

On the measurable homomorphisms

By I. G. KALMÁR (Debrecen)

In [3] R. SIKORSKI examined the necessary and sufficient conditions that every σ -homomorphism of a σ -field $\mathcal{B}$ (of subsets of Y) into a σ -quotient algebra $\mathcal{A}|\mathcal{I}$ be induced by a function of X into Y , where $(X, \mathcal{A})$ is a measurable space. According to the definition of Sikorski we say that a σ -homomorphism x of $\mathcal{B}$ into a σ -quotient algebra $\mathcal{A}|\mathcal{I}$ is induced by a function f of X into Y if $f^{-1}(E) \in \mathcal{A}$ and $[f^{-1}(E)] = x(E)$ for all $E \in \mathcal{B}$. $[f^{-1}(E)]$ denotes here the element of $\mathcal{A}|\mathcal{I}$ which contains the set $f^{-1}(E)$. This examination was done by Sikorski under some more special properties of $\mathcal{B}$, too. We prove the analogous theorems exchanged $\mathcal{A}|\mathcal{I}$ by an abstract σ -orthomodular poset.

A partially ordered set (poset) $(\mathcal{L}, \leq)$ is said to be an σ -orthomodular poset if (i) there are two elements 0 and 1 in $\mathcal{L}$ such that $0 \leq a \leq 1$ for all $a \in \mathcal{L}$, (ii) there exists a mapping $' : a \rightarrow a'$ of $\mathcal{L}$ into itself such that $(a^\perp)^\perp = a$, $a \vee a^\perp = 1$ for all $a \in \mathcal{L}$, and if $a \leq b$, then $b^\perp \leq a^\perp$, (iii) if $a_i \leq a_j^\perp$, then $\bigvee_{i=1}^{\infty} a_i$ exists in $\mathcal{L}$, (iv) if $a \leq b$, then there is $d \in \mathcal{L}$ such that $a \leq d^\perp$ and $a \vee d = b$.

Let $(Y, \mathcal{B})$ be an arbitrary measurable space and suppose $\mathcal{L} = \mathcal{L}(\vee, \wedge, ', 0, 1)$ is a σ -orthomodular poset with first and last elements 0, 1 respectively. Then for all σ -homomorphism $x : \mathcal{B} \rightarrow \mathcal{L}$ $x(\mathcal{B}) = \{x(b), b \in \mathcal{B}\}$ is a sub σ -Boolean algebra of $\mathcal{L}$.

Definition 1. The σ -homomorphism x is called *measurable* if for every such measurable space $(\Omega, \mathcal{F})$ that there exists a σ -homomorphism h of $\mathcal{F}$ onto $x(\mathcal{B})$ one can find also an $\mathcal{F}$ -measurable function $f : \Omega \rightarrow Y$ so that

$$x(E) = h(f^{-1}(E)) \quad \text{for all } E \in \mathcal{B}.$$

For brevity let $\text{Hom}(\mathcal{B}, \mathcal{L})$ denote the collection of all σ -homomorphism of $\mathcal{B}$ into $\mathcal{L}$.

Theorem 1. Let $(Y, \mathcal{B})$ be a measurable space. Then for every σ -orthomodular poset $\mathcal{L}$ every $x \in \text{Hom}(\mathcal{B}, \mathcal{L})$ is measurable if and only if for every σ -quotient algebra $\mathcal{F}|\mathcal{I}$ (of a set Ω) every element of $\text{Hom}(\mathcal{B}, \mathcal{F}|\mathcal{I})$ is induced by a function of Ω into Y .

PROOF. Let $(\Omega, \mathcal{F})$, h be the same as in Definition 1. Suppose that every σ -homomorphism of $\mathcal{B}$ into every σ -quotient algebra $\mathcal{F}|\mathcal{I}$ (of the set Ω) is induced by a function from Ω into Y . Let $x(\mathcal{B})$ be isomorphic to a σ -quotient algebra $\mathcal{F}|\mathcal{I}$, where $\mathcal{I}$ is a σ -ideal of $\mathcal{F}$. (The isomorphism: $l : \mathcal{F}|\mathcal{I} \rightarrow x(\mathcal{B})$). Let us denote the natural σ -homomorphism of $\mathcal{F}$ onto $\mathcal{F}|\mathcal{I}$ by h' (See Fig. 1.) Trivially $h = l \circ h'$.

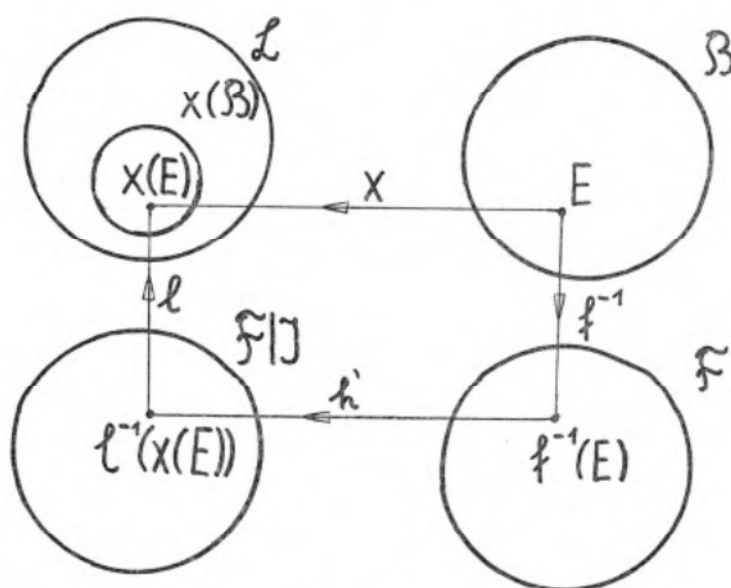


Fig. 1

Now $l^{-1} \circ x$ is a σ -homomorphism of $\mathcal{B}$ onto $\mathcal{F}|\mathcal{I}$, so there exists an $\mathcal{F}$ -measurable mapping $f: \Omega \rightarrow Y$ such that

$$l^{-1}(x(E)) = h'(f^{-1}(E)) \quad \text{for all } E \in \mathcal{B}. \quad \text{Hence}$$

$$x(E) = l \circ h'(f^{-1}(E)) = h(f^{-1}(E)).$$

Necessity. Let $x \in \text{Hom}(\mathcal{B}, \mathcal{L})$ be measurable for every σ -orthomodular poset $\mathcal{L}$ and for all $x \in \text{Hom}(\mathcal{B}, \mathcal{L})$. Suppose $(X, \mathcal{G})$ is a measurable space, $\mathcal{G}|\mathcal{I}$ is a σ -quotient algebra and $y \in \text{Hom}(\mathcal{B}, \mathcal{G}|\mathcal{I})$. Then there exists a $\mathcal{G}$ -measurable function g of X into Y such that

$$n(g^{-1}(E)) = y(E) \quad \text{for all } E \in \mathcal{B},$$

where n is the natural homomorphism of $\mathcal{G}$ onto $\mathcal{G}|\mathcal{I}$. So y is measurable. Q.e.d.

Let us consider Theorems 3.2, 4.3 and 4.4 of R. SIKORSKI [3]. With the help of our Theorem 1 we have the following three theorems.

Theorem 2. Let $\emptyset \neq E \subseteq R$. In order that every σ -homomorphism x of the Borel σ -field $\mathcal{B}(E)$ into an arbitrary σ -orthomodular poset $\mathcal{L}$ be measurable, it is necessary and sufficient that $E \in \mathcal{B}(R)$, where $\mathcal{B}(R)$ is the Borel σ -field of the real line R .

Theorem 3. Suppose that there exists an enumerable sequence $\{E_i\}$ in $\mathcal{B}$ such that $\mathcal{B}$ is the least σ -field containing all the sets E_i ($i=1, 2, \dots$). Then in order that every $x \in \text{Hom}(\mathcal{B}, \mathcal{L})$ be measurable for every σ -orthomodular poset $\mathcal{L}$, it is sufficient and necessary that $\mathcal{B}$ be isomorphic to the Borel σ -field $\mathcal{B}(\tau)$ of a Borel space τ .

Definition 2. The topological space τ is a Borel space iff $\mathcal{B}$ is homeomorphic to a Borel set of the Hilbert cube.

Theorem 4. Let Y be a separable metric space and $\mathcal{B}(Y)$ its Borel σ -field. Then in order that every $x \in \text{Hom}(\mathcal{B}, \mathcal{L})$ be measurable for arbitrary σ -orthomodular poset $\mathcal{L}$, it is necessary and sufficient that $\mathcal{B}$ be a Borel space.

Finally we shall prove two corollaries of our theorems*.

Corollary 1. Let Y be a separable complete metric space and denote $\mathcal{B}(Y)$ its Borel sets. Let $\mathcal{L}$ be a σ -orthomodular poset. Then every σ -homomorphism x from $\mathcal{B}(Y)$ into $\mathcal{L}$ is measurable.

PROOF. Since every separable complete metric space is a Borel space (see [1], IX. 6.1. Corollary 1. of Theorem 1., for example), our statement follows from Theorem 4.

In theoretical physics it is almost the case that a measurable quantity takes its values in a locally compact second countable Hausdorff space. This fact makes the following theorem interesting.

Corollary 2. If Y is locally compact second countable Hausdorff space, then every $x \in \text{Hom}(\mathcal{B}(Y), \mathcal{L})$ is measurable for every σ -orthomodular poset $\mathcal{L}$.

PROOF. It follows from [1], IX. 2.9. Corollary of Proposition 16., and IX. 6.1. Corollary of Proposition 2. that every locally compact second countable Hausdorff space is metrisable and what is more, it is a separable complete metric space. So our Corollary 1. implies Corollary 2.

References

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I. G. KALMÁR
DEPT. OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H—4010 DEBRECEN 10, PF. 12
HUNGARY

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