Full cubes in the Fibonacci sequence

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1. Introduction

Let A, B, G_0, G_1 be integers and $G_n = AG_{n-1} + BG_{n-2}$ for $n \geqq 2$. The sequence of integers G_n is a binary recurrence sequence.

If G_n is non-degenerate and w is an integer, SHOREY and STEWART [11] proved the existence of an effectively computable constant c_1 depending only on A, B, G_0, G_1 and w such that any integer solutions $n, |x| > 1, q \geqq 2$ of the Diophantine equation

$$G_n = wx^q \tag{1}$$

satisfy $\max\{n, |x|, q\} < c_1$.

Let S be the set of non-zero integers composed solely of a finite number of primes. If $(A, B) = 1$ and G_n is non-degenerate, we proved in [9] independently from [11] the existence of an effectively computable constant c_2 depending only on A, B, G_0, G_1 and S such that any integer solutions $n, |x| > 1, q > 2$ and $w \in S$ of (1) satisfy $\max\{n, |x|, q, |w|\} < c_2$.

A special but important binary recurrence sequence is the Fibonacci sequence. It is defined by $G_0 = 0, A = B = G_1 = 1$.

WYLIE [14] and COHN [1], [2] have established, applying an elementary method, all terms of the Fibonacci sequence $\{F_n\}$, which are full square and which are twice a full square. Cohn applied this result to solve completely some Diophantine equations.

LONDON and FINKELSTEIN [5], [6] established all full cube Fibonacci numbers. They reduced this problem to solving the equations $y^2 - 100 = x^3$ and $y^2 + 100 = x^3$ with $x \equiv y \equiv 0 \pmod 5$, $x > 0, y > 0$ and $x/5$ a square. To solve these equations they used results of algebraic number theory.

The main purpose of this paper is to give a new proof of the theorem of London and Finkelstein, applying Baker's method and computer investigation. Our method is applicable also in determining for small p all full p-th power in the Fibonacci sequence, having enough computer time. We give also the solutions of $F_n = qx^3$ and $F_n = q^2x^3$ for infinitely many primes q.

Theorem 1. *The Diophantine equation*

$$F_n = x^3 \tag{2}$$

has only four solutions: $n = x = 0$, $n = x = 1$, $n = 2$, $x = 1$ *and* $n = 6$, $x = 2$.

If p is a prime, let $r(p)$ denote the smallest positive integer with $p|F_{r(p)}$. From the Theorem 1 we are able to derive

Theorem 2. *Suppose that for the prime p at least one of the following conditions holds*

(i) *$r(p)$ is not a prime power.*
(ii) *$F_{r(p)}=q_1^{\alpha_1}\cdot\ldots\cdot q_v^{\alpha_v}$, with $q_1,\ldots,q_v$ distinct primes, $v\geqq 2$ and $3\nmid\alpha_1, 3\nmid\alpha_2$.*
(iii) *$r(p)$ is a power of 2, 3, 7, 13 or 17. Then the Diophantine equations*

$$F_n = px^3 \tag{3}$$

$$F_n = p^2x^3 \tag{3'}$$

have only one solution $n=x=0$, except when $p=2, p=3$, $p=13, p=233$ and $p=1597$ in which cases (3) has one further solution $x=1$ and $n=3$, $n=4$, $n=7$, $n=13$ and $n=17$ respectively.

Remarks.

Condition (i) of Theorem 2 is satisfied for infinitely many primes. Namely, there exists for any $m>12$ a prime q such that q/F_m and $q\nmid F_t$ for any $0<t<m$, [12].

From the proof it will be clear that Theorem 2 is true for a wider class of primes, but we were unable to characterize it.

In [10] we shall apply these results to solve completely some Diophantine equations.

I am indebted to Dr. Z. L. Papp and D. Szubbocsev for their assintance in preparing and writing the computer programs.

2. Elementary properties of the Fibonacci sequence

Lemma 1. *F_n is divisible by F_m if, and only if, n is divisible by m.*

Proof. [12], page 46.

Lemma 2. *Let p be an odd prime. If $p\equiv\pm 1 \pmod 5$, then $p|F_{p-1}$, and if $p\equiv\pm 2 \pmod 5$, then $p|F_{p+1}$.*

Proof. [12], page 54.

Lemma 3. *Let q be a prime divisor of F_n and $p\neq q$ a prime, then $(F_{np}|F_n, q)=1$.*

Proof. [12], page 60. See also [14].

Corollary 1. *If $p\geqq 7$ is a prime, then any prime divisor of F_p is greater than p.*

Proof. Let q be a prime divisor of F_p, then by Lemma 1 $r(q)=p$. Furthermore $q\geqq 7$, so neither $q-1$ nor $q+1$ are primes. From Lemma 1 and Lemma 2 follows $p|q-1$ or $p|q+1$, which proves the assertion.

Corollary 2. *Let q be a prime divisor of F_n and m be an integer with $(m, q)=1$. Then $(F_{nm}|F_n, q)=1$.*

PROOF. Let $m=p_1^{\alpha_1}\cdot\ldots\cdot p_r^{\alpha_r}$ be the prime decomposition of m. Then by the assumption $p_i\neq q$ for $i=1,\ldots,r$. Applying Lemma 3 several times we have the proof.

Lemma 4. *Let $q\geqq 3$ be an integer. If F_m is a full q-th power, then either $m=0, 1, 2, 6$ or there exists a prime divisor p of m such that F_p is a full q-th power.*

PROOF. If m is a prime we have nothing to prove. Assume m to be composite, and let $m=p_1^{\alpha_1}\cdot\ldots\cdot p_r^{\alpha_r}$ with primes $p_1<\ldots<p_r$.

If $p_r\geqq 7$, then by Corollary 1 any prime divisors of F_{p_r} are greater than p_r, hence they are distinct from $p_1,\ldots,p_r$. So by Corollary 2 these primes occur in F_m in the same powers as in F_{p_r}, which means F_{p_r} is a full q-th power.

Assume $m=2^{\alpha_1}3^{\alpha_2}5^{\alpha_3}$ with $\alpha_1,\alpha_2,\alpha_3\geqq 0$. An easy calculation shows that $F_8=7$, $F_9=2\cdot 17$ and $F_{25}=5^2\cdot 3001$. Again by Corollary 2 F_m can be a full q-th power only if $\alpha_1<3$, $\alpha_2<2$ and $\alpha_3<2$. Finally $F_5=5$, $F_{10}=5\cdot 11$, $F_{15}=2\cdot 5\cdot 61$ and $F_{12}=2^4\cdot 3^2$ proves the lemma completely.

Lemma 5. *For any $n\geqq 1$*

$$F_{n+1}^2-F_{n+1}F_n-F_n^2=(-1)^n \tag{4}$$

holds.

PROOF. This is due to JONES [4].

Lemma 6. *If for some integers $n>6$, x (2) is solvable, then there exist integers A, B with $x^2=A^2+4B^2$, and*

$$A^3-3A^2B-12AB^2+4B^3=\pm 1. \tag{5}$$

PROOF. By Lemma 4 we may assume n to be odd. Put $F_{n+1}=y$. Then (4) yields

$$y^2-yx^3-x^6+1=0.$$

This equation is solvable in y only if its discriminant is the square of an integer z, that is

$$5x^6=z^2+4 \tag{6}$$

If (6) is solvable, then it is solvable mod 5 too, hence $z=5v\pm 1$ with an integer v. Writing in (6) this form of z, and dividing by 5 we have

$$x^6=(v\pm 1)^2+(2v)^2. \tag{7}$$

From this follows obviously $(x, v)=1$, x odd and v even. The polynomial remaining on the right hand side of (7) can be written as a product of two linear functions in the ring of Gaussian integers

$$x^6=\bar{x}^3=\big((1+2i)v\pm 1\big)\big((1-2i)v\pm 1\big)$$

with $\bar{x}=x^2$. This ring is a unique primfactorisation domain, and its units are ± 1 and $\pm i$. These units are obviously cubes of Gaussian integers, so there exist integers

A, B_1 with $\bar{x}=A^2+B_1^2$, and

$$v\pm 1+2vi=(A+B_1 i)^3=A^3-3AB_1^2+i(3A^2B_1-B_1^3).$$

1 and i form an integer basis of the Gaussian integers, hence the last equation yields

$$v\pm 1=A^3-3AB_1^2 \tag{8}$$
$$2v=3A^2B_1-B_1^3$$

The right hand side of the second equation of (8) is divisible by 4, because v is even. A cannot be even, and both A and B_1 cannot be odd, because of the first equation of (8), hence B_1 must be odd, i.e. $B_1=2B$. Now solving (8) for v we have (5), and the lemma is proved.

3. Upper bound for the full cube Fibonacci numbers

Denote $a_0x^N+\ldots+a_N$ the minimal polynomial for the algebraic number β, while $\beta=\beta_1, \beta_2, \ldots, \beta_N$ are its conjugates. Put

$$M(\beta)=|a_0|\prod_{j=1}^{n}\max\{1, |\beta_j|\}$$

and

$$h(\beta)=\frac{1}{N}\log M(\beta).$$

In order to establish an upper bound for the full cube Fibonacci numbers we shall apply the following result of Waldschmidt [13].

Theorem A. *Let $\xi_1, \ldots, \xi_m$ be non-zero algebraic numbers, and $\eta_0, \eta_1, \ldots, \eta_m$ be algebraic numbers. For $1\leqq j\leqq m$ let $\log \xi_j$ be any determination of the logarithm of ξ_j. Let D be a positive integer, and $V_1, \ldots, V_n, W, E$ be positive real numbers, satisfying*

$$D\geqq[Q(\xi_1, \ldots, \xi_n, \eta_0, \ldots, \eta_m): Q]$$

$$V_j\geqq\max\{h(\xi_j), |\log \xi_j|/D, 1/D\}, \quad 1\leqq j\leqq m$$

$$W\geqq\max_{0\leqq j\leqq m}\{(h(\eta_j)\}$$

$$V_1\leqq\ldots\leqq V_m$$

and

$$1\leqq E\min\left\{e^{DV_1}; \min_{1\leqq j\leqq m} 4DV_j/|\log \xi_j|\right\}.$$

Finally define $V_j^+=\max\{V_j, 1\}$ for $j=m$ and $j=m-1$, with $V_0^+=1$ in the case $m=1$. If the number

$$\Lambda=\eta_0+\eta_1\log\xi_1+\ldots+\eta_m\log\xi_m$$

does not vanish, then

$$|\Lambda|>\exp\left\{-C(m)D^{m+2}V_1\cdot\ldots\cdot V_m(W+\log(EDV_m^+))(\log EDV_{m-1}^+)(\log E)^{-m-1}\right\}$$

where

$$C(1) \leqq 2^{35}, \quad C(2) \leqq 2^{53} \quad \text{and} \quad C(m) \leqq 2^{8m+31} m^{2m}.$$

Lemma 7. *Suppose A and B are integer solutions of* (5). *Then*

$$\max\{|A|, |B|\} < 5{,}172 \exp(0{,}6 \cdot 10^{38}) \tag{9}$$

PROOF. We use the ideas of the proof of a general theorem of GYŐRY and PAPP [3], so we omit the details.

If $B=0$, then $A=\pm 1$, and we have (9) at once. Suppose $B \neq 0$, and

$$|B| > \exp(40). \tag{10}$$

The roots of the equation $x^3-3x^2-12x+4=0$ are $\alpha=\alpha_1=5{,}171029785$, $\alpha_2=0{,}3115831337$ and $\alpha_3=-2{,}482612919$. Let $K=Q(\alpha)$ and $O=Z[\alpha]$ be the order of K generated by $1, \alpha, \alpha^2$. The rank of the group of units of O is 2. A system of fundamental units is

$$\varepsilon_1 = -181+508\alpha+234\alpha^2$$

$$\varepsilon_2 = -5+17\alpha-3\alpha^2$$

and the regulator of O is $R=40{,}7388$.

Write now (5) in the form

$$N_{K/Q}(A-\alpha B) = (A-\alpha_1 B)(A-\alpha_2 B)(A-\alpha_3 B) = \pm 1.$$

Put $\gamma_i = A-\alpha_i B$ $(i=1, 2, 3)$. It is well known that there exist integers a_1, a_2 with

$$A-\alpha_i B = \gamma_i = \pm \varepsilon_1^{(i)a_1} \varepsilon_2^{(i)a_2}, \tag{11}$$

where $\varepsilon_j^{(i)}$ denotes the i-th conjugate of ε_j. Applying the ideas of [3] a routine computation shows

$$|a_i| \leqq 0{,}49 \log |B| \quad (i = 1, 2).$$

Let $|\gamma_q| \leqq |\gamma_j|$ for all $j \neq q$. From (11) follows

$$(\alpha_q-\alpha_g)\gamma_h-(\alpha_q-\alpha_h)\gamma_g = (\alpha_g-\alpha_h)\gamma_q.$$

Dividing this by $(\alpha_q-\alpha_h)\gamma_g \neq 0$ we get

$$\delta_0 \delta_1^{a_1} \delta_2^{a_2} - 1 = \frac{\alpha_g-\alpha_h}{\alpha_q-\alpha_h} \gamma_q \gamma_g^{-1} \neq 0 \tag{12}$$

with $\delta_0 = \dfrac{\alpha_q-\alpha_g}{\alpha_q-\alpha_h}$ and $\delta_i = \varepsilon_i^{(h)} | \varepsilon_i^{(g)}$ $(i = 1, 2)$.

Furthermore short calculation shows $|\gamma_q| < 0{,}5139\,|B|^{-2}$ and $|\gamma_g|^{-1} \leqq 0{,}2569\,|B|^{-1}$ $(g=1, 2, 3)$. Hence we have from (12)

$$0 < |\delta_0 \delta_1^{a_1} \delta_2^{a_2} - 1| < 1{,}01\,|B|^{-3}. \tag{13}$$

Because of (10) the right hand side of (13) is less than 1/3, so (13) yields

$$0 \neq |\Lambda| = |a_3 \log \delta_3 + a_1 \log \delta_1 + a_2 \log \delta_2 + a_0 \log \delta_0| < e^{-3 \log |B|}, \tag{14}$$

where log denotes the mean value of the logarithm function,

$$\delta_3 = -1, \quad a_0 = 1 \quad \text{and} \quad 0 \neq |a_3| \leqq |a_1| + |a_2| + 1 \leqq \log |B|.$$

Theorem A can be applied to (14). In this case $m=4$, $D=6$, $W=\log\log|B|$, $V_0=\pi/6=0{,}524$, $V_1=7{,}092$, $V_2=9{,}021$, $V_3=1{,}884$, $E=4$. By its conclusion we have

$$|A| > \exp\{-1{,}873 \cdot 10^{36} \log\log|B| - 1{,}007 \cdot 10^{38}\}.$$

Combining this inequality with (13) there follows

$$|B| < \exp(0{,}6 \cdot 10^{38}). \tag{15}$$

Finally

$$|A| \leqq |\alpha_q B| + |A - \alpha_q B| < 5{,}1711|B| + 1 < 5{,}172 \exp(0{,}6 \cdot 10^{38}), \tag{15a}$$

which proves the Lemma.

Proposition. *If* (2) *is solvable in integers* n, x *then*

$$n < 3{,}76 \cdot 10^{38}. \tag{16}$$

PROOF. If (2) is solvable in n, x then by Lemma 6 there exist integer solutions A, B of (5) with

$$x^2 = A^2 + 4B^2.$$

By the proof of Lemma 7 B and A satisfy the inequalities (15) and (15a) respectively, hence

$$|x| \leqq 5{,}55 \exp(0{,}6 \cdot 10^{38}).$$

It is well known that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] > \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - 1\right].$$

Therefore

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - 1\right] < 5{,}55^3 \exp(1{,}8 \cdot 10^{38}),$$

which implies

$$n < 3{,}76 \cdot 10^{38}.$$

The Proposition is proved.

4. Preliminary results to the computer program

The following two lemmas are basic in number theory. For the proofs see for example Niven—Zuckerman [8].

Lemma 8. *If* p *is a prime and* $(a, p)=1$, *then the congruence* $x^n \equiv a \pmod p$ *has* $(n, p-1)$ *solutions or no solutions according as*

$$a^{(p-1)/(n,p-1)} \equiv 1 \pmod p \quad \text{or} \quad a^{(p-1)/(n,p-1)} \not\equiv 1 \pmod p.$$

Lemma 9. *The system of congruences*

$$x \equiv c_1 \pmod{m_1}$$
$$x \equiv c_2 \pmod{m_2}$$

are solvable if and only if $(m_1, m_2)/c_2 - c_1$. *Any two solutions are congruent modulo* $[m_1, m_2]$.

The integer a is called a cubic residue modulo p, if the congruence $x^3 \equiv a \pmod{p}$ is solvable. Put

$$\left(\frac{a}{p}\right)_3 = \begin{cases} 1, & \text{if } a \text{ is a cubic residue modulo } p \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(n,p)$ be the smallest non-negative residue of F_n modulo p. Then the sequence $\{F(n,p)\}_{n=0}^{\infty}$ is periodic for all p. Put

$$F^*(n, p) = \left(\frac{F(n,p)}{p}\right)_3.$$

Lemma 10. *If* p *is a prime, then the sequence* $\{F^*(n, p)\}_{n=0}^{\infty}$ *is periodic for all* p *and the length of its minimal period is at most* $r(p)$. *Finally* $F^*(k, p) = F^*(r(p)-k, p)$ *for all* $k=0, \ldots, r(p)$.

PROOF. If $p \not\equiv 1 \pmod 3$, then $(3, p-1)=1$, and by Euler's Theorem $a^{p-1} \equiv 1 \pmod{p}$ for all integers a relatively prime to p. Further 0 is always a cubic residue, hence $F^*(n, p)=1$ for all n.

Assume in the sequel $p \equiv 1 \pmod 3$ i.e. $(3, p-1)=3$. First we prove

$$F(r(p)-k, p) \equiv (-1)^{k-1} F(k, p) F(r(p)-1, p) \pmod{p} \tag{17}$$

for all $k=0, \ldots, r(p)$. For $k=0$ and $k=1$ (17) holds at once. Assume it is true for $k \geqq 0$ and $k+1$. Then

$$\begin{aligned} F(r(p)-(k+2), p) &= F(r(p)-k, p) - F(r(p)-(k+1), p) \equiv \\ &\equiv (-1)^{k-1} F(r(p)-1, p)[F(k, p) + F(k+1, p)] \equiv \\ &\equiv (-1)^{k+1} F(k+2, p) F(r(p)-1, p) \pmod{p}, \end{aligned}$$

and this proves (17).

Putting in (17) $k=r(p)-1$ we have

$$F(1, p) \equiv (-1)^{r(p)} F(r(p)-1, p)^2 \pmod{p}.$$

This implies

$$F(r(p)-1, p)^2 \equiv \pm 1 \pmod{p},$$

which shows that $F(r(p)-1, p)$ is a cubic residue mod p because both 1 and -1 are cubic residues and $(2, 3)=1$.

The last assertion of the Lemma follows from (17) using $\left(\frac{F(r(p)-1, p)}{p}\right)_3 = 1$ and the multiplicative property of the cubic residue.

One can prove by induction

$$F(r(p)+k, p) \equiv F(k, p)F(r(p)-1, p) \pmod{p} \tag{18}$$

for all $k \geqq 0$. $F(r(p)-1, p)$ being a cubic residue, (18) implies

$$F^*(r(p)+k, p) = F^*(k, p)$$

for all $k \geqq 0$, and the lemma is proved.

5. Description of the computer program

In this chapter p and p_{ij} denote primes congruent to 1 modulo 3. The least common multiple of the integers $a_1, \ldots, a_n$ is denoted by $[a_1, \ldots, a_n]$.

First we have determined $r(p)$ for all $p \leqq 10000$. Afterwards we have used the following sieve method.

In the i-th step are choosen integers M_i, N_i, with $N_1=2$ and N_i for $i \geqq 2$ a divisor of $\prod_{j<i} M_j$, and primes $p_{i1}, \ldots, p_{it_i}$ with $[r(p_{i1}), \ldots, r(p_{it_i})]=M_i$ and $(r(p_{i1}), \ldots, r(p_{it_i}))=N_i$. For convenience M_i is taken such that it is divisible by 10.

Computing the first period of $F^*(k, p_{ij})$ for all p_{ij} it is possible to determine the index of the possible full cube Fibonacci numbers up to M_i with the following sieve method. Put

$$S_h = \prod_{j=1}^{t_i} F^*(hN_i - 1 \pmod{r(p_{ij})}, p_{ij})$$

and

$$S_h' = \prod_{j=1}^{t_i} F^*(hN_i + 1 \pmod{r(p_{ij})}, p_{ij})$$

with $h=0, 1, \ldots, (M_i/N_i+1)/2$.

If either S_h or S_h' is 1, then the corresponding Fibonacci numbers with indices $hN_i \pm 1$ and $M_i-(hN_i \pm 1)$ are candidates for being a full cube. In the opposite case, namely if either S_h or S_h' is 0 they cannot be full cubes.

We make an effort to choose the p_{ij}-s so that apart from S_0' the others are 0. In this we succeded always, if not in the first step, then choosing new primes to the given ones.

With this method we were able to prove in each step that F_n can be a full cube only if

$$n \equiv \pm 1 \pmod{M_i}. \tag{19}$$

Consider (19) for two distinct modules M_{i_1} and M_{i_2}. The received system of congruences is solvable by Lemma 9 if and only if (M_{i_1}, M_{i_2}) divides 0, 2 or -2, but because of the choise of the M_i-s it does not divide 2 and -2. Hence the system of congruences is solvable only if $n \equiv 1 \pmod{M_{i_j}}$ and $n \equiv -1 \pmod{M_{i_j}}$ for $j=1, 2$. Again by Lemma 9 these have only one solution $n \equiv 1$ and $n \equiv -1$ respectively modulo $[M_{i_1}, M_{i_2}]$.

Applying this sieve method successively for $M_1, M_2, \ldots$ we obtain that F_n with n odd, can be a full cube only if $n=1$ or $n=[M_1, M_2, \ldots]-1$.

6. Proof of theorem

PROOF OF THEOREM 1. Apply the above described sieve method with the following values of N_i, M_i/N_i and p_{ij}.

$i = 1$; 2; $2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 19$;

7, 13, 31, 37, 211, 223, 241, 421, 571, 607, 769, 1063

$i = 2$; $2^2 \cdot 3 \cdot 5$; $3 \cdot 13 \cdot 17$;

19, 67, 181, 409, 541, 859, 883, 919, 1021, 1171, 1531, 1597, 1951

$i = 3$; $2^5 \cdot 3^2 \cdot 5 \cdot 13$; $2 \cdot 3^4 \cdot 5 \cdot 13$;

109, 127, 151, 271, 337, 601, 1087, 1459, 1621, 2029, 2269, 5407

$i = 4$; $2^3 \cdot 3 \cdot 5 \cdot 7$; $23 \cdot 61 \cdot 71$;

139, 283, 487, 643, 691, 829, 853, 1279, 1831, 1987, 2131, 4513

$i = 5$; $2^5 \cdot 3^2 \cdot 5 \cdot 7$; $11 \cdot 29 \cdot 67$;

43, 199, 307, 331, 349, 463, 661, 937, 967, 991, 1609, 1741, 2011, 2143

$i = 6$; $2^5 \cdot 3 \cdot 5 \cdot 7$; $7 \cdot 31$

433, 1471, 1489, 1567, 1861, 3037, 3529

$i = 7$; $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$; $79 \cdot 103$

619, 823, 2053, 2371, 2677, 3319, 3709, 5689

$i = 8$; $2^4 \cdot 3^2 \cdot 5 \cdot 13 \cdot 29$; $41 \cdot 43$

163, 739, 1117, 1231, 1291, 1549, 2377, 3691, 4129, 4987

$i = 9$; $2^4 \cdot 3 \cdot 5 \cdot 11 \cdot 29$; 83

499, 1327, 4813, 5479

$i = 10$; $2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 23$; 59

1297, 2713, 3067, 3541, 4483

$i = 11$; $2^4 \cdot 3 \cdot 5 \cdot 19 \cdot 23$; 89

1069, 1423, 2671, 3739, 4093, 6763

$i = 12$; $2^2 \cdot 3^3 \cdot 5 \cdot 11 \cdot 41$; 47

1033, 1129, 2539, 3853, 4231, 5641

$i = 13$; $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$; 53

2437, 3181, 3499, 6361, 6679

An easy computation shows that

$$M = [M_1, \ldots, M_{13}] = 2^6 \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot$$
$$\cdot 61 \cdot 67 \cdot 71 \cdot 79 \cdot 83 \cdot 89 \cdot 103 \approx 3{,}2 \cdot 10^{39}.$$

Hence there exist no odd integers $1<n<M-1$ with F_n a full cube. On the other hand $M-1$ is greater than the bound given in Proposition 1. This and Lemma 4 prove Theorem 1.

All computer calculations were performed on the ES 10—30 computer of the Computer Centre of the Kossuth Lajos University Debrecen.

PROOF OF THEOREM 2. In the sequel $p^u \| a$ will denote the exact power of the prime p which divides the integer a.

Let the prime decomposition of $r(p)$ be $r(p)=p_1^{z_1}:.:p_r^{z_r}$, with $p_1<\ldots<p_r$. If $p_r \geqq 3$ and $p \neq 2$ then let $P_1<P_2<\ldots<P_s=p_r$ be all primes which are not greater than p_r, and write $n=P_1^{\beta_1}:.:P_s^{\beta_s}m$ with $(P_i, m)=1$ for all $i=1, \ldots, s$. If $p_r=2$ or $p=2$ then put $P_1=2$ and $P_2=3$ and write $n=P_1^{\beta_1} \cdot P_2^{\beta_2}m$ with $(P_1 \cdot P_2, m)=1$.

First we are going to prove that the solvability of (3) or (3′) implies $m=1$. In fact assume $m>1$. Then m is odd and not less than 5. By Theorem 1 F_m has a prime divisor Q with $Q^u \| F_m$ and $3 \nmid u$. If $Q \leqq P_s$ were satisfied, then we would have by Lemma 1 $r(Q)=P_{i_1}^{\gamma_{i_1}}:.:P_{i_t}^{\gamma_{i_t}} \leqq Q \leqq P_s$. This would imply $(r(Q), m)=1$, which contradicts Lemma 3. Hence we have shown $Q>P_s$. Furthermore $p \neq Q$ since $(r(p), m)=1$. Thus by Corollary 2 $Q^u \| F_n$, and $F_n=p^i x^3$ $i=1$ or $i=2$ cannot be satisfied.

Hence we may assume $m=1$. We distinguish six cases.

Case I. p satisfies the conditions (i) or (ii) and $P_s \geqq 7$. Now either $r(p)$ is a power of P_s, and $F_{r(p)}$ has a prime divisor Q besides p with $Q^u \| F_{r(p)}$ and $3 \nmid u$, or $r(p)$ is not a power of P_s and then by Theorem 1 F_{P_s} has a prime divisor Q with $Q^u \| F_{P_s}$ and $3 \nmid u$. In both cases $Q>P_s$ is true by Corollary 1. Corollary 2 implies $Q^u \| F_n$, which proves this case.

Case II. p satisfies the conditions (i) or (ii) and $P_s=5$, Then $5 \| F_n$ or $3001 \| F_n$ according as $\beta_s=1$ or $\beta_s>1$. But $p=5$ and $p=3001$ do not satisfy the assumptions, since $r(5)=5$ and $r(3001)=5^2$.

In order to prove the following cases we shall refer to Corollary 2.

Case III. $r(p)$ is a power of 2 or 3. Put $n=2^{\beta_1} \cdot 3^{\beta_2}$. Since $F_{3^3}=2 \cdot 17 \cdot 53 \cdot 109$ and $F_{2^4}=3 \cdot 17 \cdot 47$ we have $\beta_1<4$ and $\beta_2<3$. If $\beta_1>0$, then since $F_{2 \cdot 3^2}=$ $=2^3 \cdot 17 \cdot 19$ we have $\beta_2<2$, and if $\beta_2>0$ then by $F_{2^2 \cdot 3}=2^4 \cdot 3^2$ we have $\beta_1<2$. Finally $F_6=2^3$, and there remain only $\beta_1=0$, $\beta_2<3$, and $\beta_1<4$, $\beta_2=0$. These can easily be checked.

In the following we may assume $5 \nmid n$ because of $5 \| F_n$ and $3001 \| F_n$ according as to $5 \| n$ or $5^\alpha \| n$ with $\alpha>1$.

Case IV. $r(p)$ is a power of 7. Write $n=2^{\beta_1} \cdot 3^{\beta_2} \cdot 7^{\beta_3}$. Since $F_{7^2}=13 \cdot 97 \cdot w$ with $(w, 13 \cdot 97)=1$ we have $\beta_3=1$. Further $F_{2 \cdot 7}=13 \cdot 29$ and $F_{3 \cdot 7}=2 \cdot 13 \cdot 421$ imply $\beta_1=0$ and $\beta_2=0$.

Case V. $r(p)$ is a power of 13. Put $n=2^{\beta_1}\cdot 3^{\beta_2}\cdot 7^{\beta_3}\cdot 11^{\beta_4}\cdot 13^{\beta_5}$. $\beta_5=1$ because of $F_{13^2}=233\cdot 337\cdot u_1$ with $(u_1, 233\cdot 337)=1$. $\beta_1=\beta_2=\beta_4=0$, since $F_{2\cdot 13}=233\cdot 521$, $F_{3\cdot 13}=2\cdot 233\cdot u_4$ and $F_{11\cdot 13}=89\cdot 233\cdot u_2$ with $(2\cdot 233, u_4)=(89\cdot 233, u_2)=1$. If $\beta_3>1$, then F_n is divided by the first power of 97 and 233. Finally $F_{7\cdot 13}=$ $=13^2\cdot 233\cdot u_3$ with $(u_3, 13\cdot 233)=1$.

Case VI. $r(p)$ is a power of 17. Write $n=2^{\beta_1}\cdot 3^{\beta_2}\cdot 17^{\beta_3}\cdot m$ with $(m, 2\cdot 3\cdot 17)=$ $=1$. We can exclude $m>1$ using $r(17)=3^2$ and the ideas described at the beginning of the proof. $\beta_3=1$ since $F_{17^2}=577\cdot 1597\cdot v_1$ with $(v_1, 577\cdot 1597)=1$. Finally $\beta_1=\beta_2=0$ because of $F_{2\cdot 17}=1597\cdot 3571$ and $F_{3\cdot 17}=2\cdot 1597\cdot v_2$ with $(v_2, 2\cdot 1597)=1$.

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(Received September 11, 1980)