

Convergence of vector-valued martingales with multidimensional indices

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1. Introduction

There are several theorems concerning almost sure (a.s.) convergence of multi-parameter sequences of scalar random variables. The purpose of this article is to generalize these results to the case of random variables taking values in a Banach space. After some preliminary remarks we give an extension of Gut's theorem concerning reversed martingales (cf. [7]) in section 3. Section 4 deals with Cairoli's theorem on martingales (cf. [2]). Section 5 contains a multidimensional strong law of large numbers for vector-valued random variables (the scalar case is due to Smythe, cf. [12]). Our method is a modification of Chatterji's method (see [4], Prop. 5.2 and Prop. 5.3).

2. Notation and preliminary remarks

Let $(\Omega, \mathcal{A}, P)$ be a probability space, B a real Banach space with norm $|\cdot|$. B^* is the dual of B . $X: \Omega \rightarrow B$ will be called a B -valued random variable (r.v.) if X is Bochner measurable. The expectation of X is defined by $EX = \int_{\Omega} X dP$, where the integral is Bochner integral. $L^r = L^r(\mathcal{A}, B)$ (or $L^r(\Omega, \mathcal{A}, P, B)$), $1 \leq r < \infty$, denotes the Banach space of random variables X , for which $\int_{\Omega} |X|^r dP < \infty$.

Let $(T, \equiv)$ and Z denote a directed set and the set of positive integers respectively.

Definition 2.1. Let $\mathcal{F}_t (t \in T)$ be an increasing sequence of σ -subalgebras of $\mathcal{A}$. $\{X_t, \mathcal{F}_t, t \in T\}$ is called a martingale if $X_t \in L^1(\mathcal{F}_t, B)$ and $E(X_{t_2} | \mathcal{F}_{t_1}) = X_{t_1}$ ($t_1 \equiv t_2$). The reversed martingale is defined in similar way.

For suitable references on these subjects see [3], [4], [6], [8] and [11].

Definition 2.2. Let $c(B)$ denote the set of all convergent sequences in B . If $x = (x_1, x_2, \dots) \in c(B)$, let $\|x\|_c = \sup_i |x_i|$. Let $c_0(B)$ denote the set of sequences converging to 0 (the null element of B).

Proposition 2.3. (1) $c(B)$ is a Banach space with the coordinate-wise addition, scalar multiplication and norm $\|\cdot\|_c$. $c_0(B)$ is a subspace of $c(B)$.

(2) If B is separable, then $c(B)$ is also separable.

(3) The convergence in $c(B)$ is the coordinate-wise uniform convergence.

Lemma 2.4. (1) Let X_i ($i \in \mathbb{Z}$) be B -valued random variables. If the sequence X_i converges a.s., then

$$X = (X_1, X_2, \dots)$$

is a $c(B)$ -valued random variable.

(2) Let $X = (X_1, X_2, \dots)$ be a $c(B)$ -valued r.v. and let $\mathcal{F}$ be a σ -subalgebra of $\mathcal{A}$. If $E\|X\|_c < \infty$, then $(E(X|\mathcal{F}))_i = E(X_i|\mathcal{F})$ ($i \in \mathbb{Z}$), that is the conditional expectation in $c(B)$ can be constituted coordinate-wise.

PROOF. We can suppose that B is separable.

(1) Let $y^0 = (y_1^0, y_2^0, \dots)$ be a fixed element in $c(B)$, let $y = (y_1, y_2, \dots)$ denote an arbitrary sequence of elements of B . For $\varepsilon > 0$

$$\{y: y \in c(B), \|y - y^0\|_c \leq \varepsilon\} = \left(\bigcap_{n=1}^{\infty} \{y: |y_n - y_n^0| \leq \varepsilon\} \right) \cap c(B) = G.$$

$X^{-1}(G) \in \mathcal{A}$, so the inverse image of an arbitrary sphere in $c(B)$ is measurable. The Borel σ -field of $c(B)$ is generated by the spheres of $c(B)$, because $c(B)$ is separable (cf. [10]). Thus X is measurable.

(2) Let $f \in B^*$ and for a fixed $i \in \mathbb{Z}$ we define $f_i \in [c(B)]^*$ by the following equation:

$$\begin{aligned} f_i(a) &= f(a_i), \quad a = (a_1, a_2, \dots) \in c(B). \\ f(\{E(X|\mathcal{F})\}_i) &= f_i(E(X|\mathcal{F})) = E(f_i(X)|\mathcal{F}) = \\ &= E(f(X_i)|\mathcal{F}) = f(E(X_i|\mathcal{F})) \quad \text{a.s.} \end{aligned}$$

according to Theorem 2.3 of [11]. That is $f[\{E(X|\mathcal{F})\}_i - E(X_i|\mathcal{F})] = 0$ a.s. There is a countable norm-determining set in B^* , because B is separable. Thus $\{E(X|\mathcal{F})\}_i = E(X_i|\mathcal{F})$ a.s.

We need the following results from the theory of scalar martingales.

Lemma 2.5. (1) If $\{Z_k, \mathcal{F}_k, k \in \mathbb{Z}\}$ is a scalar positive submartingale, then $\{Z_k (\log^+ Z_k)^{r-2}, \mathcal{F}_k, k \in \mathbb{Z}\}$ is also a positive submartingale ($r > 2$).

(2) Doob's maximal inequalities. If $\{X_k, \mathcal{F}_k, k = 1, 2, \dots, n\}$ is a real positive submartingale, then

$$(a) \quad E\left(\max_{1 \leq k \leq n} X_k\right) \leq \frac{e}{e-1} + \frac{e}{e-1} E(X_n \log^+ X_n)$$

and

$$(b) \quad E\left(\max_{1 \leq k \leq n} X_k^\alpha\right) \leq \left(\frac{\alpha}{\alpha-1}\right)^\alpha E(X_n^\alpha), \quad \alpha > 1.$$

(3) Cairoli's inequality:

$$t(\log^+ t)^{r-2} \log^+ [t(\log^+ t)^{r-2}] \leq (r-1)t(\log^+ t)^{r-1}, \quad r \geq 2, \quad t \geq 0.$$

PROOF. (1) $t(\log^+ t)^{r-2}$ is a convex increasing function if $r > 2$.

(2) See [5], p. 317.

(3) See [2] and [12].

Notations. Let $d \geq 1$ be an integer and let Z^d denote the positive d -dimensional integer lattice points. The notations $\bar{m} \leq \bar{n}$, where $\bar{m} = (m_1, m_2, \dots, m_d)$ and

$\bar{n} = (n_1, n_2, \dots, n_d) \in Z^d$, means that $m_i \leq n_i, i = 1, 2, \dots, d$. With this partial ordering Z^d is a directed set. $|\bar{n}|$ is used to denote $\prod_{k=1}^d n_k$. $\bar{n} \rightarrow \infty$ means that $n_i \rightarrow \infty, i = 1, 2, \dots, d$. If $\bar{n} \in Z^d$ and $m \in Z$, then $(n_1, n_2, \dots, n_d, m) \in Z^{d+1}$ is denoted by $(\bar{n}, m)$. $(1, 1, \dots, 1) \in Z^d$ is denoted by $\bar{1}$.

3. Reversed martingales

Theorem 3.1. Let $\{X_{\bar{n}}, \mathcal{F}_{\bar{n}}, \bar{n} \in Z^d\}$ be a B -valued reversed martingale and let

$$E(E(\cdot | \mathcal{F}_{\bar{n}}) | \mathcal{F}_{\bar{m}}) = E(\cdot | \mathcal{F}_{\max(\bar{m}, \bar{n})}),$$

where $\max(\bar{m}, \bar{n})$ is the coordinate-wise maximum. Let $\mathcal{F} = \bigcap_{\bar{m} \in Z^d} \mathcal{F}_{\bar{m}}$.

If $E(|X_{\bar{1}}|(\log^+ |X_{\bar{1}}|)^{d-1}) < \infty$, then there exists a random variable X which is $\mathcal{F}$ -measurable and $X_{\bar{n}} \rightarrow X$ a.s. as $\bar{n} \rightarrow \infty$.

PROOF. We consider first the case $d = 1$ and proceed by induction. For $d = 1$ the theorem is a consequence of a theorem of Chatterji (cf. Theorem 4 of [3]). Suppose that the theorem is true for $d - 1$. Let $\bar{n} \in Z^{d-1}$ and

$$Y_{\bar{n}} = (X_{(\bar{n}, 1)}, X_{(\bar{n}, 2)}, \dots).$$

We shall prove that $\{Y_{\bar{n}}, \mathcal{F}_{(\bar{n}, 1)}, \bar{n} \in Z^{d-1}\}$ is a $c(B)$ -valued reversed martingale which satisfies the assumptions of the theorem. For every fixed $\bar{n} \in Z^{d-1}$ $\{X_{(\bar{n}, j)}, \mathcal{F}_{(\bar{n}, j)}, j \geq 1\}$ is a B -valued reversed martingale and thus from the above mentioned theorem of Chatterji

$$\lim_{j \rightarrow \infty} X_{(\bar{n}, j)} = E\left\{X_{(\bar{n}, 1)} \middle| \bigcap_{j=1}^{\infty} \mathcal{F}_{(\bar{n}, j)}\right\} \quad \text{a.s.}$$

According to Lemma 2.4 $Y_{\bar{n}}$ is an $\mathcal{F}_{(\bar{n}, 1)}$ -measurable $c(B)$ -valued r.v.

To prove that $Y_{\bar{n}} \in L^1(\mathcal{F}_{(\bar{n}, 1)}, c(B))$ let $\{V_k, \mathcal{F}_k, k \geq 1\}$ be the following B -valued reversed martingale:

$$V_1 = X_{\bar{1}} \quad (\bar{1} \in Z^d),$$

$$V_k = X_{(\bar{n}, k-1)} \quad \text{for } k = 2, 3, \dots \quad (\bar{n} \in Z^{d-1}),$$

and $\mathcal{F}_k$ is the corresponding σ -subalgebra. $\{|V_k|, \mathcal{F}_k, k \geq 1\}$ is a real submartingale in the reversed ordering. According to Doob's maximal inequality

$$E\left\{\max_{1 \leq k \leq m} |V_k|\right\} \leq \frac{e}{e-1} + \frac{e}{e-1} E\{|V_1| \log^+ |V_1|\} < \infty.$$

From the monotone convergence theorem $E\{\sup_k |V_k|\} < \infty$. Thus $\|Y_{\bar{n}}\|_c = \sup_{1 \leq j < \infty} |X_{(\bar{n}, j)}|$ is integrable and this proves that $Y_{\bar{n}} \in L^1(\mathcal{F}_{(\bar{n}, 1)}, c(B))$. Since $E\{X_{(\bar{n}, j)} | \mathcal{F}_{(\bar{n}, 1)}\} = X_{(\bar{n}, j)}$ for $\bar{k} \geq \bar{n}$, it follows from Lemma 2.4 that $E\{Y_{\bar{n}} | \mathcal{F}_{(\bar{k}, 1)}\} = Y_{\bar{k}}$ for $\bar{k} \geq \bar{n}$, that is $\{Y_{\bar{n}}, \mathcal{F}_{(\bar{n}, 1)}, \bar{n} \in Z^{d-1}\}$ is a $c(B)$ -valued reversed martingale.

Now we show that $Y_{\bar{1}} \in L(\log^+ L)^{d-2}(\bar{1} \in Z^{d-1})$. $\{|X_{(\bar{1}, j)}|, \mathcal{F}_{(\bar{1}, j)}, j \geq 1\}$ ($\bar{1} \in Z^{d-1}$) is a positive submartingale. It is a consequence of Lemma 2.5 (1) that

$\{|X_{(\bar{i},j)}|(\log^+ |X_{(\bar{i},j)}|)^{d-2}, \mathcal{F}_{(\bar{i},j)}, j \geq 1\}$ is a positive submartingale. By virtue of Lemma 2.5:

$$\begin{aligned} & E\{\|Y_{\bar{1}}\|_c(\log^+ \|Y_{\bar{1}}\|_c)^{d-2}\} = \\ &= E\left\{\sup_j |X_{(\bar{i},j)}| [\log^+ (\sup_j |X_{(\bar{i},j)}|)]^{d-2}\right\} = E\left\{\sup_j [|X_{(\bar{i},j)}|(\log^+ |X_{(\bar{i},j)}|)^{d-2}]\right\} \leq \\ &\leq \frac{e}{e-1} + \frac{e}{e-1} E\{|X_{(\bar{i},1)}|(\log^+ |X_{(\bar{i},1)}|)^{d-2} \log^+ [|X_{(\bar{i},1)}|(\log^+ |X_{(\bar{i},1)}|)^{d-2}]\} \leq \\ &\leq \frac{e}{e-1} + \frac{e}{e-1} (d-1) E\{|X_{(\bar{i},1)}|(\log^+ |X_{(\bar{i},1)}|)^{d-1}\} < \infty. \end{aligned}$$

Hence by the induction assumption the $c(B)$ -valued reversed martingale $\{Y_{\bar{n}}, \mathcal{F}_{(\bar{n},1)}, \bar{n} \in Z^{d-1}\}$ converges to $Y_\infty = E\{Y_{\bar{1}} | \bigcap_{\bar{n} \in Z^{d-1}} \mathcal{F}_{(\bar{n},1)}\}$ a.s. Now using the induction assumption we can show that the components of Y_∞ form the reversed martingale

$$\{E(X_{(\bar{i},n)} | \bigcap_{\bar{m} \in Z^{d-1}} \mathcal{F}_{(\bar{m},1)}), \bigcap_{\bar{m} \in Z^{d-1}} \mathcal{F}_{(\bar{m},n)}, n = 1, 2, \dots\}.$$

This reversed martingale is convergent:

$$\begin{aligned} & \lim_{n \rightarrow \infty} E\{X_{(\bar{i},n)} | \bigcap_{\bar{m} \in Z^{d-1}} \mathcal{F}_{(\bar{m},1)}\} = \\ & E\{E(X_{(\bar{i},1)} | \bigcap_{\bar{m} \in Z^{d-1}} \mathcal{F}_{(\bar{m},1)}) | \bigcap_{\substack{\bar{m} \in Z^{d-1} \\ k=1,2,\dots}} \mathcal{F}_{(\bar{m},k)}\} = \\ & E\{X_{(\bar{i},1)} | \bigcap_{\substack{\bar{m} \in Z^{d-1} \\ k=1,2,\dots}} \mathcal{F}_{(\bar{m},k)}\} \text{ a.s. } (\bar{i} \in Z^{d-1}) \end{aligned}$$

and the limit is equal to $E\{X_{\bar{1}} | \bigcap_{\bar{n} \in Z^d} \mathcal{F}_{\bar{n}}\}$, where $\bar{1} \in Z^d$. From here we have that

$$\lim_{\bar{n} \rightarrow \infty} X_{\bar{n}} = E(X_{\bar{1}} | \bigcap_{\bar{n} \in Z^d} \mathcal{F}_{\bar{n}}) \text{ a.s.}$$

Remark. In the preceding theorem the limit can be considered as the last term of the martingale.

Lemma 3.2. Let $\{X_t, \mathcal{F}_t, t \in T\}$ be a B -valued reversed martingale, where T is a directed set. If $X_{t_0} \in L^r$ ($1 \leq r < \infty$) for some $t_0 \in T$, then X_t is convergent in L^r and in probability.

PROOF. The convergence in L^r is an immediate consequence of Prop. 4.1 of [4] and Lemma V—1—1 of [9]. According to Chebyshev's inequality convergence in L^r implies convergence in probability.

Theorem 3.3. Let $\{X_{\bar{n}}, \mathcal{F}_{\bar{n}}, \bar{n} \in Z^d\}$ be a B -valued reversed martingale. If $E(E(\cdot | \mathcal{F}_{\bar{m}}) | \mathcal{F}_{\bar{n}}) = E(\cdot | \mathcal{F}_{\max(\bar{m}, \bar{n})})$ for every $\bar{m}, \bar{n} \in Z^d$ and $E|X_{\bar{1}}|^r < \infty$ ($1 < r < \infty$), then $\lim_{\bar{n} \rightarrow \infty} X_{\bar{n}} = E(X_{\bar{1}} | \bigcap_{\bar{n} \in Z^d} \mathcal{F}_{\bar{n}})$ a.s. and in L^r .

PROOF. According to Lemma 3.2 we have to prove only a.s. convergence. For $d=1$ it is a consequence of Prop. 4.1 of [4]. We proceed by induction. Let $\bar{m}$ be an element in Z^{d-1} . Let

$$Y_{\bar{m}} = (X_{(\bar{m},1)}, X_{(\bar{m},2)} \dots).$$

It follows from Prop. 4.1 of [4] that

$$\lim_{n \rightarrow \infty} X_{(\bar{m},n)} = E\left(X_{(\bar{1},1)} \middle| \bigcap_{k=1}^{\infty} \mathcal{F}_{(\bar{m},k)}\right) \quad \text{a.s.} \quad (\bar{1}, \bar{m} \in Z^{d-1}).$$

Thus $Y_{\bar{m}}$ is a $c(B)$ -valued r.v. Using Lemma 2.5 (2b) one can prove that $Y_{\bar{1}} \in L^1(\Omega, \mathcal{A}, P, c(B))$ ($\bar{1} \in Z^{d-1}$).

Hence by the induction assumption the reversed martingale $\{Y_{\bar{m}}, \mathcal{F}_{(\bar{m},1)}, \bar{m} \in Z^{d-1}\}$ converges

$$\lim_{\bar{m} \rightarrow \infty} Y_{\bar{m}} = E(Y_{\bar{1}} \middle| \bigcap_{\bar{m} \in Z^{d-1}} \mathcal{F}_{(\bar{m},1)}) \quad \text{a.s.}$$

Finally, the proof can be completed as the proof of Theorem 3.1.

4. Martingales

Lemma 4.1. (Cf. [4].) *If $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a B -valued martingale and*

$$\sup_n E\{|X_n|(\log^+ |X_n|)^k\} < \infty$$

for a $k \in Z$, then $\sup_n |X_n| \in L^1$.

PROOF. Since $\{|X_n|, \mathcal{F}_n, n \geq 1\}$ is a real positive submartingale, then Lemma 2.4 (2a) is applicable. The monotone convergence theorem implies the required result.

Lemma 4.2. *For every $X \in L^1(\Omega, \mathcal{A}, P, B)$ the family of random variables $E(X|\mathcal{F})$ obtained when $\mathcal{F}$ varies over all the σ -subalgebras of $\mathcal{A}$ is uniformly integrable.*

PROOF. It is a simple consequence of Lemma IV—2—4 of [9] and the inequality $E(|X| |\mathcal{F}|) \geq |E(X|\mathcal{F})|$ a.s.

Theorem 4.3. *Let B have RNP (see [3], [4]) and let $\{X_t, \mathcal{F}_t, t \in T\}$ be a B -valued martingale, where T is a directed set. For X_t to be of the form $X_t = E(X|\mathcal{F}_t)$ ($t \in T$) for a r.v. $X \in L^1(\Omega, \mathcal{A}, P, B)$, it is necessary and sufficient that it be uniformly integrable. In this case $\lim_{t \in T} X_t = E(X|\sigma\{\bigcup_{t \in T} \mathcal{F}_t\})$ in L^1 , where $\sigma\{\bigcup_{t \in T} \mathcal{F}_t\}$ denotes the σ -algebra generated by $\bigcup_{t \in T} \mathcal{F}_t$.*

PROOF. This theorem is an analogue of Prop. V—1—2 of [9]. Sufficiency. We prove it under the following weaker condition: "every increasing subsequence $\{X_{t_n}, n \in Z\}$ is uniformly integrable". The proof is the same as that of the above mentioned proposition if we use Chatterji's theorem (Prop. 4.2 (a) of [4]) on B -valued martingales instead of results on real martingales.

The necessity is a simple consequence of Lemma 4.2.

Theorem 4.4. Let $\{X_{\bar{m}}, \mathcal{F}_{\bar{m}}, \bar{m} \in Z^d\}$ be a B -valued martingale. Suppose that $E\{X_{(\bar{m}, \bar{n})} | \mathcal{F}_{(\bar{k}, \infty)}\} = X_{(\bar{k}, \bar{n})}$ if $\bar{k} \leq \bar{m}$, where

$$(\bar{m}, \bar{n}) = (m_1, m_2, \dots, m_j, n_1, n_2, \dots, n_i) \in Z^d,$$

$$(\bar{k}, \bar{n}) = (k_1, k_2, \dots, k_j, n_1, n_2, \dots, n_i) \in Z^d,$$

and $\mathcal{F}_{(\bar{k}, \infty)} = \sigma\left\{\bigcup_{\bar{n} \in Z^i} \mathcal{F}_{(\bar{k}, \bar{n})}\right\}$, $(i+j=d; i, j \geq 1)$.

Let B have RNP or let $X_{\bar{m}}$ be of the form $X_{\bar{m}} = E\{X | \mathcal{F}_{\bar{m}}\}$ for a r.v. $X \in L^1$. If $\sup_{\bar{m} \in Z^d} E\{|X_{\bar{m}}|(\log^+ |X_{\bar{m}}|)^{d-1}\} < \infty$, then $\lim_{\bar{m} \rightarrow \infty} X_{\bar{m}}$ exists a.s. (and in L^1 , if $d \geq 2$).

PROOF. If $d=1$ the theorem is an immediate consequence of propositions 4.1 and 4.2 (a) of [4]. Let $d > 1$. Assume the validity of the proposition for $d-1$. Lemma 4.1 and Theorem 4.3 imply that $X_{\bar{m}} = E\{X | \mathcal{F}_{\bar{m}}\}$ ($\bar{m} \in Z^d$) for a r.v. $X \in L^1$ in both cases and $\lim_{\bar{m} \rightarrow \infty} X_{\bar{m}} = X$ in L^1 . Let

$$Y_{\bar{n}} = (X_{(\bar{n}, 1)}, X_{(\bar{n}, 2)}, \dots) \quad (\bar{n} \in Z^{d-1}).$$

According to Prop. 4.1 of [4] $\lim_{j \rightarrow \infty} X_{(\bar{n}, j)}$ exists a.s., thus $Y_{\bar{n}} \in c(B)$. It follows from Lemma 4.1 that $Y_{\bar{n}} \in L^1$. Since the martingale property in $c(B)$ can be proved coordinate-wise $\{Y_{\bar{n}}, \mathcal{F}_{(\bar{n}, \infty)}, \bar{n} \in Z^{d-1}\}$ is a $c(B)$ -valued martingale.

To show that $Y_{\bar{n}} = E(Y | \mathcal{F}_{(\bar{n}, \infty)})$ ($\bar{n} \in Z^{d-1}$) for a suitable $Y \in L^1$ put

$$\begin{aligned} X_{(\infty, n)} &= E\{X | \mathcal{F}_{(\infty, n)}\} \\ &= \lim_{\bar{m} \rightarrow \infty} E\{X | \mathcal{F}_{(\bar{m}, n)}\} \quad \text{in } L^1 \\ &= \lim_{\bar{m} \rightarrow \infty} X_{(\bar{m}, n)} \quad \text{in } L^1, \end{aligned}$$

where

$$\mathcal{F}_{(\infty, n)} = \sigma\left\{\bigcup_{\bar{m} \in Z^{d-1}} \mathcal{F}_{(\bar{m}, n)}\right\}. \quad \text{Let } Y = (X_{(\infty, 1)}, X_{(\infty, 2)}, \dots).$$

Since the components of Y form martingale thus

$$\lim_{n \rightarrow \infty} X_{(\infty, n)} = E\left\{X | \sigma\left\{\bigcup_{n=1}^{\infty} \mathcal{F}_{(\infty, n)}\right\}\right\} \quad \text{a.s.}$$

that is $Y \in c(B)$. Using Lemma 4.1 we can show that $\sup_n |X_{(\infty, n)}| \in L^1$ thus $Y \in L^1$. It follows from our σ -algebra condition that

$$E(Y | \mathcal{F}_{(\bar{m}, \infty)}) = Y_{\bar{m}} \quad (\bar{m} \in Z^{d-1}).$$

Furthermore by the Cairoli's inequality

$$\sup_{\bar{m} \in Z^{d-1}} E\{\|Y_{\bar{m}}\|_c (\log^+ \|Y_{\bar{m}}\|_c)^{d-2}\} < \infty.$$

Hence the martingale $\{Y_{\bar{m}}, \mathcal{F}_{(\bar{m}, \infty)}, \bar{m} \in Z^{d-1}\}$ converges a.s. if the theorem is true for $d-1$. Thus

$$\lim_{\substack{\bar{n} \rightarrow \infty \\ \bar{n} \in Z^d}} X_{\bar{n}} = E\left\{X | \sigma\left\{\bigcup_{\bar{m} \in Z^d} \mathcal{F}_{\bar{m}}\right\}\right\} \quad \text{a.s.}$$

Lemma 4.5. Let $\{X_t, \mathcal{F}_t, t \in T\}$ be a B -valued martingale, where T is a directed set. Suppose that $X_t = E(X|\mathcal{F}_t)$, where $X \in L^r$ ($1 < r < \infty$) or the following conditions are satisfied: B has RNP and $\sup_{t \in T} E|X_t|^r < \infty$.

Then $\{X_t, t \in T\}$ is uniformly integrable and $X_t = E(X|\mathcal{F}_t)$, $X \in L^r$. Furthermore X_t converges in L^r .

PROOF. In the first case the assertion is trivial. In the second case it follows from Prop. 4.2 (b) of [4], Lemma V—1—1 of [9] and Theorem 4.3.

Theorem 4.6. Let $\{X_{\bar{m}}, \mathcal{F}_{\bar{m}}, \bar{m} \in Z^d\}$ be a B -valued martingale. Suppose that B has RNP or $X_{\bar{m}} = E(X|\mathcal{F}_{\bar{m}})$ ($\bar{m} \in Z^d$) for $X \in L^r$ ($1 < r < \infty$). Suppose that

$$E\{X_{(\bar{m}, \bar{n})}|\mathcal{F}_{(\bar{k}, \infty)}\} = X_{(\bar{k}, \bar{n})} \quad \text{if } \bar{k} \equiv \bar{m},$$

where $(\bar{m}, \bar{n})$, $(\bar{k}, \bar{n})$ and $\mathcal{F}_{(\bar{k}, \infty)}$ are defined in Theorem 4.4.

If $\sup_{\bar{m} \in Z^d} E|X_{\bar{m}}|^r < \infty$, then $\lim_{\bar{m} \rightarrow \infty} X_{\bar{m}}$ exists a.s. and in L^r .

The proof is a modification of the proof of Theorem 4.4.

5. A strong law of large numbers

Lemma 5.1. Let $Y \in L^1(\Omega, \mathcal{A}, P, B)$. Let ξ and η be random objects. If (Y, ξ) and η are independent, then

$$E(Y|\xi, \eta) = E(Y|\xi) \quad \text{a.s.}$$

PROOF. It follows from the scalar case by the help of linear functionals.

Lemma 5.2. Let $X_1, X_2, \dots, X_n$ be independent identically distributed (i.i.d.) B -valued Bochner integrable random variables. If $S_n = \sum_{j=1}^n X_j$, then

$$E(X_k|S_n) = \frac{S_n}{n} \quad \text{a.s.} \quad (k = 1, 2, \dots, n).$$

If $\mathcal{G}_n = \sigma\{S_n, S_{n+1}, \dots\}$ ($n = 1, 2, \dots$), then $\left\{\frac{S_n}{n}, \mathcal{G}_n, n \geq 1\right\}$ is a B -valued reversed martingale.

PROOF. One can use the method of 7.8.1 Lemma and 7.8.3 Theorem of [1] and Lemma 5.1.

Theorem 5.3. Let $X_{\bar{k}}$ ($\bar{k} \in Z^d, d \geq 1$) be a sequence of i.i.d. B -valued random variables. Then

$$W_{\bar{n}} = \frac{S_{\bar{n}}}{|\bar{n}|} = \frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} X_{\bar{k}}$$

converges a.s. if and only if $E\{|X_1|(\log^+ |X_1|)^{d-1}\} < \infty$.

PROOF. 1. Let $E\{|X_1|(\log^+ |X_1|)^{d-1}\} < \infty$. Suppose that $E(X_1) = 0$. For $d = 1$ our theorem is equivalent to Prop. 3.1 of [4]. We proceed by induction. Let $d > 1$

and suppose that the theorem is true for $d-1$. Let $\bar{m} \in Z^{d-1}$ be fixed, let $V_{\bar{m}}^k = \sum_{j=1}^k X_{(\bar{m}, j)}$, $\mathcal{G}_{\bar{m}}^k = \sigma\{V_{\bar{m}}^k, V_{\bar{m}}^{k+1}, \dots\}$, ($k=1, 2, \dots$). From the preceding lemma $\left\{\frac{V_{\bar{m}}^k}{k}, \mathcal{G}_{\bar{m}}^k, k \geq 1\right\}$ is a reversed martingale. Since the sequence of the components of the vector

$$Y_{\bar{m}} = \left(\frac{V_{\bar{m}}^1}{1}, \frac{V_{\bar{m}}^2}{2}, \frac{V_{\bar{m}}^3}{3}, \dots\right)$$

converges to 0 a.s. it follows that $Y_{\bar{m}}$ is a $c_0(B)$ -valued r.v. The random variables $Y_{\bar{m}}$ ($\bar{m} \in Z^{d-1}$) are i.i.d. and their common expectation is $0 \in c_0(B)$. Since the components of $Y_{\bar{1}}$ ($\bar{1} \in Z^{d-1}$) form a reversed martingale thus $Y_{\bar{1}} \in L(\log^+ L)^{d-2}$. By the induction assumption, we then have that

$$\lim_{\substack{\bar{n} \rightarrow \infty \\ \bar{n} \in Z^{d-1}}} \frac{1}{|\bar{n}|} \sum_{\substack{\bar{m} \leq \bar{n} \\ \bar{m} \in Z^{d-1}}} Y_{\bar{m}} = 0 \quad \text{a.s. (in } c_0(B)).$$

But $\frac{1}{|(\bar{n}, j)|} S_{(\bar{n}, j)} = \frac{1}{|\bar{n}|} \left(\sum_{\bar{m} \leq \bar{n}} Y_{\bar{m}}\right)_j$, where $\bar{n}, \bar{m} \in Z^{d-1}$, $j \in Z$, and $\left(\sum_{\bar{m} \leq \bar{n}} Y_{\bar{m}}\right)_j$ denotes the j -th component of the vector $\sum_{\bar{m} \leq \bar{n}} Y_{\bar{m}}$. From here we have that

$$\lim_{(\bar{n}, j) \rightarrow \infty} \frac{1}{|(\bar{n}, j)|} S_{(\bar{n}, j)} = 0 \quad \text{a.s.}$$

2. Let $E\{|X_{\bar{1}}|(\log^+ |X_{\bar{1}}|)^{d-1}\} = \infty$. From [12] (p. 165) it follows that for $A > 0$

$$P\{|W_k| > A \text{ occurs for arbitrary large indices}\} = 1.$$

Thus the sequence W_k does not converge a.s.

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