Relations preserving semi-continuities of relations

By Á. MÜNNICH and Á. SZÁZ (Debrecen)

In this note, we give some sufficient conditions under which homeomorphisms or more general realtions of product spaces preserve lower or upper semi-continuity of certain relations.

Definition 1. Let X, Y, Z, W be topological spaces and $\mathscr{S}$ be a family of relations in $X\times Y$. A relation F from $X\times Y$ into $Z\times W$ is said to preserve lower (upper) semi-continuity of relations belonging to $\mathscr{S}$ if $F(S)$ is a lower (upper) semi-continuous relation in $Z\times W$ for each lower (upper) semi-continuous relation $S\in\mathscr{S}$.

Remark 1. Concerning lower and upper semi-continuity of relations, which are usually called multifunctions in this context, the reader is referred to [7]. Note that a relation S in $X\times Y$ have to be called lower (upper) semi-continuous if it is a lower (upper) semi-continuous relation from its domain D_s, which is considered as a subspace of X, into Y in the sense of [7]. (See also [2].)

Theorem 1. *Let X, Y, Z, W be topological spaces, Φ an open (closed) relation from X into Z and Ψ a lower (upper) semi-continuous relation from Y into W. Then the relation F from $X\times Y$ into $Z\times W$ defined by $F(x, y)=\Phi(x)\times\Psi(y)$ preserves lower (upper) semi-continuity of all relations from X into Y.*

PROOF. Let S be a lower (upper) semi-continuous relation from X into Y and $T=F(S)$. Then a straightforward computation shows that $T=\Psi\circ S\circ\Phi^{-1}$. Thus, we have $T^{-1}(V)=\Phi\big(S^{-1}(\Psi^{-1}(V))\big)$ for any $V\subset W$. Hence, it is clear that $T^{-1}(V)$ is open (closed) in Z if V is open (closed) in W, which is a little more than the lower (upper) semi-continuity of T, since the domain of T may be a proper subset of Z.

Corollary 1.1. *Let X, Y, Z, W be topological spaces, φ a continuous function from Z onto X and Ψ a lower (upper) semi-continuous relation from Y into W. Then the relation F from $X\times Y$ into $Z\times W$ defined by $F(x, y)=\varphi^{-1}(x)\times\Psi(y)$ preserves lower (upper) semi-continuity of all relations in $X\times Y$.*

PROOF. Let S be a lower (upper) semi-continuous relation in $X\times Y$ with domain D. Since φ is a continuous function from Z onto X, the restriction $\Phi=\varphi^{-1}|D$ is an open and closed relation from D onto $\varphi^{-1}(D)$. Thus, by Theorem 1, $F(S)$ is a lower (upper) semi-continuous relation in $\varphi^{-1}(D)\times W$, and thus also in $Z\times W$.

Corollary 1.2. *Let X, Y, Z, W be topological spaces, φ a homeomorphism of X onto Z and ψ a homeomorphism of Y onto W. Then the function f defined on $X\times Y$*

by $f(x, y)=(\varphi(x), \psi(y))$ *is a homeomorphism of* $X\times Y$ *onto* $Z\times W$ *which preserves both lower and upper semi-continuity of all relations in* $X\times Y$.

Example 1. Let f be the function defined on $[0, 1]^2$ by $f(x, y)=(y, x)$. Then f is a homeomorphism of $[0, 1]^2$ onto itself which preserve neither lower nor upper semi-continuity of all relations from $[0, 1]$ into itself.

To see this, define the relation S from $[0, 1]$ into itself such that $S(x)=[1/2, 1]$ if $0\leqq x\leqq 1/2$ and $S(x)=[0, 1]$ if $1/2<x\leqq 1$. Then S is lower semi-continuous, but $f(S)=S^{-1}$ is not lower semi-continuous, and S^{-1} is upper semi-continuous, but $f(S^{-1})=S$ is not upper semi-continuous.

Namely, $S^{-1}(A)$ is open and $(S^{-1})^{-1}(A)=S(A)$ is closed in $[0, 1]$ for any $A\subset[0, 1]$, but $S^{-1}(0)=]1/2, 1]$ is not closed and $(S^{-1})^{-1}([0, 1/2[)=S([0, 1/2[)=$ $=[1/2, 1]$ is not open in $[0, 1]$.

Theorem 2. *Let* X, Y, Z, W *be topological spaces and* F *be a lower semi-continuous open relation from* $X\times Y$ *into* $Z\times W$ *such that* $F(\{x\}\times Y)\subset\{z\}\times W$ *whenever* $(z, w)\in F(x, y)$. *Then* F *preserves lower semi-continuity of all relations from* X *into* Y.

Proof. Let S be a lower semi-continuous relation from X into Y and $T=F(S)$. To prove that T is also lower semi-continuous, suppose that $(z_0, w_0)\in T$ and Ω is a neighborhood of w_0 in W. Pick $(x_0, y_0)\in S$ such that $(z_0, w_0)\in F(x_0, y_0)$. Since F is lower semi-continuous, there are neighborhoods U and V of x_0 and y_0 in X and Y, respectively, such that $U\times V\subset F^{-1}(Z\times\Omega)$. Furthermore, since S is lower semi-continuous, there is a neighborhood $U_1\subset U$ of x_0 in X such that $U_1\subset S^{-1}(V)$. Finally, since F is an open relation, there are neighborhoods Σ and Ω_1 of z_0 and w_0 in Z and W, respectively, such that $\Sigma\times\Omega_1\subset F(U_1\times V)$. Thus, it remains only to prove that $\Sigma\subset T^{-1}(\Omega)$, which shows that T is lower semi-continuous at z_0.

If $z\in\Sigma$, then $(z, w_0)\in\Sigma\times\Omega_1\subset F(U_1\times V)$. Thus, there exist $x\in U_1$ and $y\in V$ such that $(z, w_0)\in F(x, y)$. Since $x\in U_1\subset S^{-1}(V)$, there exists $y_1\in V$ such that $x\in S^{-1}(y_1)$, i.e., $(x, y_1)\in S$. Moreover, since $(x, y_1)\in U\times V\subset F^{-1}(Z\times\Omega)$, there exist $z_1\in Z$ and $w_1\in\Omega$ such that $(x, y_1)\in F^{-1}(z_1, w_1)$, i.e., $(z_1, w_1)\in F(x, y_1)$. Hence, since $F(\{x\}\times Y)\subset\{z\}\times W$, it follows that $z_1=z$. Consequently, we have $(z, w_1)=(z_1, w_1)\in F(x, y_1)\subset F(S)=T$, and hence $z\in T^{-1}(w_1)\subset T^{-1}(\Omega)$.

Remark 2. For a relation F from $X\times Y$ into $Z\times W$, the following properties are pairwise equivalent:

(i) $F(\{x\}\times Y)\subset\{z\}\times W$ whenever $(z, w)\in F(x, y)$;

(ii) $\varrho\circ F\circ p^{-1}$, where p and ϱ denote the projections of $X\times Y$ onto X and $Z\times W$ onto Z, respectively, is a function;

(iii) there exist a function φ from X into Z and a relation Ψ_x from Y into W for each $x\in X$ such that $F(x, y)=\varphi(x)\times\Psi_x(y)$ for all $(x, y)\in X\times Y$.

However, we could not use this fact to simplify the above proof.

Corollary 2.1. *Let* X, Y, Z, W *be topological spaces and* f *be a continuous open mapping of* $Z\times W$ *onto* $X\times Y$ *such that* $f^{-1}(\{x\}\times Y)=\{z\}\times W$ *whenever* $f(z, w)=$ $=(x, y)$. *Then the relation* f^{-1} *from* $X\times Y$ *onto* $Z\times W$ *preserves lower semi-continuity of all relations in* $X\times Y$.

Proof. Let S be a lower semi-continuous relation in $X\times Y$ with domain D. Then, by the assumptions on f, it is clear that $f^{-1}(D\times Y)=E\times W$ for some $E\subset Z$ and the restriction $F=f^{-1}|D\times Y$ is a lower semi-continuous open relation from

$D \times Y$ onto $E \times W$ such that $F(\{x\} \times Y) = \{z\} \times W$ whenever $(z, w) \in F(x, y)$. Thus, by Theorem 2, $f^{-1}(S) = F(S)$ is a lower semi-continuous relation in $E \times W$, and thus also in $Z \times W$.

Corollary 2.2. *Let X, Y, Z, W be topological spaces and f be a homeomorphism of $X \times Y$ onto $Z \times W$ such that $f(\{x\} \times Y) = \{z\} \times W$ whenever $f(x, y) = (z, w)$. Then f preserves lower semi-continuity of all relations in $X \times Y$.*

Example 2. Let f be the function defined on $[0, 1]^2$ by $f(x, y) = (x, \psi_x(y))$, where $\psi_x(y) = (x + 1/2)y$ if $0 \leqq y \leqq 1/2$ and $\psi_x(y) = (3/2 - x)y + x - 1/2$ if $1/2 < y \leqq 1$. Then f is a homeomorphism of $[0, 1]^2$ onto itself such that $f(\{x\} \times [0, 1]) = \{x\} \times [0, 1]$ for all $0 \leqq x \leqq 1$, but f does not preserve upper semi-continuity of all relations from $[0, 1]$ into itself.

To see this, define the relation S from $[0, 1]$ into itself by $S(x) = [0, 1] \setminus \{1/2\}$. Then S is upper semi-continuous, but $T = f(S)$ is not upper semi-continuous. Namely, we have $T(x) = [0, 1] \setminus \{(1/2)x + 1/4\}$ for all $0 \leqq x \leqq 1$, whence it is clear that for any $x_0 \in [0, 1]$, $T(x_0)$ is a neighborhood of itself in $[0, 1]$, but $T(x) \not\subset T(x_0)$ for each $x \in [0, 1] \setminus \{x_0\}$, and thus T can not be upper semi-continuous at x_0.

Theorem 3. *Let X, Y, Z, W be topological spaces such that Y is regular and W is compact, and let F be a closed relation from $X \times Y$ into $Z \times W$. Then F preserves upper semi-continuity of all closed-valued relations from X into Y.*

PROOF. Let S be an upper semi-continuous closed-valued relation from X into Y and $T = F(S)$. Then, by [2, (3)], S is closed in $X \times Y$. Thus, by the assumption, T is closed in $Z \times W$. Hence, by [2, (7)], T is also upper semi-continuous.

Corollary 3.1. *Let X, Y, Z, W be topological spaces such that Y is regular and W is compact, and let f be a continuous function from $Z \times W$ onto $X \times Y$ such that $f^{-1}(\{x\} \times Y) = \{z\} \times W$ whenever $f(z, w) = (x, y)$. Then the relation f^{-1} from $X \times Y$ onto $Z \times W$ preserves upper semicontinuity of all closed-valued relations in $X \times Y$.*

PROOF. A similar argument as in the proof of Corollary 2.1 can be applied.

Corollary 3.2. *Let X, Y, Z, W be topological spaces such that Y is regular and W is compact, and let f be a homeomorphism of $X \times Y$ onto $Z \times W$ such that $f(\{x\} \times Y) = \{z\} \times W$ whenever $f(x, y) = (z, w)$. Then f preserves both lower and upper semi-continuity of all closed-valued relations in $X \times Y$.*

Remark 3. Note that, by Example 2, closed-valuedness is an essential requirement in the above assertions.

References

[1] W. M. FLEISCHMAN *(editor)*, Set-Valued Mappings, Selections, and Topological Properties of 2^x, *Lecture Notes in Math.* **171**, *Berlin*, 1970.

[2] S. P. FRANKLIN, Closed and image-closed relations, *Pacific J. Math.* **19** (1966), 433—439.

[3] J. L. KELLEY, General Topology, *New York*, 1955.

[4] B. L. MCALLISTER, Hyperspaces and Multifunctions, The first half century (1900—1950), *Nieuw Arch. Wisk.* **26** (1978), 309—329.

[5] Á. MÜNNICH and Á. SZÁZ, An alternative theorem for continuous relations and its applications, *Publ. Inst. Math. Beograd* **33** (1983), 163—168.

[6] Á. MÜNNICH and Á. SZÁZ, Regularity, normality, paracompactness and semicontinuity, *Math. Nachr.* **113** (1983).

[7] R. E. SMITHSON, Multifunctions, *Nieuw Arch. Wisk.* **20** (1972), 31—53.

(Received October 13, 1980.)