

On some partial differential inequalities for mappings of $\mathcal{C}^1$ class in $\mathbb{C}^n$

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Abstract. In this paper the authors obtain some partial differential inequalities involving mappings of $\mathcal{C}^1$ class defined on the unit ball of $\mathbb{C}^n$. Some interesting applications are also presented.

1. Introduction

Let $\mathbb{C}^n$ denote the n -dimensional space of complex variables $z = (z_1, \dots, z_n)' = (x_1 + iy_1, \dots, x_n + iy_n)'$ with the Euclidean inner product $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$ and the Euclidean norm $\|z\| = \sqrt{\langle z, z \rangle}$. A' means the transpose of the matrix A . The open Euclidean ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r and the open unit ball is abbreviated by B_1 . If G is a domain in $\mathbb{C}^n$ and $f = (f_1, \dots, f_n)' : G \rightarrow \mathbb{C}^n$, then we say that f belongs to the class $\mathcal{C}^1(G)$ if for each $j, k = 1, \dots, n$ the functions $u_j = \Re f_j$, $v_j = \Im f_j$, have all first order partial derivatives in respect to the real variables x_k, y_k and they are continuous in G . For $f \in \mathcal{C}^1(G)$

$$D_z f(a) = \left[\frac{\partial f_j}{\partial z_k}(a) \right]_{1 \leq j, k \leq n},$$

$$D_{\bar{z}} f(a) = \left[\frac{\partial f_j}{\partial \bar{z}_k}(a) \right]_{1 \leq j, k \leq n},$$

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where, as usually

$$\begin{aligned}\frac{\partial}{\partial z_k} &= \frac{1}{2} \left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right), \\ \frac{\partial}{\partial \bar{z}_k} &= \frac{1}{2} \left(\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right).\end{aligned}$$

Of course, if f is a holomorphic mapping on G , then $D_{\bar{z}}f = 0$ and $D_z f(a)$ is equal to the Frechet derivative $Df(a)$ of f at the point a . For our purpose we use the following one-dimensional lemma included in the paper [3] of P.T. MOCANU:

Lemma 1. *Let U be the open unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and f a complex function from the class $\mathcal{C}^1(U - \{0\})$. If there exists $\zeta_0 \in U - \{0\}$ such that*

$$\Re f(\zeta_0) = 0 = \min\{\Re f(\zeta) : 0 < |\zeta| \leq |\zeta_0|\},$$

then the following relation holds

$$(1) \quad \Im \left[\zeta_0 \frac{\partial f}{\partial \zeta}(\zeta_0) - \bar{\zeta}_0 \frac{\partial f}{\partial \bar{\zeta}}(\zeta_0) \right] = 0.$$

2. Main results

Theorem 1. *Let $p \in \mathcal{C}^1(B - \{0\})$. If there exists an $a \in B - \{0\}$ such that*

$$(2) \quad \Re \langle p(a), a \rangle = 0 = \min\{\Re \langle p(z), z \rangle : 0 < \|z\| \leq \|a\|\},$$

then

$$(3) \quad \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a) = ma$$

and

$$(4) \quad \Im \langle [D_z p(a)]a - [D_{\bar{z}} p(a)]\bar{a} - p(a), a \rangle = 0,$$

where m is a real number.

PROOF. Let $r = \|a\| \in (0, 1)$ and v be an arbitrary vector of the real tangent space $T_a(\partial B_r)$ at the point a . Since ∂B_r is a real surface at least of $\mathcal{C}^1$ class, it is well known that there exists an $\varepsilon > 0$ and a once differentiable function $\gamma : (-\varepsilon, \varepsilon) \rightarrow \partial B_r$ such that $\gamma(0) = a$ and $\frac{d\gamma}{dt}(0) = v$. Let

$$\alpha(t) = \Re \langle p(\gamma(t)), \gamma(t) \rangle, \quad t \in (-\varepsilon, \varepsilon).$$

Then α is the differentiable function on $(-\varepsilon, \varepsilon)$ and it satisfies the following relation

$$\alpha(0) = \Re \langle p(a), a \rangle = \min\{\alpha(t) : t \in (-\varepsilon, \varepsilon)\},$$

hence $\frac{d\alpha}{dt}(0) = 0$. On the other hand

$$\begin{aligned} \frac{d\alpha}{dt}(0) &= \frac{d}{dt} \Re \langle p(\gamma(t)), \gamma(t) \rangle_{|t=0} \\ &= \Re \left\langle [D_z p(a)] \frac{d\gamma}{dt}(0) + [D_{\bar{z}} p(a)] \overline{\frac{d\gamma}{dt}(0)}, a \right\rangle + \Re \left\langle p(a), \frac{d\gamma}{dt}(0) \right\rangle \\ &= \Re \langle [D_z p(a)]v + [D_{\bar{z}} p(a)]\bar{v}, a \rangle + \Re \langle p(a), v \rangle. \end{aligned}$$

So

$$\Re \left\langle \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a), v \right\rangle = 0.$$

Since the above relation holds for every arbitrary tangent vector v , we conclude that

$$\left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a)$$

is a normal vector to ∂B_r at the point a . On the other hand an outward normal vector to ∂B_r at the point a is also the vector a , hence, we can find a real number m such that the equality (3) holds. Now let us consider the following complex function

$$f(\zeta) = \zeta^{-1} \left\langle p(\zeta a \|a\|^{-1}), a \|a\|^{-1} \right\rangle, \quad \zeta \in U - \{0\}.$$

Since $p \in \mathcal{C}^1(B - \{0\})$, so $f \in \mathcal{C}^1(U - \{0\})$ and f satisfies, at $\zeta_0 = \|a\|$, the conditions

$$\Re f(\zeta_0) = 0, \Re f(\zeta) = |\zeta|^{-2} \Re \left\langle p(\zeta a \|a\|^{-1}), \zeta a \|a\|^{-1} \right\rangle,$$

so using the relation (2), we deduce that $\Re f(\zeta) \geq 0$ for $0 < |\zeta| \leq |\zeta_0|$. Therefore f fulfils all assumptions of Lemma 1, so the equality (1) holds. It is not difficult to see that

$$\zeta_0 \frac{\partial f}{\partial \zeta}(\zeta_0) = \left\langle [D_z p(a)]a, a \|a\|^{-2} \right\rangle - \left\langle p(a), a \|a\|^{-2} \right\rangle$$

and

$$\bar{\zeta}_0 \frac{\partial f}{\partial \bar{\zeta}}(\zeta_0) = \left\langle [D_{\bar{z}} p(a)]\bar{a}, a \|a\|^{-2} \right\rangle.$$

The relation (4) follows from (1) and from the above relations. $\square$

Remark 1. The example of the mapping $p(z) = z - \bar{z}$ and the points $a \in B - \{0\}$ with $a = \bar{a}$, shows that the set of the mappings which fulfil all assumptions of the above theorem, is nonempty (in this case $m = 0$).

An immediate application of Theorem 1 is given in the following result:

Corollary 1. Let p be a mapping from $C^1(B - \{0\})$. Assume that there exists and is positive the limit

$$\lim_{z \rightarrow 0} \|z\|^{-2} \Re \langle p(z), z \rangle = d.$$

If for all $z \in B - \{0\}$

$$(5) \quad \Re \left\langle [D_z p(z)]p(z) + [D_{\bar{z}} p(z)]\overline{p(z)}, z \right\rangle + \|p(z)\|^2 \neq 0,$$

then for all $z \in B - \{0\}$

$$\Re \langle p(z), z \rangle > 0.$$

PROOF. Suppose that the thesis does not hold. Then the real value function

$$h(z) = \begin{cases} \|z\|^{-2} \Re \langle p(z), z \rangle & \text{for } z \in B - \{0\} \\ d & \text{for } z = 0 \end{cases}$$

is continuous on B and there exists a point $a \in B - \{0\}$ such that

$$h(a) = 0 = \min\{h(z) : \|z\| \leq \|a\|\} = \min\{h(z) : 0 < \|z\| \leq \|a\|\}.$$

Therefore

$$0 = \|a\|^2 h(a) = \min\{\|z\|^2 h(z) : 0 < \|z\| \leq \|a\|\},$$

because the relation

$$0 < \|z\|^2 h(z) \leq h(z), \quad 0 < \|z\| \leq \|a\|,$$

imply the relation

$$0 \leq \min\{\|z\|^2 h(z) : 0 < \|z\| \leq \|a\|\} \leq \min\{h(z) : 0 < \|z\| \leq \|a\|\}.$$

Consequently (2) is fulfilled. Theorem 1 gives that the equality (3) holds. Let us create the inner product of the vector $p(a)$ and both sides of (3).

$$\left\langle \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a), p(a) \right\rangle = m \langle a, p(a) \rangle.$$

Then, in view of the properties of inner product and adjoint operators, we obtain

$$\Re \left\langle [D_z p(a)]p(a) + [D_{\bar{z}} p(a)]\overline{p(a)}, a \right\rangle + \|p(a)\|^2 = m \Re \langle a, p(a) \rangle = 0.$$

This is a contradiction with (5), so our supposition is false and the thesis is true.

Corollary 2. *Let q be a mapping from the class $\mathcal{C}^1(B - \{0\})$ with the property $\|q(z)\| \leq \|z\|$ for $z \in B - \{0\}$. If an $a \in B - \{0\}$, $a = \bar{a}$, is the fixed point of q , then it is the zero of the mapping*

$$\left[\overline{D_z q(a)} \right]' + [D_{\bar{z}} q(a)]'$$

or the fixed point of the mapping

$$s \left(\left[\overline{D_z q(a)} \right]' + [D_{\bar{z}} q(a)]' \right)$$

with some $s \neq 0$.

PROOF. Let us put

$$p(z) = z - q(z), \quad z \in B - \{0\}.$$

Then $p(a) = 0$ and

$$\Re \langle p(z), z \rangle = \|z\|^2 - \Re \langle q(z), z \rangle \geq \|z\|^2 - |\langle q(z), z \rangle| \geq \|z\|^2 - \|q(z)\| \|z\| \geq 0.$$

Therefore (2) holds and by Theorem 1 we obtain that there exists a real number m such that (3) is fulfilled. From this we have

$$\left[\overline{D_z q(a)} \right]' a + [D_{\bar{z}} q(a)]' a = (1 - m)a$$

because $\bar{a} = a$ and $p(a) = 0$. From this there follows the thesis.

Remark 2. *If q is holomorphic, then $D_{\bar{z}} q = 0$ and the assumption $a = \bar{a}$ is not necessary.*

Theorem 2. *Let $p \in \mathcal{C}^1(B - \{0\})$. If in a point $a \in B - \{0\}$*

$$(6) \quad |\langle p(a), a \rangle| = M \|a\|^2 = \max\{|\langle p(z), z \rangle| : \|z\| \leq \|a\|\} > 0,$$

then there exist a real number m such that

$$(7) \quad e^{2i\theta} \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a) = m a e^{i\theta},$$

where $\theta = \arg \langle p(a), a \rangle$.

PROOF. Let $r = \|a\| \in (0, 1)$ and v be an arbitrary vector of $T_a(\partial B_r)$. Then there exists an $\varepsilon > 0$ and an at least differentiable function $\gamma : (-\varepsilon, \varepsilon) \rightarrow \partial B_r$ such that $\gamma(0) = a$ and $\frac{d\gamma}{dt}(0) = v$. If we put

$$\beta(t) = |\langle p(\gamma(t)), \gamma(t) \rangle|^2, \quad t \in (-\varepsilon, \varepsilon),$$

then, from (6) we have

$$\beta(0) = M^2 \|a\|^4 = \max\{\beta(t) : t \in (-\varepsilon, \varepsilon)\},$$

hence $\frac{d\beta}{dt}(0) = 0$.

An easily computation yields

$$(8) \quad \frac{d\beta}{dt}(0) = 2\Re \left\{ \left\{ \langle [D_z p(a)]v + [D_{\bar{z}} p(a)]\bar{v}, a \rangle + \langle p(a), v \rangle \overline{\langle p(a), a \rangle} \right\} \right\},$$

so

$$0 = \Re \left\langle \lambda \left\{ \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a) \right\}, v \right\rangle,$$

where $\lambda = M \|a\|^2 e^{-i\theta}$.

From this we obtain (7) on the similar way as in the proof of Theorem 1. $\square$

Remark 3. The example of the mapping $p(z) = z + \bar{z}$ and the points $a \in B - \{0\}$ with $a = \bar{a}$ shows that the set of the mappings which fulfil all assumptions of the above theorem is nonempty.

An application of the above result is given in the following:

Corollary 3. Let p be a mapping from $C^1(B - \{0\})$. Assume that there exists the limit

$$\lim_{z \rightarrow 0} \|z\|^{-2} |\langle p(z), z \rangle| = d$$

and it belongs to the interval $[0, M)$. If for all $z \in B - \{0\}$ and all $\theta \in [0, 2\pi)$

$$(9) \quad \Im \left\langle e^{-2i\theta} [D_z p(z)]p(z) - [D_{\bar{z}} p(z)]\overline{p(z)}, z \right\rangle \neq 0,$$

then

$$(10) \quad |\langle p(z), z \rangle| < M \|z\|^2$$

for all $z \in B - \{0\}$.

PROOF. Suppose that the thesis does not hold. Then the real value function

$$h(z) = \begin{cases} \|z\|^{-2} |\langle p(z), z \rangle| & \text{for } z \in B - \{0\} \\ d & \text{for } z = 0 \end{cases}$$

is continuous on B and there exists a point $a \in B - \{0\}$ such that

$$h(a) = M = \max\{h(z) : \|z\| \leq \|a\|\}.$$

From this there follows (6), hence we can find a real number m and such that the equality (7) is fulfilled.

Let us create the inner product of the vector $p(a)$ and of the both sides of (7).

$$\left\langle p(a), e^{2i\theta} \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a) \right\rangle = m e^{i\theta} \langle p(a), a \rangle.$$

Therefore

$$\Im \left\langle p(a), e^{2i\theta} \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} \right\rangle = 0,$$

because $\theta = \arg \langle p(a), a \rangle$. From the above, in view of the properties of inner product and adjoint operators, we obtain

$$\Im \left\langle e^{-2i\theta} [D_z p(a)] p(a) - [D_{\bar{z}} p(a)] \overline{p(a)}, a \right\rangle = 0.$$

This is a contradiction with (9), so our supposition is false and the thesis is true. $\square$

Remark 4. If p is a holomorphic mapping, the results from Theorem 1 and Theorem 2 are similar to the results from [1] and [2].

References

- [1] G. KOHR, On some partial differential inequalities for holomorphic mappings in $\mathbb{C}^n$ (to appear).
- [2] G. KOHR, Certain partial differential inequalities and applications for holomorphic mappings defined on the unit ball of $\mathbb{C}^n$ (to appear).
- [3] P. T. MOCANU, Alpha-convex nonanalytic functions, *Mathematica* **29** (52) (1987), 49–55.

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