

On equivalence of variational problems subject to constraints

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1 §. Introduction

Let us consider two n -dimensional parameter-invariant variational problems subject to constraints. The fundamental functions are of the form $F(x, \dot{x})$ and $F^*(x, \dot{x})$; $x := (x^1, x^2, \dots, x^n)$, $\dot{x} := (\dot{x}^1, \dot{x}^2, \dots, \dot{x}^n)$.

The variational problems subject to constraints are formulated as follows: it is required to find curves C and C^* respectively joining two given fixed points. These curves must satisfy some equations of constraint (see [4]) of the form ¹⁾

$$(1.1) \quad G_{(\varrho)}(x, \dot{x}) := A_{(\varrho)k}(x) \dot{x}^k = 0 \quad (\varrho = 1, 2, \dots, m).$$

Furthermore C must afford an extreme value to the integral $\mathfrak{I} = \int_{t_0}^{t_1} F(x, \dot{x}) dt$ relative to other curves joining the same points which also satisfy the conditions (1.1). C^* must do the same with respect to $\mathfrak{I}^* = \int_{t_0}^{t_1} F^*(x, \dot{x}) dt$. It is known (cf. H. Rund [3] page 338) that the extremals of these problems are those solutions of the Euler—Lagrange equations

$$(1.2) \quad (a) \mathcal{E}_i(F + \lambda^{\varrho} G_{(\varrho)}) = 0 \quad \text{and} \quad (b) \mathcal{E}_i(F^* + \lambda^{*\varrho} G_{(\varrho)}) = 0$$

respectively, which satisfy the conditions (1.1). In these equations λ^{ϱ} and $\lambda^{*\varrho}$ are unknown constants.

Definition. Two variational problems subject to constraints (1.1) with fundamental functions $F(x, \dot{x})$ and $F^*(x, \dot{x})$ respectively, are called equivalent, if the relations

$$(1.3) \quad \mathcal{E}_i(F^* + \lambda^{*\varrho} G_{(\varrho)}) - \mu(x, \dot{x}) \mathcal{E}_i(F + \lambda^{\varrho} G_{(\varrho)}) \equiv \Phi_i^{\varrho}(x, \dot{x}) G_{(\varrho)}(x, \dot{x});$$

$$\Phi_i^{\varrho}(x, \dot{x}) \neq 0, \quad \mu(x, \dot{x}) \neq 0$$

¹⁾ Here and in the following Greek indices run from 1 to m , and Einstein-summation convention is applied for Latin and Greek indices too.

hold identically in $(x^i, \dot{x}^i, \ddot{x}^i)$ with constants $\lambda^e, \lambda^{*e}, \mathcal{E}_i$ being the Euler—Lagrange operators:

$$(1.3a) \quad \mathcal{E}_i := \partial_i - \frac{d}{dt} \partial_i \quad \left(\partial_i := \frac{\partial}{\partial x^i}; \partial_i := \frac{\partial}{\partial \dot{x}^i} \right).$$

$\mu(x, \dot{x})$ and $\Phi_i^e(x, \dot{x})$ have to be homogeneous functions of degree zero in $\dot{x}^i$.

It is clear from the definition of the equivalence, that if a curve $x(t)$ satisfies $G(x, \dot{x})=0$ and $\mathcal{E}_i(F + \lambda^e G)=0$, then $\mathcal{E}_i(F^* + \lambda^{*e} G)=0$ holds also, and conversely. In this paper we investigate the form of the functions $\mu(x, \dot{x})$ and $F(x, \dot{x})$, $F^*(x, \dot{x})$ satisfying (1.3). At the end of the paper we make some geometrical remarks.

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2§. The independence of μ of $\dot{x}^i$

The relation (1.3) has the explicit form

$$(2.1) \quad \begin{aligned} & \partial_i F^* - \frac{d}{dt} \partial_i F^* + \lambda^{*e} (\partial_i A_k - \partial_k A_i) \dot{x}^k - \mu(x, \dot{x}) \times \\ & \times \left[\partial_i F - \frac{d}{dt} \partial_i F + \lambda^e (\partial_i A_k - \partial_k A_i) \dot{x}^k \right] \equiv \Phi_i^e(x, \dot{x}) A_k \dot{x}^k. \end{aligned}$$

Performing the derivations with respect to t and multiplying with (-1) we get:

$$(2.2) \quad \begin{aligned} & (\partial_i \partial_k F^* - \mu(x, \dot{x}) \partial_i \partial_k F) \dot{x}^k + \\ & + [\partial_i \partial_k F^* - \mu(x, \dot{x}) \partial_i \partial_k F - (\lambda^{*e} - \mu(x, \dot{x}) \lambda^e) (\partial_i A_k - \partial_k A_i) + \Phi_i^e A_k] \times \\ & \times \dot{x}^k - \partial_i F^* + \mu(x, \dot{x}) \partial_i F \equiv 0. \end{aligned}$$

In the case without constraints H. RUND has already proved that if the relations $\mathcal{E}_i(F^*(x, \dot{x})) \equiv \mu(x, \dot{x}) \mathcal{E}_i(F(x, \dot{x}))$ hold identically, then μ is necessarily independent of $\dot{x}^j$ (see [2]). Since in the identity (2.2) the coefficient of $\dot{x}^k$ is the same as in the case without constraints, so we obtain analogously to H. RUND [2] the relations

$$(2.3) \quad F^*(x, \dot{x}) = \mu(x, \dot{x}) F(x, \dot{x}) + \psi(x, \dot{x}),$$

where

$$(2.3a) \quad \partial_i \partial_k \psi \equiv - (F \partial_i \partial_k \mu + (\partial_k \mu) \partial_i F + (\partial_i \mu) \partial_k F).$$

It is evident that $\psi(x, \dot{x})$ must be homogeneous of first degree in $\dot{x}^i$. The method of H. RUND used in [2] can also be applied for equations of type (2.1). Let us substitute (2.3) into (2.1). With respect to (2.3a) it reduces to

$$(2.4) \quad \begin{aligned} & F \partial_i \mu + \partial_i \psi - \\ & - [F \partial_i \partial_k \mu + (\partial_i \mu) \partial_k F + (\partial_k \mu) \partial_i F + \partial_i \partial_k \psi - (\lambda^{*e} - \mu(x, \dot{x}) \lambda^e) (\partial_i A_k - \partial_k A_i)] \dot{x}^k \equiv \\ & \equiv \Phi_i^e A_k \dot{x}^k. \end{aligned}$$

From the homogeneity of ψ it follows

$$(2.5) \quad \psi(x, \dot{x}) = \psi_j(x, \dot{x}) \dot{x}^j \quad (\psi_j := \partial_j^* \psi(x, \dot{x})),$$

which also implies that

$$(2.6) \quad \partial_i \psi = \partial_i \psi_j \dot{x}^j.$$

With the aid of (2.5) and (2.6) we can write (2.4) in the following form:

$$(2.7) \quad \begin{aligned} \dot{x}^k (\partial_i \psi_k - \partial_k \psi_i) &\equiv \Phi_i^e A_k \dot{x}^k + \\ &+ [F \partial_i^* \partial_k \mu + (\partial_i^* \mu) \partial_k F + (\partial_k \mu) \partial_i^* F] \dot{x}^k - (\lambda^{*e} - \mu(x, \dot{x}) \lambda^e) (\partial_i A_k - \partial_k A_i) \dot{x}^k - F \partial_i \mu. \end{aligned}$$

Let us differentiate this with respect to $\dot{x}^j$. It follows with respect to $\partial_j^* A_k \equiv 0$

$$(2.8) \quad \begin{aligned} (\partial_i \psi_j - \partial_j \psi_i) + \dot{x}^k (\partial_i \partial_j^* \psi_k - \partial_k \partial_j^* \psi_i) &\equiv \partial_j^* \Phi_i^e A_k \dot{x}^k + \Phi_i^e A_j + \\ &+ [F \partial_i^* \partial_k \partial_j^* \mu + (\partial_j^* F) \partial_i^* \partial_k \mu + (\partial_i^* \partial_j^* \mu) \partial_k F + (\partial_i^* \mu) \partial_k \partial_j^* F + (\partial_k \partial_j^* \mu) \partial_i^* F + (\partial_k \mu) \partial_i^* \partial_j^* F] \dot{x}^k + \\ &+ F \partial_i^* \partial_j \mu + (\partial_i^* \mu) \partial_j F + (\partial_j \mu) \partial_i^* F - (\lambda^{*e} - \mu(x, \dot{x}) \lambda^e) (\partial_i A_j - \partial_j A_i) + \\ &+ \lambda^e \partial_j \mu (\partial_i A_k - \partial_k A_i) \dot{x}^k - F \partial_i \partial_j^* \mu - (\partial_i \mu) \partial_j^* F. \end{aligned}$$

We differentiate (2.3a) with respect to x^j and we calculate the identity $\partial_i^* \partial_k \partial_j^* \psi \equiv \partial_j^* \partial_k \psi_i$. Substituted this in (2.8); we have

$$(2.9) \quad \begin{aligned} &(\partial_i^* \mu) \partial_j F + \dot{x}^k [(\partial_k \mu) \partial_i^* \partial_j^* F - (\partial_j \mu) \partial_i^* \partial_k F] + \\ &+ \lambda^e \partial_j \mu (\partial_i A_k - \partial_k A_i) \dot{x}^k + \Phi_i^e A_j + \partial_j^* \Phi_i^e A_k \dot{x}^k \equiv \\ &\equiv \partial_i \psi_j - \partial_j \psi_i + (\partial_i \mu) \partial_j^* F - (\partial_j \mu) \partial_i^* F - F (\partial_i \partial_j \mu - \partial_j \partial_i \mu) + \\ &+ (\lambda^{*e} - \mu(x, \dot{x}) \lambda^e) (\partial_i A_j - \partial_j A_i). \end{aligned}$$

The right hand side is skew-symmetric in (i, j) , thus the symmetric part in (i, j) of the left hand side is identically zero:

$$(2.10) \quad \begin{aligned} &\frac{1}{2} [(\partial_i^* \mu) \partial_j F + (\partial_j^* \mu) \partial_i F] + \dot{x}^k (\partial_k \mu) \partial_i^* \partial_j^* F - \frac{\dot{x}^k}{2} [(\partial_j \mu) \partial_i^* \partial_k F + (\partial_i \mu) \partial_j^* \partial_k F] + \\ &+ \frac{\lambda^e}{2} \dot{x}^k [\partial_j \mu (\partial_i A_k - \partial_k A_i) + \partial_i \mu (\partial_j A_k - \partial_k A_j)] + \\ &+ \frac{1}{2} (\Phi_i^e A_j + \Phi_j^e A_i) + \frac{1}{2} (\partial_j^* \Phi_i^e + \partial_i^* \Phi_j^e) A_k \dot{x}^k \equiv 0. \end{aligned}$$

After multiplication by 2, we write (2.10) in the following form

$$\begin{aligned}
 (2.11) \quad & + \partial_j^* \mu [\partial_i^* F - \partial_i^* \partial_k^* F \dot{x}^k - \partial_i^* \partial_k^* F \ddot{x}^k + \lambda^e (\partial_i^* A_k - \partial_k^* A_i) \dot{x}^k] + \\
 & + 2(\dot{x}^k \partial_k^* \mu + \ddot{x}^k \partial_k^* \mu) \partial_i^* \partial_j^* F + \Phi_i^e A_j + \Phi_j^e A_i + (\partial_j^* \Phi_i^e + \partial_i^* \Phi_j^e) A_k \dot{x}^k \equiv \\
 & \equiv -\ddot{x}^k [(\partial_i^* \mu) \partial_j^* \partial_k^* F + (\partial_j^* \mu) \partial_i^* \partial_k^* F - 2(\partial_k^* \mu) \partial_i^* \partial_j^* F]. \\
 & \equiv -\ddot{x}^k [(\partial_i^* \mu) \partial_j^* \partial_k^* F + (\partial_j^* \mu) \partial_i^* \partial_k^* F - 2(\partial_k^* \mu) \partial_i^* \partial_j^* F].
 \end{aligned}$$

This can be expressed as

$$\begin{aligned}
 (2.12) \quad & \partial_i^* \mu \mathcal{E}_j(F + \lambda^e G) + \partial_j^* \mu \mathcal{E}_i(F + \lambda^e G) + \\
 & + 2 \frac{d\mu}{dt} \partial_i^* \partial_j^* F + \Phi_i^e A_j + \Phi_j^e A_i + (\partial_j^* \Phi_i^e + \partial_i^* \Phi_j^e) A_k \dot{x}^k \equiv \\
 & \equiv -\ddot{x}^k [(\partial_i^* \mu) \partial_j^* \partial_k^* F + (\partial_j^* \mu) \partial_i^* \partial_k^* F - 2(\partial_k^* \mu) \partial_i^* \partial_j^* F].
 \end{aligned}$$

Since Φ_i^e are covariant vectors, the left hand side of (2.12) is a tensor, and this is true also for the coefficient of $\ddot{x}^k$ on the right hand side. But $\ddot{x}^k$ itself is not a vector, and hence (2.12) can be invariant iff this coefficient vanishes (see [2] page 24), so that

$$(2.13) \quad 2(\partial_k^* \mu) \partial_i^* \partial_j^* F \equiv (\partial_i^* \mu) \partial_j^* \partial_k^* F + (\partial_j^* \mu) \partial_i^* \partial_k^* F.$$

It is well-known from the Finsler-Geometry that

$$(2.14) \quad g_{ij} = l_i l_j + h_{ij},$$

where

$$(2.14a) \quad l_i = \partial_i^* F; \quad h_{ij} = F \partial_i^* \partial_j^* F.$$

We can write (2.13) after multiplication by $F(x, \dot{x}^i)$ in the following form

$$(2.15) \quad 2 \partial_k^* \mu h_{ij} \equiv \partial_i^* \mu h_{jk} + \partial_j^* \mu h_{ik}.$$

By contraction of (2.14) by g^{ij} with respect to (2.14a) we get

$$(2.16) \quad h_j^j \equiv F g^{ih} \partial_i^* \partial_j^* F = \delta_j^h - l^h l_j.$$

In particular, contracting over h and j , we find

$$(2.17) \quad h_j^j \equiv F g^{ij} \partial_i^* \partial_j^* F = \delta_j^j - l^j l_j = n - 1.$$

We now multiply (2.15) by g^{ij} and since μ is homogeneous of degree zero in $\dot{x}^i$, using (2.17), it is found that

$$(2.18) \quad 2(n-1) \partial_k^* \mu = \partial_i^* \mu (\delta_k^i - l^i l_k) + \partial_j^* \mu (\delta_k^j - l^j l_k) = 2 \partial_k^* \mu.$$

This gives

$$(2.18a) \quad 2(n-2) \partial_k^* \mu = 0.$$

Excluding the special case $n=2$, we therefore infer that

$$(2.19) \quad \partial_k^* \mu = 0.$$

Thus μ is independent of $\dot{x}^k$.

We can summarize the result in

Theorem 1. *If (1.3) and (1.1) hold, then for $n > 2$, μ is necessarily independent of $\dot{x}^k$.*

3§. The explicit form of F and F^*

Since according to Theorem 1 μ is independent of $\dot{x}^i$, it follows from (2.3a) on account of the homogeneity of ψ in $\dot{x}^i$, that $\psi(x, \dot{x})$ must be linear in $\dot{x}^k$:

$$(3.1) \quad \psi(x, \dot{x}) = S_k(x) \dot{x}^k.$$

Substituting this into (2.3) we find

$$(3.2) \quad F^*(x, \dot{x}) = \mu(x) F(x, \dot{x}) + S_k(x) \dot{x}^k.$$

Furthermore, let us substitute (3.1) and (3.2) in (2.4). Thus we have

$$(3.3) \quad F \partial_i \mu - \frac{d\mu}{dt} \partial_i^* F + (\partial_i S_k - \partial_k S_i) \dot{x}^k + \\ + (\lambda^{*e} - \mu(x) \lambda^e) (\partial_i A_{(e)}^k - \partial_k A_{(e)}^i) \dot{x}^k \equiv \Phi_i^e(x, \dot{x}) A_{(e)}^k(x) \dot{x}^k.$$

From this we get the following theorem:

Theorem 2. *If $\Phi_i^e = \partial_i^* \Phi^e(x, \dot{x})$, $A_{(e)}^i = \partial_i A_{(e)}(x)$ and $S_i = \partial_i S(x)$ then the fundamental function $F(x, \dot{x})$ has the form:*

$$(3.4) \quad F(x, \dot{x}) = \left(\frac{d\mu}{dt} \right)^{-1} \left(2N(x, \dot{x}) - \frac{1}{2} \Phi^e(x, \dot{x}) \frac{d A_{(e)}(x)}{dt} \right),$$

where $\mu = \mu(x)$, and $\Phi^e(x, \dot{x})$, $N(x, \dot{x})$ are positively homogeneous functions of first and second degree respectively in $\dot{x}^i$.

PROOF. First we transform the relation (3.3). Differentiating (3.3) with respect to $\dot{x}^j$ we have

$$(3.5) \quad (\partial_i \mu) \partial_j^* F - (\partial_j \mu) \partial_i^* F - \frac{d\mu}{dt} \partial_i^* \partial_j^* F + \partial_i S_j - \partial_j S_i + \\ + (\lambda^{*e} - \mu(x) \lambda^e) (\partial_i A_{(e)}^j - \partial_j A_{(e)}^i) - \Phi_i^e A_{(e)}^j - \partial_j^* \Phi_i^e A_{(e)}^k \dot{x}^k \equiv 0.$$

Now we take the skew-symmetric part in (i, j) of (3.5)

$$(3.6) \quad (\partial_i \mu) \partial_j^* F - (\partial_j \mu) \partial_i^* F + \partial_i S_j - \partial_j S_i + (\lambda^{*e} - \mu(x) \lambda^e) (\partial_i A_{(e)}^j - \partial_j A_{(e)}^i) - \\ - \frac{1}{2} (\Phi_i^e A_{(e)}^j - \Phi_j^e A_{(e)}^i) - \frac{1}{2} (\partial_j^* \Phi_i^e - \partial_i^* \Phi_j^e) A_{(e)}^k(x) \dot{x}^k \equiv 0.$$

By the assumptions of Theorem 2 this reduces to

$$(3.7) \quad (\partial_i \mu) \partial_j^* F - (\partial_j \mu) \partial_i^* F - \frac{1}{2} (\partial_i^* \Phi^e \partial_j A_{(e)} - \partial_j^* \Phi^e \partial_i A_{(e)}) \equiv 0.$$

This can be written in the following form

$$(3.8) \quad \partial_j \left(F \partial_i \mu + \frac{1}{2} \Phi^e \partial_i A_{(e)} \right) - \partial_i \left(F \partial_j \mu + \frac{1}{2} \Phi^e \partial_j A_{(e)} \right) \equiv 0.$$

From this it follows that $F \partial_j \mu + \frac{1}{2} \Phi^e \partial_j A_{(e)}$ has the form

$$(3.9) \quad F \partial_j \mu + \frac{1}{2} \Phi^e \partial_j A_{(e)} \equiv \partial_j N(x, \dot{x}).$$

Multiplying (3.9) by $\dot{x}^j$, we obtain with respect to the homogeneity of the functions Φ^e and N in $\dot{x}^i$:

$$(3.10) \quad F \frac{d\mu}{dt} + \frac{1}{2} \Phi^e \frac{dA_{(e)}}{dt} = 2N(x, \dot{x}).$$

Since $\frac{d\mu}{dt} \neq 0$, so from (3.10) we get the statement of Theorem 2.

We still calculate the form of F^* in case if Theorem 2 holds. Substituting (3.4) in (3.2) and using $S_i = \partial_i S(x)$ we have

$$(3.11) \quad F^*(x, \dot{x}) = \mu(x) \left(\frac{d\mu}{dt} \right)^{-1} \left(2N(x, \dot{x}) - \frac{1}{2} \Phi^e(x, \dot{x}) \frac{dA_{(e)}(x)}{dt} \right) + \frac{dS(x)}{dt}.$$

4§. Geometrical remarks

Proposition 1. If $S_i = \partial_i S(x)$, $\mu = \mu(x)$ and no constraints exist, then the extremals of $\int F^*(x, \dot{x}) dt$ and $\int \mu(x) F(x, \dot{x}) dt$ are identical.

PROOF. Because of (3.2)

$$(4.1) \quad F^*(x, \dot{x}) = \mu(x) F(x, \dot{x}) + \frac{dS(x)}{dt}.$$

F^* and μF differ by a total differential. The addition of a total differential to the integrand evidently cannot affect any extremals, which completes the proof.

Remark: In [1] A. MOÓR has shown that if the relations $\mathcal{E}_i(F^*(x, \dot{x})) - \mu \mathcal{E}_i(F(x, \dot{x})) \equiv 0$ hold, where $\mu = \text{const.}$, then $S_i = \partial_i S(x)$ and $F^* = \mu F + \frac{dS}{dt}$ is also satisfied. So in this case $\int F^* dt$ and $\int F dt$ have identical extremals.

Proposition 2. If along the curve $x^i = x^i(t)$ $A_k \dot{x}^k = 0$ and $\Phi_i^e(x, \dot{x}) A_{(e)}^i(x) = 0$ then $\mu(x(t)) = \text{const.}$

PROOF. The symmetric-part in (i, j) of (3.5) always vanishes

$$(4.2) \quad \frac{d\mu}{dt} \partial_i \partial_j F + \frac{1}{2} (\Phi_i^e A_{(e)}^j + \Phi_j^e A_{(e)}^i) + \frac{1}{2} (\partial_j \Phi_i^e + \partial_i \Phi_j^e) A_{(e)}^k \dot{x}^k \equiv 0.$$

Let us multiply this by g^{ij} . Using (2.17) we obtain

$$(4.3) \quad \frac{1}{F}(n-1)\frac{d\mu}{dt} + \Phi_i^e A^i + \partial_j \Phi_i^e g^{ij} A_k \dot{x} \equiv 0.$$

From this, and from the conditions of Proposition 2 it follows

$$(4.4) \quad \frac{1}{F}(n-1)\frac{d\mu}{dt} = 0.$$

So

$$(4.5) \quad \frac{d\mu}{dt} \equiv 0,$$

and thus $\mu(x(t)) = \text{const.}$

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