' \in A' \cap B'$, then $X, Y \in \mathfrak{f}(x')$, and since $\mathfrak{f}(x')$ is a filter, we get $X \cap Y \in \mathfrak{f}(x')$, accordingly $X \cap Y \in \mathfrak{z}(A' \cap B')$. ■

(1.7) **Corollary.** *If $\mathfrak{z}$ is a topogenous g -mapping, then $\mathfrak{z}^p$ is perfect and topogenous.* ■

2. g -mappings and semi-topogenous orders

Let $<$ be a semi-topogenous order on the set X [1]. A subset X_0 of X is called $<$ -dense, if $x < V \subset X$ implies $X_0 \cap V \neq \emptyset$ for any $x \in X$ (or equivalently, if $x \in X$, then $x < X - X_0$ does not hold).

We shall prove that a g -mapping $\mathfrak{z}_{<}$ can be assigned to any semi-topogenous order $<$ on E' , provided $g(E)$ is $<$ -dense.

(2.1) **Proposition.** *If $<$ is a semi-topogenous order on E' such that $g(E)$ is $<$ -dense, then the definition*

$$(2.1.1) \quad \mathfrak{z}_{<}(A') = \{X \subset E : A' < E' - g(E - X)\} \quad (A' \subset E') \text{ yields a } g\text{-mapping } \mathfrak{z}_{<},$$

which will be called the g -mapping deduced from $<$.

Before the verification of (2.1) we give a lemma explaining (2.1.1):

(2.2) **Lemma.** *Under the conditions of (2.1) $X \in \mathfrak{z}_{<}(A')$ iff there exists a set $X' \subset E'$ such that $A' < X'$ and $g^{-1}(X') \subset X$.*

PROOF. In fact, if there is such a set X' , then $E - X \subset E - g^{-1}(X')$, thus $g(E - X) \subset g(E - g^{-1}(X')) \subset E' - X'$, therefore $A' \prec' X' \subset E' - g(E - X)$. Conversely, if $X \in \mathfrak{z}_{<'}(A')$, then putting $X' = E' - g(E - X)$, we have $A' \prec' X'$ and $g^{-1}(X') = E - g^{-1}(g(E - X)) \subset X$. ■

PROOF OF (2.1). Let us consider the system of axioms defining a semi-topogenous order (see [1], (0₁)–(0₃)), and prove the validity of (M0)–(M3).

(M0): $A' \prec' E' - g(E - E)$, thus $E \in \mathfrak{z}_{<'}(A')$. If $X \subset Y$ and $X \in \mathfrak{z}_{<'}(A')$, then by (2.2) $Y \in \mathfrak{z}_{<'}(A')$.

(M1): $\emptyset \prec' \emptyset$ implies $\emptyset \in \mathfrak{z}_{<'}(\emptyset)$. If $\emptyset \in \mathfrak{z}_{<'}(A')$ and $x' \in A'$, then $x' \prec' E' - g(E)$ (see (0₃)), but this contradicts the density of $g(E)$.

(M2) If $X \in \mathfrak{z}_{<'}(A')$ and X' is a set required by (2.2), then because of (0₂) we have $g^{-1}(A') \subset g^{-1}(X') \subset X$.

(M3): If $A' \subset B'$ and $X \in \mathfrak{z}_{<'}(B')$, then $B' \prec' E' - g(E - X)$, thus (0₃) and (2.1.1) imply $X \in \mathfrak{z}_{<'}(A')$. ■

We show that the notion of a topogenous or perfect g -mapping introduced in part I very exactly corresponds to the concept of a topogenous or perfect semi-topogenous order [1].

(2.3) **Proposition.** Let $\prec'$ be a semi-topogenous order on E' , and $g(E)$ be $\prec'$ -dense.

(2.3.1) If $\prec'$ is topogenous, then $\mathfrak{z}_{<}'$ is topogenous, too.

(2.3.2) $\mathfrak{z}_{<}'^p = \mathfrak{z}_{<}'^p$ consequently if $\prec'$ is perfect, then $\mathfrak{z}_{<}'$ is also of this kind.

PROOF. (2.3.1): If $X \in \mathfrak{z}_{<'}(A')$ and $Y \in \mathfrak{z}_{<'}(B')$, then $A' \prec' E' - g(E - X)$ and $B' \prec' E' - g(E - Y)$, consequently $A' \cap B' \prec' (E' - g(E - X)) \cap (E' - g(E - Y)) = E' - g(E - (X \cup Y))$, and $A' \cup B' \prec' (E' - g(E - X)) \cup (E' - g(E - Y)) \subset E' - g(E - (X \cap Y))$. In view of these we get $X \cap Y \in \mathfrak{z}_{<'}(A' \cap B')$ and similary $X \cup Y \in \mathfrak{z}_{<'}(A' \cup B')$.

(2.3.2): $g(E)$ is obviously $\prec'^p$ -dense, and $X \in \mathfrak{z}_{<}'^p(A') \Leftrightarrow x' \prec' E' - g(E - X)$ for any $x' \in A' \Leftrightarrow A' \prec'^p E' - g(E - X) \Leftrightarrow X \in \mathfrak{z}_{<}'^p(A')$ (see [1], (4.7)).

If $\prec'$ is perfect, then $\prec'^p = \prec'$, therefore $\mathfrak{z}_{<}'^p = \mathfrak{z}_{<}'$. This means that $\mathfrak{z}_{<}'$ is perfect (see (1.5)). ■

It is easy to show that if $E = E'$ and g is the identity of E , then by the definition (2.1.1) the correspondence described in (0.1) is obtained. After this we shall study the effect of the mapping g having on the properties of the correspondence $\prec' \rightarrow \mathfrak{z}_{<}'$. The connection existing between the behaviour of $\prec' \rightarrow \mathfrak{z}_{<}'$ and g can be formulated in the following general duality theorem:

(2.4) **Theorem.** If E' consists of at least two elements, then

(2.4.1) $\prec' \rightarrow \mathfrak{z}_{<}'$ is surjective iff g is injective;

(2.4.2) $\prec' \rightarrow \mathfrak{z}_{<}'$ is injective iff g is surjective;

(2.4.3) $\prec' \rightarrow \mathfrak{z}_{<}'$ is one-to-one iff g is of this kind.

The case of a set E' containing one element x' is trivial, namely the unique semi-topogenous order on E' is $<' = <_{0, E'}$ and the unique g -mapping is $\mathfrak{z}_{<'}$.

The proof of theorem (2.4) can be read from (2.5), (2.7), (2.12) and (2.13), finally (2.4.1) and (2.4.2) imply (2.4.3).

(2.5) *Examples.* Let E' be a set consisting of at least two elements, and suppose that g is not an injection. We shall study two different cases.

(2.5.1) g is not a surjection.

Put $\mathfrak{f}(\xi(\nu)) = \{X \subset E: g^{-1}(g(x)) \subset X\}$ for any $x \in E$, and $\mathfrak{f}(x') = \{X \subset E: x_0 \in X\}$ for every $x' \in E' - g(E)$, where x_0 is a fixed element of E such that $g^{-1}(g(x_0))$ contains at least two elements. The filters $\mathfrak{f}(x')$ ($x' \in E'$) satisfy condition (1.6.1), therefore we have a perfect topogenous g -mapping $\mathfrak{z}$ defined by (1.6.1), but this g -mapping cannot be deduced from some semi-topogenous order $<'$ on E' such that $g(E)$ is $<'$ -dense.

(In fact, assume that $<'$ is such an order. Then $\{x_0\} \in \mathfrak{z}(x')$ for an $x' \in E' - g(E)$. Because of the choice of x_0 this means $x' <' E' - g(E - x_0) = E' - g(E)$, which is impossible, since $g(E)$ is $<'$ -dense.

(2.5.2) g is a surjection.

Suppose that x_0 and y_0 are fixed elements of E such that $g^{-1}(g(x_0))$ has at least two members, and $g(x_0) \neq g(y_0)$. Put $\mathfrak{f}(g(x)) = \{X \subset E: g^{-1}(g(x)) \subset X\}$ for $y_0 \neq x \in E$, and $\mathfrak{f}(g(y_0)) = \{X \subset E: V \subset X\}$, where $V = g^{-1}(g(y_0)) \cup \{x_0\}$. Since the filters $\mathfrak{f}(x')$ ($x' \in E'$) have property (1.6.1), we can define a perfect topogenous g -mapping $\mathfrak{z}$ in accordance with (1.6.2). $\mathfrak{z}$ cannot be deduced from some semi-topogenous order, that is there does not exist $<'$ on E' such that $\mathfrak{z} = \mathfrak{z}_{<'}$ (in this case $g(E) = E'$ is obviously $<'$ -dense). Indeed, $V \in \mathfrak{z}(g(y_0))$, consequently there is a set X' such that $g(y_0) <' X'$ and $g^{-1}(X') \subset V$. At the same time $g^{-1}(X') \in \mathfrak{z}(g(y_0)) = \mathfrak{f}(g(y_0))$, therefore $V \subset g^{-1}(X')$. From these $g^{-1}(X') = V$, but because of the properties of x_0 this is an impossibility. ■

Further we shall consider a semi-topogenous order on E' for any g -mapping $\mathfrak{z}$, from which $\mathfrak{z}$ can be deduced, provided g is an injection.

(2.6) **Proposition.** Let $\mathfrak{z}$ be a g -mapping. Then a semi-topogenous order $<_{\mathfrak{z}}$ can be defined on E' by the following formula:

$$(2.6.1) \quad A' <_{\mathfrak{z}} B' \Leftrightarrow A' \subset B' \text{ and } g^{-1}(B') \in \mathfrak{z}(A').$$

PROOF. $\emptyset <_{\mathfrak{z}} \emptyset$ and $E' <_{\mathfrak{z}} E'$ obviously hold. By the definition $A' <_{\mathfrak{z}} B'$ implies $A' \subset B'$. Finally let us suppose $A' \subset A'_1 <_{\mathfrak{z}} B'_1 \subset B'$. Then $A' \subset A'_1 \subset B'_1 \subset B'$ and $g^{-1}(B') \supset g^{-1}(B'_1) \in \mathfrak{z}(A'_1) \subset \mathfrak{z}(A')$ (see (M3)), hence $A' \subset B'$ and $g^{-1}(B') \in \mathfrak{z}(A')$ (cf. (M0)), that is $A' <_{\mathfrak{z}} B'$. ■

(2.7) **Theorem.** If $\mathfrak{z}$ is an arbitrary g -mapping, then $<_{\mathfrak{z}}$ is the finest of all semi-topogenous orders $<'$ on E' such that $g(E)$ is $<'$ -dense and $\mathfrak{z}_{<} \subset \mathfrak{z}$. In particular, if g is an injection, then $\mathfrak{z}_{<_{\mathfrak{z}}} = \mathfrak{z}$, i.e. $\mathfrak{z}$ can be deduced from $<_{\mathfrak{z}}$.

PROOF. If $x' \in E'$ and $x' <_{\mathfrak{z}} B'$, then $g^{-1}(B') \in \mathfrak{z}(x')$, hence by (M1) $\emptyset \neq g(g^{-1}(B')) = g(E) \cap B'$. This shows that $g(E)$ is $<_{\mathfrak{z}}$ -dense. If $A' <_{\mathfrak{z}} X'$ and

$g^{-1}(X') \subset X$, then we have $g^{-1}(X') \in \mathfrak{z}(A')$, and from (M0) $X \in \mathfrak{z}(A')$ follows that is $\mathfrak{z} \prec_{t_3} \mathfrak{z}$. Let $\prec'$ be a semi-topogenous order on E' such that $g(E)$ is $\prec'$ -dense, and $\mathfrak{z} \prec' \mathfrak{z}$. If $A' \prec' B'$, then $A' \subset B'$ and $g^{-1}(B') \in \mathfrak{z}_{\prec'}(A') \subset \mathfrak{z}(A')$, therefore $A' \prec_{t_3} B'$, so that $\prec' \subset \prec_{t_3}$. Finally assume that g is an injection, and $X \in \mathfrak{z}(A')$. Then for $X' = A' \cup g(X)$ we have $A' \subset X'$ and $g^{-1}(X') = X$, since $g^{-1}(A') \subset X$. This gives $A' \prec_{t_3} X'$, thus $X \in \mathfrak{z}_{\prec_{t_3}}(A')$. ■

(2.8) **Proposition.** Let $\mathfrak{z}$ be a g -mapping.

(2.8.1) If $\mathfrak{z}$ is topogenous, then $\prec_{t_3}$ is also topogenous.

(2.8.2) $\prec_{t_3}^p = \prec_{t_3^p}$, in particular $\prec_{t_3}$ is perfect for every perfect $\mathfrak{z}$.

PROOF. (2.8.1): Suppose $A' \prec_{t_3} B'$ and $C' \prec_{t_3} D'$. Then $A' \subset B'$, $C' \subset D'$, $g^{-1}(B') \in \mathfrak{z}(A')$ and $g^{-1}(D') \in \mathfrak{z}(C')$. Obviously $A' \cap C' \subset B' \cap D'$, $A' \cup C' \subset B' \cup D'$, and from the topogeneity of $\mathfrak{z}$ the relations $g^{-1}(B' \cap D') = g^{-1}(B') \cap g^{-1}(D') \in \mathfrak{z}(A' \cap C')$, $g^{-1}(B' \cup D') = g^{-1}(B') \cup g^{-1}(D') \in \mathfrak{z}(A' \cup C')$ follow. Thus we have $A' \cap C' \prec_{t_3} B' \cap D'$, $A' \cup C' \prec_{t_3} B' \cup D'$, consequently $\prec_{t_3}$ is topogenous.

(2.8.2): $A' \prec_{t_3}^p B' \Leftrightarrow x' \prec_{t_3} B'$ for any $x' \in A'$ (see [1], (4.7)) $\Leftrightarrow x' \in B'$ and $g^{-1}(B') \in \mathfrak{z}(x')$ for every $x' \in A' \Leftrightarrow A' \subset B'$ and $g^{-1}(B') \in \mathfrak{z}^p(A') \Leftrightarrow A' \prec_{t_3^p} B'$. Therefore if $\mathfrak{z}$ is perfect, then by (1.5) $\mathfrak{z} = \mathfrak{z}^p$, and thus $\prec_{t_3}^p = \prec_{t_3}$. In view of [1], (4.9) this means that $\prec_{t_3}$ is perfect. ■

The semi-topogenous order $\prec_{t_3}$ was defined on E' . In another way one can determine a semi-topogenous order $\prec_{i_3}$ on E for an arbitrary g -mapping $\mathfrak{z}$. This order will be used for the examination of the case of a surjective g .

(2.9) **Proposition.** If $\mathfrak{z}$ is a g -mapping, we have a semi-topogenous order $\prec_{i_3}$ on E given by the following definition:

(2.9.1) $A \prec_{i_3} B \Leftrightarrow B \in \mathfrak{z}(\xi(A))$.

PROOF. $\emptyset \prec_{i_3} \emptyset$ and $E \prec_{i_3} E$ are obvious. If $A \prec_{i_3} B$, then $B \in \mathfrak{z}(\xi(A))$, therefore by (M2) $A \subset g^{-1}(g(A)) \subset B$. Finally suppose $A \subset A_1 \prec_{i_3} B_1 \subset B$. Then $B \supset B_1 \in \mathfrak{z}(\xi(A_1))$, $\xi(A) \subset \xi(A_1)$, consequently because of (M0) and (M3) $B \in \mathfrak{z}(\xi(A))$, that is $A \prec_{i_3} B$ holds. ■

Let us observe that if g is the identity of $E = E'$, then $\prec_{i_3} = \prec_{t_3}$ for any g -mapping $\mathfrak{z}$, and the correspondence $\mathfrak{z} \rightarrow \prec_{i_3}$ is identical with (0.2).

(2.10) **Theorem.** Let $\prec'$ be a semi-topogenous order on E' , and $g(E)$ be $\prec'$ -dense. Then we have $g^{-1}(\prec') = \prec_{i_3 \prec'}$.

PROOF. In fact, $A g^{-1}(\prec') B \Leftrightarrow \xi(A) \prec' E' - \xi(E - B)$ (see [1], (5.1)) $\Leftrightarrow B \in \mathfrak{z}_{\prec'}(g(A)) \Leftrightarrow A \prec_{i_3 \prec'} B$. ■

(2.11) **Proposition.** If $\mathfrak{z}$ is a g -mapping, then

(2.11.1) $\prec_{i_3}$ is topogenous, if $\mathfrak{z}$ is one.

(2.11.2) $\prec_{i_3}^p = \prec_{i_3^p}$, therefore if $\mathfrak{z}$ is perfect, then $\prec_{i_3}$ is perfect, too.

PROOF. (2.11.1): Supposing $A \prec_{i_3} B$ and $C \prec_{i_3} D$, we get $B \in \mathfrak{z}(\xi(A))$ and $D \in \mathfrak{z}(\xi(C))$. $\xi(A \cap C) \subset \xi(A) \cap \xi(C)$ and $g(A \cup C) = g(A) \cup g(C)$, thus $B \cap D \in$

$\in \mathfrak{z}(g(A \cap C))$ (cf. (M3)), and $B \cup D \in \mathfrak{z}(g(A \cup C))$. This means that $A \cap C <_{i_3} B \cap D$ and $A \cup C <_{i_3} B \cup D$.

(2.11.2): $A <_{i_3}^p B \Leftrightarrow x <_{i_3} B$ for every $x \in A \Leftrightarrow B \in \mathfrak{z}(g(x))$ for any $x \in A \Leftrightarrow B \in \mathfrak{z}^p(g(A)) \Leftrightarrow A <_{i_3}^p B$. If $\mathfrak{z}$ is perfect, then $\mathfrak{z} = \mathfrak{z}^p$, consequently $<_{i_3}^p = <_{i_3}$ implies that $<_{i_3}$ is perfect. ■

(2.12) **Theorem.** Let g be a surjection and $\mathfrak{z}$ be a g -mapping. If there exists a semi-topogenous order $<'$ on E' such that $\mathfrak{z} = \mathfrak{z}_{<'}$, then $<' = g(<_{i_3})$.

PROOF. If $\mathfrak{z} = \mathfrak{z}_{<'}$, then $<_{i_3} = g^{-1}(<')$, in accordance with (2.10). By [1], (6.36) this implies

$$<' = g(g^{-1}(<')) = g(<_{i_3}). \quad \blacksquare$$

(2.13) *Example.* Suppose that E' is a set having at least two elements, and g is not a surjection. Put $<'_1 = <_{\emptyset, E'}$, and let $<'_2$ be the biperfect topogenous order generated by the system $\mathfrak{S} = \{\emptyset, g(E), E'\}$ (see [1], (2.1)). Then $<'_1 \neq <'_2$, $g(E)$ is $<'_1$ - and $<'_2$ -dense, and obviously $\mathfrak{z}_{<'_1} = \mathfrak{z}_{<'_2}$. This gives that in this situation the correspondence $<' \rightarrow \mathfrak{z}_{<'}$ cannot be injective. ■

References

- [1] Á. CSÁSZÁR, Foundations of General Topology, Oxford—London—New York—Paris, 1963.
- [2] S. GACSÁLYI, On Hacque's E -mappings and on semi-topogenous orders, *Publ. Math. (Debrecen)* **12** (1965), 265—270.
- [3] M. HACQUE, Sur les E -structures, *C. R. Acad. Sci. Paris* **254** (1962), 1905—1907 and 2120—2122.
- [4] M. HACQUE, Étude des E -structures, *Séminaire Choquet (Initiation à l'Analyse)* 1re année, 1962, no. 6 (Mimeographed).
- [5] K. MATOLCSY, Syntopogenous g -families and their application in extension theory, *Publ. Math. (Debrecen)* **31** (in print).

(Received August 7, 1980)