

The neutrix distribution product $x_+^\lambda \circ x_+^\mu$

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In the following we define the ordinary locally summable functions x_+^λ, x_-^λ for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda & \text{for } x > 0, \\ 0 & \text{for } x < 0, \end{cases}$$

$$x_-^\lambda = \begin{cases} 0 & \text{for } x > 0, \\ (-x)^\lambda & \text{for } x < 0. \end{cases}$$

We define the distributions x_+^λ, x_-^λ for $\lambda < -1$ and $\lambda \neq -2, -3, \dots$ inductively by

$$x_+^\lambda = (\lambda + 1)^{-1} (x_+^{\lambda+1})',$$

$$x_-^\lambda = -(\lambda + 1)^{-1} (x_-^{\lambda+1})'.$$

We define the distributions $|x|^\lambda, \operatorname{sgn} x \cdot |x|^\lambda$ for $\lambda \neq -1, -2, \dots$ by

$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \quad \operatorname{sgn} x \cdot |x|^\lambda = x_+^\lambda - x_-^\lambda.$$

It follows that

$$(|x|^\lambda)' = \lambda \operatorname{sgn} x \cdot |x|^{\lambda-1},$$

$$(\operatorname{sgn} x \cdot |x|^\lambda)' = \lambda |x|^{\lambda-1}.$$

Further, if $\lambda > -r-1, \lambda \neq -1, -2, \dots, -r$ and φ is an arbitrary test function with compact support, then

$$(x_+^\lambda, \varphi) = \int_0^1 x^\lambda \left[\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m \right] dx + \int_1^\infty x^\lambda \varphi(x) dx + \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m! (\lambda + m + 1)}$$

see GELFAND and SHILOV [7].

The following definition was given in [2].

Definition 1. Let f and g be two distributions for which on the open interval (a, b) , f is the r -th derivative of an ordinary summable function F in $L^p(a, b)$ and $g^{(r)}$ is an ordinary summable function in $L^q(a, b)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = gf = \sum_{i=0}^r \binom{r}{i} (-1)^i [F g^{(i)}]^{(r-i)},$$

where

$$\binom{r}{i} = \frac{r!}{i!(r-i)!}.$$

The next definition was given by van der CORPUT [1].

Definition 2. A neutrix N is a commutative additive group of functions $v(\xi)$ defined on a domain N' with values in an additive group N'' , where further if for some v in N , $v(\xi) = \gamma$ for all ξ in N' , then $\gamma = 0$. The functions in N are called negligible functions. Now let N' be a set contained in a topological space with a limit point b which does not belong to N' . If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi) - \beta$ is negligible in N , then β is called the neutrix limit of f as ξ tends to b and we write

$$N\text{-}\lim_{\xi \rightarrow \infty} f(\xi) = \beta,$$

where the limit β must be unique if it exists.

Now let ϱ be a fixed infinitely differentiable function having the properties

- (i) $\varrho(x) = 0$ for $|x| \geq 1$,
- (ii) $\varrho(x) \geq 0$,
- (iii) $\varrho(x) = \varrho(-x)$,
- (iv) $\int_{-1}^1 \varrho(x) dx = 1$.

We define the function δ_n by

$$\delta_n(x) = n(\varrho nx)$$

for $n=1, 2, \dots$. It is obvious that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-distribution δ . For an arbitrary distribution g we define the function g_n by

$$g_n(x) = g * \delta_n(x) = \int_{-1/n}^{1/n} g(x-t) \delta_n(t) dt$$

for $n=1, 2, \dots$. The sequence $\{g_n\}$ is regular and converges to g .

The following definition now extends definition 1 to a wider class of distributions and was given in [4].

Definition 3. Let f and g be arbitrary distributions and let

$$g_n = g * \delta_n.$$

We say that the neutrix product fog of f and g exists and is equal to h on the open interval (a, b) if

$$N\text{-}\lim_{n \rightarrow \infty} (fg_n, \varphi) = N\text{-}\lim_{n \rightarrow \infty} (f, g_n \varphi) = (h, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b) , where N is the neutrix having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' the real numbers with negligible functions linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n$$

for $\lambda > 0$ and $r = 1, 2, \dots$ and all functions of n that converge to zero as n tends to infinity.

The following theorems are immediate consequences of theorems given in [3] and [4].

Theorem 1. *Let f and g be distributions. If the product fg exists on the open interval (a, b) then the neutrix products $f \circ g$ and $g \circ f$ exist and*

$$f \circ g = g \circ f = fg$$

on this interval.

Theorem 2. *Let f and g be distributions and suppose that the neutrix products $f \circ g$ and $f \circ g'$ (or $f' \circ g$) exist on the open interval (a, b) . Then the neutrix product $f' \circ g$ (or $f \circ g'$) exists and*

$$(f \circ g)' = f' \circ g + f \circ g'$$

on this interval.

The next theorem was proved in [5].

Theorem 3. *The neutrix products $x_+^\lambda \circ x_-^\mu$ and $x_-^\lambda \circ x_+^\mu$ exist and*

$$x_+^\lambda \circ x_-^\mu = x_-^\lambda \circ x_+^\mu = 0$$

for $\lambda + \mu \neq -1, -2, \dots$.

We now prove the following theorem.

Theorem 4. *The neutrix product $x_+^\mu \circ x_+^\lambda$ exists and*

$$(1) \quad x_+^\lambda \circ x_+^\mu = x_+^{\lambda+\mu}$$

for $\lambda, \mu, \lambda + \mu \neq -1, -2, \dots$.

PROOF. We will first of all suppose that $\lambda > -1$ and put

$$(x_+^\mu)_n = x_+^\mu * \delta_n(x).$$

Then it follows that

$$\begin{aligned} \prod_{i=1}^r (\mu + i)(x_+^\mu)_n &= x_+^{\mu+r} * \delta_n^{(r)}(x) = \\ &= \begin{cases} \int_{-1/n}^{1/n} (x-t)^{\mu+r} \delta_n^{(r)}(t) dt & \text{for } x > 1/n, \\ \int_{-1/n}^x (x-t)^{\mu+r} \delta_n^{(r)}(t) dt & \text{for } -1/n \leq x \leq 1/n, \\ 0 & \text{for } x < -1/n, \end{cases} \end{aligned}$$

where r is a non-negative integer chosen so that $\mu+r>0$. Thus

$$\begin{aligned} \prod_{i=1}^r (\mu+i) \int_0^1 x^{\lambda+m} (x_+^\mu)_n dx &= \int_0^{-1/n} x^{\lambda+m} \int_{-1/n}^x (x-t)^{\mu+r} \delta_n^{(r)}(t) dt dx + \\ &+ \int_{1/n}^1 x^{\lambda+m} \int_{-1/n}^{1/n} (x-t)^{\mu+r} \delta_n^{(r)}(t) dt dx = \\ &= n^{-\lambda-\mu-m-1} \int_0^1 u^{\lambda+m} \int_{-1}^u (u-v)^{\mu+r} \varrho^{(r)}(v) dv du + \\ &+ n^{-\lambda-\mu-m-1} \int_1^n u^{\lambda+m} \int_{-1}^1 (u-v)^{\mu+r} \varrho^{(r)}(v) dv du = I_1 + I_2, \end{aligned}$$

where the substitutions $nx=u$ and $nt=v$ have been made.

Since $\lambda+\mu \neq -1, -2, \dots$, we see immediately that I_1 is negligible for $m=0, 1, 2, \dots$. Further

$$\int_1^n u^{\lambda+m} \int_{-1}^1 (u-v)^{\mu+r} \varrho^{(r)}(v) dv du = \int_{-1}^1 \varrho^{(r)}(v) \int_1^n u^{\lambda+\mu+r+m} (1-v/u)^{\mu+r} du dv$$

and

$$\begin{aligned} &\int_1^n u^{\lambda+\mu+r+m} (1-v/u)^{\mu+r} du = \\ &= \int_1^n u^{\lambda+\mu+r+m} \left[1 - (\mu+r)v/u + \frac{(\mu+r)(\mu+r-1)}{2!} \frac{v^2}{u^2} - \dots \right] du = \\ &= n^{\lambda+\mu+r+m+1} \left[\frac{1}{\lambda+\mu+r+m+1} - \frac{(\mu+r)v}{(\lambda+\mu+r+m)n} + \frac{(\mu+r)(\mu+r-1)v^2}{2!(\lambda+\mu+r+m-1)n^2} - \dots \right] - \\ &\quad - \left[\frac{1}{\lambda+\mu+r+m+1} - \frac{(\mu+r)v}{\lambda+\mu+r+m} + \dots \right]. \end{aligned}$$

It follows that

$$\begin{aligned} N-\lim_{n \rightarrow \infty} I_2 &= (-1)^r \prod_{i=1}^r (\mu+i) [r! (\lambda+\mu+m+1)]^{-1} \int_{-1}^1 v^r \varrho^{(r)}(v) dv = \\ &= \prod_{i=1}^r (\mu+i) / (\lambda+\mu+m+1) \end{aligned}$$

and so

$$\int_0^1 x^{\lambda+m} (x_+^\mu)_n dx = (\lambda+\mu+m+1)^{-1}$$

for $m=0, 1, 2, \dots$.

Now let φ be an arbitrary test function with compact support contained in the interval (a, b) . Then

$$\begin{aligned} (x_+^\lambda, (x_+^\mu)_n \varphi) &= \int_0^\infty x^\lambda (x_+^\mu)_n \varphi(x) dx = \int_0^1 x^\lambda (x_+^\mu)_n \left[\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m \right] dx + \\ &+ \sum_{m=1}^{r-1} \frac{\varphi^{(m)}(0)}{m!} \int_0^1 x^{\lambda+m} (x_+^\mu)_n dx + \int_1^\infty x^\lambda (x_+^\mu)_n \varphi(x) dx. \end{aligned}$$

By Taylor's theorem

$$\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m = \frac{x^r}{r!} \varphi^{(r)}(\xi x),$$

where $0 \leq \xi \leq 1$ and so

$$\begin{aligned} (x_+^\lambda, (x_+^\mu)_n \varphi) &= \frac{1}{r!} \int_0^1 x^\lambda [x^r (x_+^\mu)_n] \varphi^{(r)}(\xi x) dx + \\ &+ \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} \int_0^1 x^{\lambda+m} (x_+^\mu)_n dx + \int_1^\infty x^\lambda (x_+^\mu)_n \varphi(x) dx. \end{aligned}$$

Since the sequence of continuous functions $\{x^r (x_+^\mu)_n\}$ converges uniformly to the continuous function $x^{\mu+r}$ ($\mu+r > 0$) on the closed interval $[0, 1]$ and the sequence of continuous functions $\{(x_+^\mu)_n\}$ converges uniformly to the continuous function x^μ on the closed interval $[a, b] \cap [1, \infty)$, it follows that

$$\begin{aligned} N\text{-}\lim_{n \rightarrow \infty} (x_+^\lambda, (x_+^\mu)_n \varphi) &= \lim_{n \rightarrow \infty} \frac{1}{r!} \int_0^1 x^\lambda [x^r (x_+^\mu)_n] \varphi^{(r)}(\xi x) dx + \\ &+ N\text{-}\lim_{n \rightarrow \infty} \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} \int_0^1 x^{\lambda+m} (x_+^\mu)_n dx + \lim_{n \rightarrow \infty} \int_1^\infty x^\lambda (x_+^\mu)_n \varphi(x) dx = \\ &= \frac{1}{r!} \int_0^1 x^{\lambda+\mu} x^r \varphi^{(r)}(\xi x) dx + \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m! (\lambda + \mu + m + 1)} + \int_1^\infty x^{\lambda+\mu} \varphi(x) dx = \\ &= \int_0^1 x^{\lambda+\mu} \left[\varphi(x) - \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} x^m \right] dx + \int_1^\infty x^{\lambda+\mu} \varphi(x) dx + \\ &+ \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m! (\lambda + \mu + m + 1)} = (x_+^{\lambda+\mu}, \varphi). \end{aligned}$$

This proves that the neutrix product $x_+^\lambda \circ x_+^\mu$ exists and

$$x_+^\lambda \circ x_+^\mu = x_+^{\lambda+\mu}$$

for $\lambda > -1$ and $\mu, \lambda + \mu \neq -1, -2, \dots$.

Now suppose that the neutrix product $x_+^\lambda \circ x_+^\mu$ exists and satisfies equation (1) for $-k-1 < \lambda-k$ and $\mu, \lambda+\mu \neq -1, -2, \dots$, where k is a positive integer. Then it follows from theorem 2 that the neutrix product $x_+^\lambda \circ x_+^\mu$ exists and satisfies

equation (1) for $-k-2 < \lambda < -k-1$ and $\mu, \lambda + \mu \neq -1, -2, \dots$. Since equation (1) is certainly satisfied for $-1 < \lambda < 0$ and $\mu, \lambda + \mu \neq -1, -2, \dots$ the result of the theorem follows by induction. This completes the proof of the theorem.

Corollary 4.1. *The neutrix product $x_-^\lambda \circ x_-^\mu$ exists and*

$$x_-^\lambda \circ x_-^\mu = x_-^{\lambda+\mu}$$

for $\lambda, \mu, \lambda + \mu \neq -1, -2, \dots$.

PROOF. The result follows immediately from the theorem on replacing x in equation (1) by $-x$.

Corollary 4.2. *The neutrix products $|x|^\lambda \circ |x|^\mu$, $(\operatorname{sgn} x \cdot |x|^\lambda) \circ (\operatorname{sgn} x \cdot |x|^\mu)$, $|x|^\lambda \circ (\operatorname{sgn} x \cdot |x|^\mu)$ and $(\operatorname{sgn} x \cdot |x|^\lambda) \circ |x|^\mu$ exist and*

$$|x|^\lambda \circ |x|^\mu = (\operatorname{sgn} x \cdot |x|^\lambda) \circ (\operatorname{sgn} x \cdot |x|^\mu) = |x|^{\lambda+\mu},$$

$$|x|^\lambda \circ (\operatorname{sgn} x \cdot |x|^\mu) = (\operatorname{sgn} x \cdot |x|^\lambda) \circ |x|^\mu = \operatorname{sgn} x \cdot |x|^{\lambda+\mu}$$

for $\lambda, \mu, \lambda + \mu \neq -1, -2, \dots$.

PROOF. Since the neutrix product is obviously distributive with respect to addition we have

$$\begin{aligned} |x|^\lambda \circ |x|^\mu &= (x_+^\lambda + x_-^\lambda) \circ (x_+^\mu + x_-^\mu) = \\ &= x_+^\lambda \circ x_+^\mu + x_+^\lambda \circ x_-^\mu + x_-^\lambda \circ x_+^\mu + x_-^\lambda \circ x_-^\mu = |x|^{\lambda+\mu} \end{aligned}$$

on using theorems 3 and 4 and corollary 4.1.

The other results follows similarly.

Corollary 4.3. *The neutrix products $(x+i0)^\lambda \circ (x+i0)^\mu$ and $(x-i0)^\lambda \circ (x-i0)^\mu$ exist and*

$$(x+i0)^\lambda \circ (x+i0)^\mu = (x+i0)^{\lambda+\mu},$$

$$(x-i0)^\lambda \circ (x-i0)^\mu = (x-i0)^{\lambda+\mu}$$

for $\lambda, \mu, \lambda + \mu \neq -1, -2, \dots$.

PROOF. The distributions $(x+i0)^\lambda$ and $(x-i0)^\mu$ are defined by

$$(x+i0)^\lambda = x_+^\lambda + e^{i\lambda\pi} x_-^\lambda,$$

$$(x-i0)^\lambda = x_+^\lambda + e^{-i\lambda\pi} x_-^\lambda$$

for $\lambda \neq -1, -2, \dots$, see GELFAND and SHILOV [7]. The results follow immediately from theorems 3 and 4 and corollary 4.1.

Corollary 4.4. *The neutrix products $x_+^\lambda \circ \delta^{(r)}$ and $\delta^{(r)} \circ x_+^\lambda$ exist and*

$$x_+^\lambda \circ \delta^{(r)} = \delta^{(r)} \circ x_+^\lambda = 0$$

for $\lambda \neq 0, 1, \dots, r, -1, -2, \dots$ and $r = 0, 1, 2, \dots$.

PROOF. From the theorem we have

$$x_+^\lambda \circ H = x_+^\lambda$$

for $\lambda \neq -1, -2, \dots$, where $H = x_+^0$. It follows from theorems 2 and 4 that the neutrix product $x_+^\lambda \circ \delta$ exists and

$$x_+^\lambda \circ \delta = 0$$

for $\lambda \neq -1, -2, \dots$. A simple induction argument now proves that

$$x_+^\lambda \circ \delta^{(r)} = 0$$

for $\lambda \neq -1, -2, \dots$ and $r = 0, 1, 2, \dots$.

The existence of the neutrix product $\delta^{(r)} \circ x_+^\lambda$ follows similarly. The result of this corollary was given in [6].

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