

The neutrix distribution product $x_+^\lambda \circ x_-^\mu$

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The ordinary summable function x_+^λ is defined for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & \text{for } x > 0, \\ 0, & \text{for } x < 0. \end{cases}$$

The distribution x_+^λ is defined for $-r-1 < \lambda < -r$ and $r=1, 2, \dots$ by

$$x_+^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} \frac{d^r}{dx^r} x_+^{\lambda+r},$$

where Γ denotes the gamma function.

The ordinary summable function x_-^μ is defined for $\mu > -1$ by

$$x_-^\mu = \begin{cases} |x|^\mu, & \text{for } x < 0, \\ 0, & \text{for } x > 0. \end{cases}$$

The distribution x_-^μ is defined for $-r < \mu < -r$ and $r=1, 2, \dots$ by

$$x_-^\mu = \frac{(-1)^r \Gamma(\mu+1)}{\Gamma(\mu+r+1)} \frac{d^r}{dx^r} x_-^{\mu+r}.$$

The following definition for the product fg of two distributions f and g was given in [2].

Definition 1. Let f and g be two distributions for which on the open interval (a, b) , f is the r -th derivative of an ordinary summable function F in $L^p(a, b)$ and $g^{(r)}$ is an ordinary summable function in $L^q(a, b)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = gf = \sum_{i=0}^r r c_i (-1)^i [F g^{(i)}]^{(r-i)},$$

where

$$r c_i = \frac{r!}{i! (r-i)!}.$$

The following theorem follows immediately from this definition.

Theorem 1. *The product $x_+^\lambda x_-^\mu$ exists and*

$$x_+^\lambda x_-^\mu = 0$$

for $\lambda + \mu > -1$.

Now let ϱ be a fixed infinitely differentiable function having the properties

- (i) $\varrho(x) = 0$, for $|x| \geq 1$,
- (ii) $\varrho(x) \geq 0$,
- (iii) $\varrho(x) = \varrho(-x)$,
- (iv) $\int_{-1}^1 \varrho(x) dx = 1$.

Define the function δ_n by

$$\delta_n(x) = n\varrho(nx)$$

for $n = 1, 2, \dots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac deltafunction δ . For an arbitrary distribution f define the function f_n by

$$f_n(x) = f * \delta_n = \int_{-1/n}^{1/n} f(x-t)\delta_n(t) dt$$

for $n = 1, 2, \dots$. It is obvious that $\{f_n\}$ is a sequence of infinitely differentiable functions converging to f .

The following definition was given in [3] and extends definition 1 so that a wider class of distribution products could be defined, the resulting product not necessarily being commutative.

Definition 2. Let f and g be arbitrary distributions and let

$$g_n = g * \delta_n.$$

We say that the product $f \circ g$ of f and g exists and is equal to h on the open interval (a, b) if

$$\lim_{n \rightarrow \infty} (fg_n, \varphi) = \lim_{n \rightarrow \infty} (f, g_n \varphi) = (h, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b) .

That definition 2 is an extension of definition 1 is shown by the following theorem which was proved in [3].

Theorem 2. *Let f and g be distributions. If the product fg exists on the open interval (a, b) then the product $f \circ g$ and $g \circ f$ exist and*

$$f \circ g = g \circ f = fg$$

on this interval.

The next definition was given by VAN DER CORPUT, see [1].

Definition 3. A neutrix N is a commutative additive group of functions $v(\xi)$ defined on a domain N' with values in an additive group N'' , where further if for some v in N , $v(\xi) = \gamma$ for all ξ in N' , then $\gamma = 0$. The functions in N are called negli-

gible functions. Now let N' be a set contained in a topological space with a limit point b which does not belong to N' . If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi) - \beta$ is negligible in N , then β is called the neutrix limit or N -limit of f as ξ tends to b and we write

$$N\text{-}\lim_{\xi \rightarrow b} f(\xi) = \beta,$$

where the limit β must be unique if it exists.

The following definition was given in [4].

Definition 4. Let f and g be arbitrary distributions and let

$$g_n = g * \delta_n.$$

We say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} (fg_n, \varphi) = N\text{-}\lim_{n \rightarrow \infty} (f, g_n \varphi) = (h, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b) , where N is the neutrix having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' the real numbers with negligible functions linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \dots$ and all functions $f(n)$ for which $\lim_{n \rightarrow \infty} f(n) = 0$.

The following theorems were given in [4].

Theorem 3. Let f and g be distributions for which the product $f \circ g$ exists by definition 2. Then the neutrix product $f \circ g$ exists and defines the same distribution.

Theorem 4. Let f and g be distributions and suppose that the neutrix products $f \circ g$ and $f \circ g'$ exist on the open interval (a, b) . Then the neutrix product $f' \circ g$ exists on the interval (a, b) and

$$(f \circ g)' = f' \circ g + f \circ g'$$

on the interval (a, b) .

Theorem 5. The neutrix products $x_+^\lambda \circ x_-^{\lambda-r}$ and $x_-^{\lambda-r} \circ x_+^\lambda$ exist and

$$x_+^\lambda \circ x_-^{\lambda-r} = x_-^{\lambda-r} \circ x_+^\lambda = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!} \delta^{r-1}(x)$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $r = 1, 2, \dots$

We now prove the following theorem.

Theorem 6. The neutrix products $x_+^\lambda \circ x_-^\mu$ and $x_-^\lambda \circ x_+^\mu$ exist and

$$(1) \quad x_+^\lambda \circ x_-^\mu = x_-^\lambda \circ x_+^\mu = 0$$

for $\lambda + \mu \neq -1, -2, \dots$

PROOF. We first of all note that if $\lambda + \mu > -1$ then equations (1) follow from theorems 1, 2, and 3.

Now suppose that $\lambda > -1$ and $-k < \mu < -k+1$, where k is a positive integer. Then

$$x_-^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+k)} \frac{d^{k-1}}{dx^{k-1}} x_-^{\mu+k-1},$$

$x_-^{\mu+k-1}$ being an ordinary summable function and so

$$(x_-^\mu)_n = x_-^\mu * \delta_n = \frac{\Gamma(\mu+1)}{\Gamma(\mu+k)} \int_x^{1/n} (t-x)^{\mu+k-1} \delta_n^{(k-1)}(t) dt.$$

Thus

$$\begin{aligned} & \frac{\Gamma(\mu+k)}{\Gamma(\mu+1)} \int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^m dx = \\ &= \int_0^{1/n} x^{\lambda+n} \int_x^{1/n} (t-x)^{\mu+k-1} \delta_n^{(k-1)}(t) dt dx \\ &= \int_0^{1/n} \delta_n^{(k-1)}(t) \int_0^t x^{\lambda+m} (t-x)^{\mu+k-1} dx dt \\ &= \int_0^{1/n} t^{\lambda+\mu+m+k} \delta_n^{(k-1)}(t) \int_0^1 v^{\lambda+m} (1-v)^{\mu+k-1} dv dt \\ &= B(\lambda+m+1, \mu+k) \int_0^{1/n} t^{\lambda+\mu+m+k} \delta_n^{(k-1)}(t) dt, \end{aligned}$$

where the substitution $x=tv$ has been made and B denotes the beta function.

Making the substitution $nt=s$ we have

$$\int_0^{1/n} t^{\lambda+\mu+m+k} \delta_n^{(k-1)}(t) dt = n^{-\lambda-\mu-m-1} \int_0^1 s^{\lambda+\mu+m+k} \varrho^{(k-1)}(s) ds$$

and it follows that the functions

$$\int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^m dx$$

are negligible, or zero, for $m=0, 1, 2, \dots$ and $\lambda+\mu \neq -1, -2, \dots$. Further, choosing a positive integer $p > -\lambda-\mu-1$, we see that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |x_+^\lambda (x_-^\mu)_n x^p| dx = 0.$$

Now let φ be an arbitrary test function. Then

$$(2) \quad \varphi(x) = \sum_{m=0}^{p-1} \frac{x^m}{m!} \varphi^{(m)}(0) + \frac{x^p}{p!} \varphi^{(p)}(\xi x)$$

where $0 \leq \xi \leq 1$ and so

$$(x_+^\lambda, (x_-^\mu)_n \varphi(x)) = \sum_{m=0}^{p-1} \frac{\varphi^{(m)}(0)}{m!} \int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^m dx + \frac{1}{p!} \int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^p \varphi^{(p)}(\xi x) dx.$$

Since

$$\left| \int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^p \varphi^{(p)}(\xi x) dx \right| \leq \sup_x \{|\varphi^{(p)}(x)|\} \cdot \int_{-\infty}^{\infty} |x_+^\lambda (x_-^\mu)_n x^p| dx$$

it follows from what we have just proved that

$$N\text{-}\lim_{n \rightarrow \infty} (x_+^\lambda, (x_-^\mu)_n \varphi(x)) = 0 = (0, \varphi).$$

Thus

$$(3) \quad x_+^\lambda \circ x_-^\mu = 0$$

when $\lambda > -1$ and $\mu, \lambda + \mu \neq -1, -2, \dots$.

Now assume that equation (3) holds when $-k < \lambda < -k+1$ and $\mu, \lambda + \mu \neq -1, -2, \dots$, where k is a positive integer. It follows from theorem 4 that

$$(x_+^\lambda \circ x_-^\mu)' = 0 = \lambda x_+^{\lambda-1} \circ x_-^\mu - \mu x_+^\lambda \circ x_-^{\mu-1},$$

provided that $\mu \neq 0$ and it follows from our assumption that

$$(4) \quad x_+^{\lambda-1} \circ x_-^\mu = 0$$

when $-k < \lambda < -k+1$ and $\mu-1, \lambda+\mu \neq -1, -2, \dots$. To deal with the particular case $\mu=0$ we notice that the derivative of x_-^0 is $-\delta(x)$. Now it is easily proved that

$$x_+^\lambda \circ \delta(x) = 0$$

for $\lambda \neq 0, -1, -2, \dots$ and this result was in fact given in [5]. It follows from theorem 4 that

$$(x_+^\lambda \circ x_-^0)' = 0 = \lambda x_+^{\lambda-1} \circ x_-^0 - x_+^\lambda \circ \delta(x)$$

and equation (4) follows from our assumption when $-k < \lambda < -k+1$ and $\mu=0$. This establishes equation (3) by induction when $\lambda, \mu, \lambda+\mu \neq -1, -2, \dots$.

We will now consider the neutrix product $x_+^\lambda \circ x_-^{-r}$ for $\lambda, \lambda-r \neq -1, -2, \dots$ and $r=1, 2, \dots$. By definition

$$\frac{d^r}{dx^r} \ln x_- = -(r-1)! x_-^{-r},$$

$\ln x_-$ being an ordinary summable function and so

$$(x_-^{-r})_n = x_-^{-r} * \delta_n = -\frac{1}{(r-1)!} \int_x^{1/n} \ln(t-x) \delta_n^{(r)}(t) dt.$$

Thus if $\lambda > -1$

$$\begin{aligned}
 -(r-1)! \int_{-\infty}^{\infty} x_+^{\lambda} (x_-^{-r})_n x^m dx &= \int_0^{1/n} x^{\lambda+m} \int_x^{1/n} \ln(t-x) \delta_n^{(r)}(t) dt dx \\
 &= \int_0^{1/n} \delta_n^{(r)}(t) \int_0^t x^{\lambda+m} \ln(t-x) dx dt \\
 &= \int_0^{1/n} t^{\lambda+m+1} \delta_n^{(r)}(t) \int_0^1 v^{\lambda+m} [\ln(1-v) + \ln t] dv dt \\
 &= \left\{ \int_0^1 v^{\lambda+m} \ln(1-v) dv \right\} \left\{ \int_0^{1/n} t^{\lambda+m+1} \delta_n^{(r)}(t) dt \right\} + \frac{1}{\lambda+m+1} \int_0^{1/n} t^{\lambda+m+1} \ln t \delta_n^{(r)}(t) dt,
 \end{aligned}$$

where the substitution $x=tv$ has been made.

Making the substitution $nt=s$ we see that the functions

$$\int_{-\infty}^{\infty} x^{\lambda} (x_-^{-r})_n x^m dx$$

are negligible, or zero, for $n=0, 1, 2, \dots$ and $\lambda-r \neq -1, -2, \dots$. Further, choosing a positive integer $p > r-\lambda-1$, we see that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |x_+^{\lambda} (x_-^{-r})_n x^p| dx = 0.$$

Now let φ be an arbitrary test function. Then equation (2) holds and it follows that

$$N-\lim_{n \rightarrow \infty} (x_+^{\lambda}, (x_-^{-r})_n \varphi(x)) = 0 = (0, \varphi).$$

Thus

$$(5) \quad x_+^{\lambda} \circ x_-^{-r} = 0$$

when $\lambda > -1$, $\lambda-r \neq -1, -2, \dots$ and $r=1, 2, \dots$.

We will now assume that equation (5) holds when $-k < \lambda < -k+1$ and $r=1, 2, \dots$, where k is a positive integer. It follows from theorem 4 that

$$(x_+^{\lambda} \circ x_-^{-r})' = 0 = \lambda x_+^{\lambda-1} \circ x_-^{-r} + r x_+^{\lambda} \circ x_-^{-r-1}$$

and it follows from our assumption that

$$x_+^{\lambda-1} \circ x_-^{-r} = 0$$

when $-k < \lambda < -k+1$ and $r=1, 2, \dots$. This establishes equation (5) by induction for $\lambda, \lambda-r \neq -1, -2, \dots$ and $r=1, 2, \dots$.

We will now consider the neutrix product $\ln x_+ \circ x_-^{\mu}$ for $\mu \neq -1, -2, \dots$. It follows from theorems 1, 2 and 3 that this product exists and

$$(6) \quad \ln x_+ \circ x_-^{\mu} = 0$$

for $\mu > -1$. Let us therefore suppose that $-k < \mu < -k+1$, where k is a positive integer. Then

$$\begin{aligned} \frac{\Gamma(\mu+k)}{\Gamma(\mu+1)} \int_{-\infty}^{\infty} \ln x_+ (x_-^\mu)_n x^m dx &= \int_0^{1/n} x^m \ln x \int_x^{1/n} (t-x)^{\mu+k-1} \delta_n^{(k-1)}(t) dt dx \\ &= \int_0^{1/n} \delta_n^{(k-1)}(t) \int_0^t x^m (t-x)^{\mu+k-1} \ln x dx dt \\ &= \int_0^{1/n} t^{\mu+m+k} \delta_n^{(k-1)}(t) \int_0^1 v^m (1-v)^{\mu+k-1} (\ln t + \ln v) dv dt \\ &= B(m+1, \mu+k) \int_0^{1/n} t^{\mu+m+k} \ln t \delta_n^{(k-1)}(t) dt + \\ &\quad + \left\{ \int_0^1 v^m (1-v)^{\mu+k-1} \ln v dv \right\} \left\{ \int_0^{1/n} t^{\mu+m+k} \delta_n^{(k-1)}(t) dt \right\}, \end{aligned}$$

where the substitution $x=tv$ has been made.

Making the substitution $nt=s$ we see that the functions

$$\int_{-\infty}^{\infty} \ln x_+ (x_-^\mu)_n x^m dx$$

are negligible, or zero, for $m=0, 1, 2, \dots$. Further, choosing a positive integer $p > -\mu-1$, we see that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |\ln x_+ (x_-^\mu)_n x^p| dx = 0.$$

Now let φ be an arbitrary test function. Then equation (2) holds and it follows that

$$N\text{-}\lim_{n \rightarrow \infty} (\ln x_+, (x_-^\mu)_n \varphi(x)) = 0 = (0, \varphi),$$

when $-k < \mu < -k+1$. This establishes equation (6) for $\mu \neq -1, -2, \dots$.

It now follows from theorem 4 that

$$\begin{aligned} (\ln x_+ \circ x_-^\mu)' &= 0 = x_+^{-1} \circ x_-^\mu - \mu \ln x_+ \circ x_-^{\mu-1} \\ &= x_+^{-1} \circ x_-^\mu \end{aligned}$$

for $\mu \neq 0, -1, -2, \dots$. Let us therefore assume that the neutrix product $x_+^{-r} \circ x_-^\mu$ exists and

$$(7) \quad x_+^{-r} \circ x_-^\mu = 0$$

for some positive integer r and $\mu, \mu-r \neq -1, -2, \dots$. Then by theorem 4

$$\begin{aligned} (x_+^{-r} \circ x_-^\mu)' &= 0 = -r x_+^{-r-1} \circ x_-^\mu - \mu x_+^{-r} \circ x_-^{\mu-1} \\ &= -r x_+^{-r-1} \circ x_-^\mu \end{aligned}$$

from our assumption. This establishes equation (7) by induction for $\mu, \mu-r \neq -1, -2, \dots$.

We have therefore proved the neutrix product $x_+^\lambda \circ x_-^\mu$ exists and

$$x_+^\lambda \circ x_-^\mu = 0$$

for $\lambda + \mu \neq -1, -2, \dots$. By replacing x by $-x$ in this equation we see that the neutrix product $x_-^\lambda \circ x_+^\mu$ exists and

$$x_-^\lambda \circ x_+^\mu = 0$$

for $\lambda + \mu \neq -1, -2, \dots$. This completes the proof of the theorem.

We can now define further neutrix products of distributions. For example let $\sin x_+^{\frac{1}{2}}$ be the ordinary summable function defined by

$$\sin x_+^{\frac{1}{2}} = \sum_{m=1}^{\infty} x_+^{m-\frac{1}{2}} / (2m-1)!.$$

Then for a non-negative integer r the function

$$\sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}} / (2m-1)!$$

is r times continuously differentiable and it follows from definition 1 and theorem 2 that

$$\left\{ \sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}} / (2m-1)! \right\} \circ x_-^{-r-\frac{1}{2}} = x_-^{-r-\frac{1}{2}} \circ \left\{ \sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}} / (2m-1)! \right\} = 0.$$

Since the neutrix product is obviously distributive with respect to addition we see that

$$\begin{aligned} \sin x_+^{\frac{1}{2}} \circ x_-^{-r-\frac{1}{2}} &= \sum_{m=1}^r (x_+^{m-\frac{1}{2}} \circ x_-^{-r-\frac{1}{2}}) / (2m-1)! + \left\{ \sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}} / (2m-1)! \right\} \circ x_-^{-r-\frac{1}{2}} = \\ &= \sum_{m=1}^r \frac{(-1)^m \pi}{2(r-m)! (2m-1)!} \delta^{(r-m)}(x) = x_-^{-r-\frac{1}{2}} \circ \sin x_+^{\frac{1}{2}}, \end{aligned}$$

where theorems 5 and 6 have been used, for $r=1, 2, \dots$ and

$$\sin x_+^{\frac{1}{2}} \circ x_-^{-\frac{1}{2}} = x_-^{-\frac{1}{2}} \circ \sin x_+^{\frac{1}{2}} = 0.$$

Now define the function $x_+^{-\frac{1}{2}} \cos x_+^{\frac{1}{2}}$ by

$$x_+^{-\frac{1}{2}} \cos x_+^{\frac{1}{2}} = 2(\sin x_+^{\frac{1}{2}})' = \sum_{m=0}^{\infty} x_+^{m-\frac{1}{2}} / (2m)!.$$

Then it can be proved similarly that

$$(x_+^{-\frac{1}{2}} \cos x_+^{\frac{1}{2}}) \circ x_-^{-r-\frac{1}{2}} = x_-^{-r-\frac{1}{2}} \circ (x_+^{-\frac{1}{2}} \cos x_+^{\frac{1}{2}}) = \sum_{m=0}^r \frac{(-1)^m \pi}{2(r-m)! (2m)!} \delta^{(r-m)}(x)$$

for $r=0, 1, 2, \dots$.

The Bessel function $J_\nu(x)$, see for example SNEDDON [6], is defined by

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}x\right)^{\nu+2m}}{m! \Gamma(m+\nu+1)}$$

for $\nu \neq -1, -2, \dots$. We will define the distribution $J_\nu(x_+)$ by

$$J_\nu(x_+) = \sum_{m=0}^{\infty} \frac{(-1)^m \frac{1}{2}^{\nu+2m} x_+^{\nu+2m}}{m! \Gamma(m+\nu+1)},$$

for $\nu \neq -1, -2, \dots$, where the x_+^ν for $\nu < -1$ are interpreted in the distributional sense. It follows as above that

$$\begin{aligned} J_\nu(x_+) \circ x_-^{r-\nu} &= x_-^{r-\nu} \circ J_\nu(x_+) = \\ &= -\frac{1}{2} \pi \operatorname{cosec}(\nu\pi) \sum_{m=0}^k \frac{(-1)^m \frac{1}{2}^{\nu+2m}}{m! (r-2m-1)! \Gamma(m+\nu+1)} \delta^{(r-2m-1)}(x), \end{aligned}$$

where

$$k = \begin{cases} \frac{1}{2}(r-1), & r \text{ odd}, \\ \frac{1}{2}(r-2) & r \text{ even} \end{cases}$$

for $r=1, 2, \dots$ and $\nu \neq 0, \pm 1, \pm 2, \dots$.

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