

## A remark on lattices satisfying the maximum condition

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In this note we are going to prove the following <sup>1)</sup>

**Proposition.** *Let  $a_0 < a_1 < \dots < a_r$  and  $b_0 < b_1 < \dots < b_s$  be two chains of a lattice satisfying the maximum condition, and let  $a_1, \dots, a_r, b_1, \dots, b_s$  be join-irreducible elements. Let moreover be  $a_0 \parallel b_0$ . Then  $a_j \cap b_k = a_0 \cap b_0$  for any pair of indices  $(j, k)$  ( $j = 0, 1, \dots, r; k = 0, 1, \dots, s$ ).*

First we establish the following

**Lemma.**

$$\left. \begin{array}{l} a < c \\ b < c \\ a \neq b \end{array} \right\} \implies a \cup b = c.$$

**PROOF.** We have  $a \parallel b$ , because, say,  $a < b (< c)$  would contradict  $a < c$ . Now,  $a, b < a \cup b$  because, say,  $a = a \cup b \iff a \cong b$ . At the same time  $a \cup b \leq c$ , and here  $a \cup b < c$  cannot hold, because

$$a, b < a \cup b < c$$

would contradict our hypothesis. Thus  $a \cup b = c$ .

The Proposition is now capable of the following

**PROOF.**

$$(1) \quad a_j \parallel b_0 \quad (j = 0, 1, \dots, r).$$

Indeed, let  $j \in [1, r]$ . Then  $a_j \neq b_0$ , since  $a_j$  is comparable with  $a_0$ , while  $b_0$  is not.

We cannot have  $a_j < b_0$ , since  $a_j < b_0 \implies a_0 < b_0$ , in contradiction to  $a_0 \parallel b_0$ .

Suppose now  $b_0 < a_j$  where  $j > 0$  is the smallest index for which this inequality holds. By the maximum condition there exists an element  $h \in L$  satisfying  $b_0 \leq h < a_j$ . Clearly  $h \neq a_{j-1}$ , since  $b_0 \neq a_{j-1}$  and  $b_0 < h = a_{j-1}$  would contradict the choice of  $j$ . The Lemma now yields

$$a_j = h \cup a_{j-1},$$

in contradiction to the join-irreducibility of  $a_j$ . This establishes (1).

<sup>1)</sup> This proposition was suggested by Exercise 16. on p. 57 of the book [1], of which we are also using the terminology and notations. It seemed however necessary to add the condition  $a_0 \parallel b_0$ , while finiteness has been replaced by the maximum condition, and join-irreducibility has been postulated only for  $a_j (j > 0)$  and  $b_k (k > 0)$ .

Let us remark that in deriving (1) we have used the incomparability of  $b_0$  with  $a_0$ , but nothing else about  $b_0$ . Accordingly we can replace  $b_0$  by any element  $h$  of  $L$  provided  $a_0 \parallel h$ , thus obtaining  $a_j \parallel h$  ( $j=0, 1, \dots, r$ ).

Having established (1), we see that by symmetry

$$(2) \quad a_0 \parallel b_k \quad (k = 0, 1, \dots, s)$$

also holds.

Consider now the chains  $a_1 < \dots < a_r$  and  $b_0 < b_1 < \dots < b_s$ . Since  $a_1 \parallel b_0$  by (1), we can replace  $a_0$  by  $a_1$  in (2):

$$a_1 \parallel b_k \quad (k = 0, 1, \dots, s).$$

Again, consider  $a_0 < a_1 < \dots < a_r$  and  $b_1 < \dots < b_s$ . Since  $a_0 \parallel b_1$  by (2), we can replace  $b_0$  by  $b_1$  in (1):

$$a_j \parallel b_1 \quad (j = 0, 1, \dots, r).$$

Given  $a_2 < \dots < a_r$  and  $b_2 < \dots < b_s$ , we infer with the help of  $a_2 \parallel b_0$  that

$$a_2 \parallel b_k \quad (k = 0, 1, \dots, s).$$

Starting with  $a_0 < a_1 < \dots < a_r$  and  $b_2 < \dots < b_s$ , and taking into account  $a_0 \parallel b_2$ , we obtain

$$a_j \parallel b_2 \quad (j = 0, 1, \dots, r).$$

Continuing this process, we finally reach  $a_r$  and  $b_0 < b_1 < \dots < b_s$  as well as  $a_0 < a_1 < \dots < a_r$  and  $b_s$ , and we see that

$$(3) \quad a_j \parallel b_k \quad (j = 0, 1, \dots, r; k = 0, 1, \dots, s).$$

Let us now show that

$$(4) \quad a_j \cap b_k \equiv a_0 \quad (j = 0, 1, \dots, r; k = 0, 1, \dots, s).$$

We cannot have  $a_0 < a_j \cap b_k$ , since this would imply  $a_0 < b_k$ , thus contradicting  $a_0 \parallel b_k$ .

Suppose now  $a_0 \parallel (a_j \cap b_k)$ . Replacing  $b_0$  by  $a_j \cap b_k$  in (1), we obtain  $a_j \parallel (a_j \cap b_k)$  in contradiction to  $a_j \cap b_k \equiv a_j$ . This establishes (4), and by symmetry

$$(5) \quad a_j \cap b_k \equiv b_0.$$

(4) and (5) together yield

$$a_j \cap b_k \equiv a_0 \cap b_0.$$

The reverse inequality being trivial, this completes the proof of the proposition.

### Reference

- [1] G. Szász, *Introduction to Lattice Theory*. New York and London, 1963.

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