

Characterization of integervalued R -additive functions

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1. Let $\mathfrak{N}$ be the set of natural numbers. For an arbitrary subset $S \subseteq \mathfrak{N}$ let $\bar{S} = S \cup \{0\}$, and $|S|$ be the cardinality of S . We shall write $\mathfrak{N}_0 = \mathfrak{N} \cup \{0\}$ ($= \bar{\mathfrak{N}}$).

Let $R_i \subseteq \mathfrak{N}$ ($i=0, 1, \dots$). We shall say that $\{\bar{R}_0, \bar{R}_1, \dots\}$ is a finite direct decomposition (FDD) of $\mathfrak{N}_0$ if every $n \in \mathfrak{N}$ can be stated uniquely as

$$(1.1) \quad n = r_{i_1} + \dots + r_{i_s}, \quad r_{i_k} \in R_{i_k} \quad (k = 1, \dots, s).$$

We shall say that this FDD is an R -system, if:

- a) R_i is finite and nonempty,
- b) the smallest element of R_i is smaller than the smallest element of R_j for every $i < j$.

We shall say that an R -system is monotonic if the smallest element of R_j is larger than the greatest element of R_i for every $i < j$.

The sets $R_i = \{q^i, 2q^i, \dots, (q-1)q^i\}$ ($i=0, 1, \dots$) generate the q -ary number-system, this is a monotonic R -system.

Let $f(n)$ be an arbitrary complex-valued function defined on $\mathfrak{N}_0$. We shall say that f is an R -additive function if $f(0)=0$ and for $n \in \mathfrak{N}$ written in the form (1.1) we get

$$(1.2) \quad f(n) = f(r_{i_1}) + \dots + f(r_{i_s}).$$

This is a straightforward generalization of q -additivity that has been introduced by A. O. GELFOND [1].

For the sake of brevity let

$$(1) \quad R = \bigcup_{i=0}^{\infty} R_i,$$

$$(2) \quad A_i = \{\lambda_0 + \dots + \lambda_{i-1} \mid \lambda_j \in \bar{R}_j\} \setminus \{0\},$$

$$T_i = \{\lambda_i + \lambda_{i+1} + \dots + \lambda_h \mid h = i+1, i+2, \dots; \lambda_j \in \bar{R}_j\} \setminus \{0\}$$

The following assertion has been proved in [2] and [3].

Theorem A. *Let us be given an R -system and let f be an integervalued R -additive function. Assume that*

- (a) *the R -system is monotonic, or there exists a suitable prime p for which $p \nmid m$ holds for infinitely many $m \in R$;*
- (b) *there exists an index i_0 such that $n \mid f(n)$ for every $n \in T_{i_0}$.*

Then $f(n) = c \cdot n$ holds for every $n \in T_{i_0}$, where c is a suitable integer.

2. We shall assume that $f, g, f_1, \dots, f_s, g_1, \dots, g_s, h_1, \dots, h_s$ are integer-valued functions, furthermore that $M_1 < M_2 < \dots < M_s$ are fixed nonnegative integers. Let us consider the conditions

$$(A_s) \quad \sum_{i=1}^s f_i(n + M_i) \equiv \sum_{i=1}^s f_i(M_i) \pmod{n} \quad (\forall n \in \mathfrak{N});$$

$$(B_s) \quad \sum_{i=1}^s g_i(n + M_i) \equiv 0 \pmod{n} \quad (\forall n \in \mathfrak{N}).$$

It is clear that A_1 with $M_1=0$ is the same as condition (b) in Theorem A.

We shall say that $(f_1, \dots, f_s)$ is a trivial solution of (A_s) if $f_i(n) = c_i n$ ($i=1, \dots, s$) with integer constants c_i . Similarly $(g_1, \dots, g_s)$ is a trivial solution of (B_s) if $g_i(n) = a_i n$ ($i=1, \dots, s$) with integer constants a_i . It is obvious that in the latter case $\sum_{i=1}^s a_i M_i = 0$.

Theorem 1. *If (A_s) has a nontrivial solution then so has (B_s) , and vice versa.*

First we shall prove

Lemma 1. *Let f be an integer-valued R -additive function. If*

$$(2.1) \quad f(n) \equiv A \pmod{n}$$

holds for every $n \in T_{i_0}$, then $A=0$.

PROOF. Let P be a large integer, $P > |A|$. There exists an infinite sequence $r_{i_1}, r_{i_2}, \dots$ of integers, $r_{i_k} \in R_{i_k}$ ($i=1, 2, \dots$) such that $i_0 \leq i_1 < i_2 < \dots$, and

$$r_{i_k} \equiv z_{i_k} \pmod{P}, \quad f(r_{i_k}) \equiv w \pmod{P} \quad (k=1, 2, \dots),$$

where z and w are suitable residues mod P . Then the integer

$$N = \sum_{k=1}^P r_{i_k}$$

belongs to T_{i_0} , $N \equiv Pz \equiv 0 \pmod{P}$, $f(N) \equiv 0 \pmod{P}$, consequently $P|A$, which involves that $A=0$. $\square$

PROOF OF THEOREM 1. Let $(f_1, \dots, f_s)$ and $(h_1, \dots, h_s)$ be some solutions of (A_s) ; $H = \sum_{i=1}^s h_i(M_i)$, $F = \sum_{i=1}^s f_i(M_i)$. Then $(g_1, \dots, g_s)$ defined by

$$(2.2) \quad g_i(n) = Hf_i(n) - Fh_i(n) \quad (i=1, \dots, s)$$

is a solution of (B_s) . If $(f_1, \dots, f_s)$ is a nontrivial solution and $(h_1, \dots, h_s)$ is a trivial solution such that $H \neq 0$ (there exists such a solution), then $(g_1, \dots, g_s)$ is a nontrivial solution.

To prove the second part of the assertion it is enough to show that for every solution $(g_1, \dots, g_s)$ satisfying (B_s) the relation

$$(2.3) \quad \sum_{i=1}^s g_i(M_i) = 0$$

holds. Indeed, under (2.3) the functions $(g_1, \dots, g_s)$ give a solution satisfying (A_s) .

Let i_0 be such an index for which $M_i \in A_{i_0}$ ($i=1, \dots, s$). From (B_s) we get that

$$\sum_{i=1}^s g_i(n) \equiv - \sum_{i=1}^s g_i(M_i) \pmod{n}$$

holds for every $n \in T_{i_0}$, and so by Lemma 1 applied for $f = \sum g_i$ we get that (2.3) holds. $\square$

If $(g_1, \dots, g_s)$ is a solution of (B_s) , then $(g_1(n) + c_1 n, \dots, g_s(n) + c_s n)$ with arbitrary integer constants $c_1, \dots, c_s$ is a solution of (A_s) . This is a straightforward consequence of (2.3). Furthermore, if $(f_1, \dots, f_s)$ is a solution of (A_s) and the integer constants $c_1, \dots, c_s$ are so chosen that $\sum_{i=1}^s c_i M_i = 0$, then $(f_1(n) - c_1 n, \dots, f_s(n) - c_s n)$ is a solution of (A_s) . Consequently it is enough to characterize the solutions of (B_s) only.

Theorem 2. Let $F, g_1, \dots, g_s$ be integer-valued R -additive functions, $0 < M_1 < M_2 < \dots < M_s$ be fixed integers. Assume that
(α) the R -system is monotonic, or there exists a suitable prime p such that $p \nmid m$ holds for infinitely many $m \in R$, and
(β) that the relation

$$(2.4) \quad \sum_{i=1}^s g_i(n + M_i) \equiv F(n) \pmod{n}$$

holds for every positive integer n .

Then

$$\sum_{i=1}^s g_i(n + M_i) = F(n) + cn$$

identically, with a suitable integer c .

PROOF. Let i_0 be so large that $M_i \in A_{i_0}$ holds for every i . Then for $n \in T_{i_0}$ we get that $g_i(n + M_i) = g_i(n) + g_i(M_i)$, consequently from (2.4) we deduce that

$$\sum_{i=1}^s g_i(M_i) + \sum_{i=1}^s g_i(n) \equiv F(n) \pmod{n}.$$

Taking into account Lemma 1 with $f = F - g_1 - \dots - g_s$, we get that

$$\sum_{i=1}^s g_i(M_i) = 0,$$

and so from Theorem A that $f(n) = cn$, i.e.

$$(2.5) \quad \sum_{i=1}^s g_i(N) = F(N) + cN \quad (\forall N \in T_{i_0}).$$

Let now $b \in \mathfrak{N}_0$ be an arbitrary integer. Let i_1 be so large that $i_1 \geq i_0$, $b \in A_{i_1}$, $b + M_i \in A_{i_1}$ ($i=1, \dots, s$). Let us consider now (2.4) with $n = N + b$, $N \in T_{i_1}$. Taking

into account (2.5) we get immediately that

$$\sum_{i=1}^s g_i(b+M_i) \equiv F(b) - cN \pmod{(N+b)},$$

i.e. that

$$(2.6) \quad \sum_{i=1}^s g_i(b+M_i) \equiv F(b) + cb \pmod{(N+b)} \quad (\forall N \in T_{i_1}).$$

Since T_{i_1} contains arbitrary large elements, we have that

$$\sum_{i=1}^s g_i(b+M_i) = F(b) + cb.$$

The proof is finished. $\square$

Corollary 1. *Let f be an integervalued R -additive function, $M \geq 0$ be a fixed integer. Assume that condition (a) in Theorem A holds, and that*

$$(2.7) \quad f(n+M) \equiv f(M) \pmod{n}$$

is satisfied for every $n \in \mathfrak{N}$.

Then $f(n) = cn$ for every $n \in \mathfrak{N}_0$.

PROOF. Let the integers c and d be so chosen that $df(M) = cM$. Let $g(n) = df(n) - cn$. From (2.7) we get that $g(n+M) \equiv 0 \pmod{n}$ holds identically, and so from Theorem 2 ($s=1$, $F=0$) that $g(n) = 0$. Consequently $f(n) = \frac{c}{d}n$ c/d is an integer, since f is an integervalued function. $\square$

Remark. Theorem 2 contains the following special case. If f and g are integervalued R -additive functions, $M \geq 0$ being an integer, furthermore condition (a) in Theorem A and the relation

$$g(n+M) \equiv f(n) \pmod{n} \quad (\forall n \in \mathfrak{N})$$

hold, then

$$(2.8) \quad g(n+M) = f(n) + cn.$$

We could give a complete description of the solutions of (2.8) in the case $R_0 = \{1, 2, \dots, k\}$ [4].

3.

Theorem 3. *Let the R -system be arbitrary, f_i ($i=1, \dots, s$) be arbitrary integervalued R -additive functions and φ_i ($i=1, \dots, s$) be arbitrary integervalued functions defined on $\mathfrak{N}_0$. Assume that*

$$(C_s) \quad \sum_{i=1}^s f_i(n+M_i) \equiv \sum_{i=1}^s \varphi_i(M_i) \pmod{n}$$

holds for every $n, M_1, \dots, M_s \in \mathfrak{N}$.

Then $f_i(n) = c_i n$, $\varphi_i(n) = f_i(n) + a_i$, where c_i, a_i are integers and $\sum_{i=1}^s a_i = 0$.

Lemma 2. Let f be an intervalued R -additive function.
Assume that

$$(3.1) \quad f(n+K) - f(n+M) \equiv f(K) - f(M) \pmod{n}$$

holds for every $n, K, M \in \mathfrak{N}$.

Then $f(n) = cn$, c being an integer.

PROOF OF LEMMA 2. Let $g(n) = f(n) - f(1) \cdot n$. Then $g(1) = 0$, and the requirements stated for f in Lemma 2 hold for g too. Let $n \in \mathfrak{N}$ be arbitrary. Let i_0 and $i_1 > i_0$ be so chosen, that $n+1 \in A_{i_0}$, N_1 and N_2 be so that $N_1 \in T_{i_0} \cap A_{i_1}$ and $N_2 \in T_{i_1}$. Let us consider (3.1) with $M = N_2$ and $K = N_2 + 1$. Then we have

$$(3.2) \quad g(n+1) - g(n) \equiv 0 \pmod{(n+N_1)}.$$

Observing that N_1 can be an arbitrary large number we get that $g(n+1) = g(n)$. From $g(1) = 0$ we get that $g(n) = 0$ identically. $\square$

PROOF OF THEOREM 3. Let $M_1, \dots, M_s$ be fixed. Let i_0 be so large that $M_i \in A_{i_0}$ ($i = 1, \dots, s$). Then (C_s) involves that

$$\sum_{i=1}^s f_i(n) \equiv \sum_{i=1}^s \varphi_i(M_i) - \sum_{i=1}^s f_i(M_i) \pmod{n} \quad \forall n \in T_{i_0}.$$

Lemma 1 gives that the right hand side is zero, i.e.

$$(3.3) \quad \sum_{i=1}^s \varphi_i(M_i) = \sum_{i=1}^s f_i(M_i).$$

(3.3) holds for every choice of $M_1, \dots, M_s$, therefore $\varphi_i(M) - f_i(M) = \varphi_i(1) - f_i(1)$ is a constant, i.e.

$$(3.4) \quad \varphi_i(n) = f_i(n) + a_i, \quad a_i \text{ integer.}$$

After substituting (3.4) into (C_s) we get that

$$(3.5) \quad \sum_{i=1}^s f_i(n+M_i) \equiv \sum_{i=1}^s f_i(M_i) + \sum_{i=1}^s a_i \pmod{n}$$

holds for every choice of $n, M_1, \dots, M_s \in \mathfrak{N}$. Hence we get immediately that

$$f_i(n+K) - f_i(n+M) \equiv f_i(K) - f_i(M) \pmod{n}$$

holds for every $n, K, M \in \mathfrak{N}$, i.e. the condition (3.1) holds for $f_i = f$. From Lemma 2 we get that $f_i(n) = c_i n$. The assertion $\sum a_i = 0$ is an immediate consequence of (3.5). $\square$

The following assertion is a straightforward consequence of Lemma 2.

Corollary 2. *Let f be an arbitrary integervalued R -additive function. Assume that*

$$f(n+M) \equiv f(n)+f(M) \pmod{n}$$

holds for every $n, M \in \mathfrak{N}$.

Then $f(n)=cn$, c being an integer constant.

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