

Simplicial faces in pure and factorial state spaces of operator algebras

By ADEL M. ZAKI (Abha)

1. Introduction

It is known by [10] and [1] that for a unital C^* -algebra A , the pure and factorial state spaces of A (see §2) can be written as a union of w^* -closed faces of the state space. In this work, we investigate the question when the pure and factorial state spaces of a C^* -algebra A can be written as unions of w^* -closed simplicial faces of the quasi-state space $Q(A)$.

The answer in the case of the factorial state space $\overline{F(A)}$ is easy. In fact, A is an abelian C^* -algebra if, and only if, $\overline{F(A)}$ is a union of w^* -closed simplicial faces of $Q(A)$. By a closed face we shall always mean a w^* -closed face.

For the pure state space $\overline{P(A)}$, we prove that the following conditions are equivalent.

1. $\overline{P(A)}$ is a union of simplicial faces of $Q(A)$.
2. If ψ_1, ψ_2 are two distinct equivalent pure states of A , then $(1/2)(\psi_1 + \psi_2) \notin \overline{P(A)}$
3. $F(A) \cap \overline{P(A)} = P(A)$.

Moreover, we show that if $\overline{P(A)}$ is a union of simplicial faces of $Q(A)$, A is postliminal and for all irreducible representations π of A ,

$$\pi(A)/LC(H_\pi),$$

where $LC(H_\pi)$ denotes the compact operators on a Hilbert space H_π , is abelian.

2. Preliminaries

Let A be a C^* -algebra. If S is a subset of the dual space A^* , we denote by $\overline{S}$ the closure of S in the w^* -topology. We denote the state space of A by $S(A)$. $S(A)$ is convex and w^* -compact, if A is unital. Let $P(A)$ be the set of extreme points of $S(A)$, which we call the pure states of A . The pure state space of A is $\overline{P(A)}$. A state of A is said to be factorial if the von Neumann algebra generated by $\pi_\phi(A)$ is a factor, where π_ϕ is the GNS representation associated with ϕ . The factor state space of A is $\overline{F(A)}$, where $F(A)$ is the set of factorial states. The type I factorial states will be denoted by $F_I(A)$. The quasi-state space of A is the set of all positive linear functionals on A with norm less than or equal to 1.

Recall that, two pure states ϕ_1, ϕ_2 are said to be equivalent if $\pi_{\phi_1}, \pi_{\phi_2}$ are unitarily equivalent. Let F be a w^* -closed face of $S(A)$, where A is a unital C^* -algebra. Then, F is a Choquet simplex if, and only if, F does not contain two distinct equivalent pure states of A ([3; Th 2.5] and [2; cor 3]).

Let A be an arbitrary C^* -algebra. Consider the following condition which will have a special significance throughout this work: “ $\overline{P(A)}$ is a union of closed simplicial faces of $S(A)$ ”.

Suppose that A is non-unital and let $\tilde{A}$ be the C^* -algebra obtained from A by adjoining an identity. The restriction map $r : S(\tilde{A}) \rightarrow Q(A)$ is an affine homeomorphism of $S(\tilde{A})$ onto $Q(A)$ which maps $\overline{P(\tilde{A})}$ onto $\overline{P(A)}$ and $\overline{F(\tilde{A})}$ onto $\overline{F(A)}$ (see, for example [11]). Since $\overline{P(\tilde{A})}$ and $\overline{F(\tilde{A})}$ are unions of closed faces of $S(\tilde{A})$ [1;10], it follows that $\overline{P(A)}$ and $\overline{F(A)}$ are unions of closed faces of $Q(A)$. Furthermore, $\overline{P(A)}$ (respectively $\overline{F(A)}$) is a union of closed simplicial faces of $Q(A)$ if, and only if, $\overline{P(\tilde{A})}$ (respectively $\overline{F(\tilde{A})}$) is a union of closed simplicial faces of $S(\tilde{A})$.

3. Main results

We start this section by the following definition:

Definition 3.1. A C^* -algebra A is said to satisfy the condition $(*)$ if, and only if, whenever ψ_1, ψ_2 are two distinct equivalent pure states of A then

$$(1/2)(\psi_1 + \psi_2) \notin \overline{P(A)}.$$

The following result shows the connection between the above condition $(*)$ and simplicial faces of the state space of A .

Proposition 3.2. *Let A a unital C^* -algebra. Then A satisfies $(*)$ if, and only if, $\overline{P(A)}$ is a union of closed simplicial faces of $S(A)$.*

PROOF. ($\longrightarrow$) Let A satisfy $(*)$. Let $\phi \in \overline{P(A)}$ and let F_ϕ be the smallest closed face of $S(A)$ which contains ϕ . Using [10], we have $\overline{P(A)}$ is a union of closed faces of $S(A)$ and hence $F_\phi \subseteq \overline{P(A)}$.

Suppose that F_ϕ is not a Choquet simplex. Then using [3; Th 2.5] and [2; cor 3] F_ϕ contains two distinct equivalent pure states of A , ψ_1, ψ_2 say. Since F_ϕ is convex, then

$$\psi = (1/2)(\psi_1 + \psi_2) \in F_\phi$$

and hence $\psi \in \overline{P(A)}$, which contradicts $(*)$. Thus F_ϕ is Choquet simplex. Finally, $\overline{P(A)}$ is the union of the simplicial faces F_ϕ ($\phi \in \overline{P(A)}$).

($\longleftarrow$) Suppose $(*)$ does not hold. Then there exist equivalent pure states ψ_1, ψ_2 such that $\psi_1 \neq \psi_2$ and

$$(1/2)(\psi_1 + \psi_2) \in \overline{P(A)}$$

Let $\phi = (1/2)(\psi_1 + \psi_2)$. Then $\psi_1, \psi_2 \in F_\phi$. If F is a closed face of $S(A)$ such that $\phi \in F \subseteq \overline{P(A)}$ then $F_\phi \subseteq F$, so $\psi_1, \psi_2 \in F$ and F is not a Choquet simplex (see [3; Th 2.5] and [2; cor 3]). Thus $\overline{P(A)}$ is not a union of closed simplicial faces of $S(A)$.

Remark. Note that, when A is unital, $\overline{P(A)}$ is a union of simplicial faces of $S(A)$ if, and only if, it is a union of simplicial faces of $Q(A)$.

Proposition 3.3. *Let A be a non-unital C^* -algebra. Then the following are equivalent.*

- (i) $\overline{P(A)}$ is a union of closed simplicial faces of $Q(A)$
- (ii) $\tilde{A}$ satisfies $(*)$.
- (iii) A satisfies $(*)$.

PROOF. (i) $\longleftrightarrow$ (ii)

As observed in section 2, $\overline{P(A)}$ is a union of closed simplicial faces of $Q(A)$ if, and only if, $\overline{P(\tilde{A})}$ is a union of closed simplicial faces of $S(\tilde{A})$, and the latter condition is equivalent, to $\tilde{A}$ satisfying $(*)$ (see proposition 3.2).

(iii) $\longrightarrow$ (ii)

Let ϕ_1 and ϕ_2 be distinct equivalent pure states of $\tilde{A}$. Since the restriction map $r : S(\tilde{A}) \longrightarrow Q(A)$ is (1-1), then $r(\phi_1) \neq r(\phi_2)$. Moreover,

$r(\phi_1)$ and $r(\phi_2)$ are both in $P(A)$, since ϕ_1 and ϕ_2 are distinct and equivalent. It is routine to check that $r(\phi_1)$ and $r(\phi_2)$ are equivalent (see, for example [11]). Hence by assumption,

$$\begin{aligned} (1/2)(r(\phi_1) + r(\phi_2)) &\notin \overline{P(A)} \\ (1/2)(\phi_1 + \phi_2) &\notin r^{-1}(\overline{P(A)}) = \overline{P(\tilde{A})} \end{aligned}$$

(ii) $\longrightarrow$ (iii)

Suppose $\tilde{A}$ satisfies $(*)$ and let ψ_1 and ψ_2 be distinct equivalent pure states of A . We show that

$$(1/2)(\psi_1 + \psi_2) \notin \overline{P(A)}$$

Let $\tilde{\psi}_i$ be the unique pure state extension of ψ_i to $\tilde{A}$ ($i = 1, 2$). Then $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are distinct and equivalent. By assumption, we have

$$(1/2)(\tilde{\psi}_1 + \tilde{\psi}_2) \notin \overline{P(\tilde{A})}.$$

Since r is $(1-1)$, then

$$r((1/2)(\tilde{\psi}_1 + \tilde{\psi}_2)) \notin r(\overline{P(\tilde{A})}) = \overline{P(A)}$$

and thus we get (iii).

Remark. Combining proposition 3.3 with the remark after proposition 3.2, we see that, for any C^* -algebra A , A satisfies $(*)$ if, and only if, $\overline{P(A)}$ is a union of simplicial faces of $Q(A)$.

The next results illustrates the relation between commutativity and closed simplicial faces.

Proposition 3.4. *Let A be an arbitrary C^* -algebra. Suppose that $\overline{P(A)}$ can be written as a union of closed simplicial faces of $Q(A)$. Then*

- (i) A is of type I.
- (ii) For all irreducible representations π of A on a Hilbert space H_π , $\pi(A) \supseteq LC(H_\pi)$ and $\pi(A)/LC(H_\pi)$ is abelian.

PROOF. Let π be an irreducible representation of A on a Hilbert space H_π . For (i), it is enough to prove that

$$\pi(A) \supseteq LC(H_\pi) \quad (\text{see [9]}).$$

It is known by [5; 4.1.10] that either

$$\pi(A) \supseteq LC(H_\pi) \quad \text{or} \quad \pi(A) \cap LC(H_\pi) = (0)$$

Suppose that $\pi(A) \cap LC(H_\pi) = (0)$, then π is not one dimensional and so there exist distinct equivalent pure states ψ_1 and ψ_2 of $\pi(A)$. Therefore, $\psi_1 \circ \pi$ and $\psi_2 \circ \pi$ are distinct equivalent pure states of A . Then

$$S(\pi(A)) = S(\pi(A)/\pi(A) \cap LC(H_\pi)) \subseteq \overline{VS(\pi(A))}$$

(See GLIMM's results in [7]), where $VS(\pi(A))$ denotes the set of vector states of $\pi(A)$. So

$$(1/2)(\psi_1 + \psi_2) \in \overline{VS(\pi(A))} = \overline{P(\pi(A))} \quad ([7, \text{ Th 2}])$$

and

$$(1/2)(\psi_1 + \psi_2) \circ \pi = (1/2)(\psi_1 \circ \pi + \psi_2 \circ \pi) \in \overline{P(A)},$$

which contradicts the fact that A satisfies $(*)$. Thus

$$\pi(A) \supseteq LC(H_\pi).$$

Now, we prove that $\pi(A)/LC(H_\pi)$ is abelian, for all irreducible representations π of A . Assume the contrary, then there exists some π with $\pi(A)/LC(H_\pi)$ not abelian. Hence, there exist distinct equivalent pure states ψ_1 and ψ_2 of $\pi(A)/LC(H_\pi)$. Then $\psi_1 \circ \pi$ and $\psi_2 \circ \pi$ are distinct equivalent pure states of A . Now by [8, lemma 9].

$$(1/2)(\psi_1 + \psi_2) \in S(\pi(A)/LC(H_\pi)) \subseteq \overline{P(\pi(A))}$$

Hence

$$(1/2)(\psi_1 + \psi_2) \circ \pi \in \overline{P(A)}.$$

This contradicts the fact that A satisfies $(*)$.

Next, we are going to find another condition equivalent to $\overline{P(A)}$ being a union of closed simplicial faces.

Proposition 3.5. *Let A be a C^* -algebra. The following conditions are equivalent:*

- (i) $F(A) \cap \overline{P(A)} = P(A)$ that is, $P(A)$ is relatively closed in $F(A)$
- (ii) A satisfies $(*)$.

PROOF. (i) $\longrightarrow$ (ii) Suppose that condition $(*)$ does not hold for A . Therefore, there exist two distinct equivalent pure states of A , ψ_1, ψ_2 say, such that $(1/2)(\psi_1 + \psi_2) \in \overline{P(A)}$. Furthermore, using [4; 2.1 (ii)], we get

$$(1/2)(\psi_1 + \psi_2) \in F(A)$$

Finally, since $(1/2)(\psi_1 + \psi_2)$ is not pure, we get a contradiction.

(ii) $\longrightarrow$ (i) Let ϕ be in $F(A) \cap \overline{P(A)}$. By proposition 3.4, A is necessarily of type I . Then $\phi \in F_I(A) \cap \overline{P(A)}$, (where $F_I(A)$ denotes the set of factorial states of type I). Note that, by [4, §2], we have

$$\phi = \sum_{i=1}^{\infty} \lambda_i \phi_i \quad \text{where} \quad \lambda_i > 0, \quad \sum_{i=1}^{\infty} \lambda_i = 1$$

and $\{\phi_i\}$ are equivalent pure states of A . On the other hand, since $\phi \in \overline{P(A)}$, there exists a closed simplicial face F of $Q(A)$ which lies in $\overline{P(A)}$ and contains ϕ .

We prove that ϕ is pure. Suppose not, then without loss of generality, we can assume that $\phi_1 \neq \phi_2$. To reach a contradiction, it is sufficient to show that $\phi_1, \phi_2 \in F$ (see [3, Th. 2.5] and [2., cor 3]). Note that $\phi = \lambda_1 \phi_1 + \psi$ where ψ is the rest of the infinite sum. By considering an (approximate) identity for A we obtain that

$$\|\psi\| = 1 - \lambda_1$$

Consider

$$\psi_{\circ} = \frac{1}{1 - \lambda_1} \psi \in S(A).$$

So $\phi = \lambda_1 \phi_1 + (1 - \lambda_1) \psi_{\circ}$, which implies that $\phi_1 \in F$. Similarly, we can prove that ϕ_2 is in F . This contradicts the fact that F is simplicial and so ϕ must be pure.

We end this section by summarizing the above results in the following theorem.

Theorem 3.6. *Let A be an arbitrary C^* -algebra. Then the following are equivalent:*

- (i) $\overline{P(A)}$ a union of closed simplicial faces of $Q(A)$.
- (ii) whenever ψ_1 and ψ_2 are distinct equivalent pure states of A , then

$$(1/2)(\psi_1 + \psi_2) \notin \overline{P(A)}$$

- (iii) $\overline{P(A)} \cap F(A) = P(A)$.

4. Examples and related results

In this section, we consider some examples of type I C^* -algebras with and without property $(*)$.

Let A be a C^* -algebra. A point $\pi_0 \in \hat{A}$ is said to be singular [8, p160], if there is an $E \in A$ with $\pi(E)$ a projection for all π in some

neighbourhood N of π_0 , with $\pi_0(E)$ one dimensional, and such that for each neighbourhood M of π_0 contained in N , there exists π in M so that $\dim \pi(E) > 1$. If π_0 is not singular π_0 is called regular.

Let D be the C^* -algebra of all bounded sequences $x = (x_n)_{n \geq 1}$ of 2×2 complex matrices with coordinatewise operations and

$$\|x\| = \sup_n \|x_n\|$$

Let A be the C^* -subalgebra of D consisting of all $x = (x_n)$ such that x_n converges in norm to a matrix of the form

$$\begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix}, \quad \text{as } n \rightarrow \infty.$$

By [6, Th 1.1. and following], $\hat{A}$ is homeomorphic to $N \cup \{\infty\}$, each $n \in N$ corresponding to a 2-dimensional representation π_n where $\pi_n(x) = x_n$ and ∞ to the 1-dimensional representation π_λ given by $\pi_\lambda(x) = \lambda(x)$. Define $E \in A$ such that

$$E = (E_n) \quad \text{and} \quad E_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all } n.$$

That is, E is the identity of A . Notice that

$$\pi_\lambda(E) = 1 \quad \text{and} \quad \pi_n(E) = E_n$$

is a projection of dimension 2, so that π_λ is singular. On the other hand,

$$\pi_\lambda(A) = \{\lambda(a) : a \in A\}$$

is 1-dimensional. Now applying [8, Th 5], we have

$$\overline{P(A)} = P(A) = \bigcup_{\phi \in P(A)} \{\phi\}$$

and so we can see directly that A satisfies $(*)$ and that $\overline{P(A)}$ is a union of closed simplicial faces of $S(A)$ (in a trivial way).

We show next how tensoring with $M_2(C)$ can destroy property $(*)$.

Let C be the C^* -algebra of all sequences $x = (x_n)_{n \geq 1}$ of 4×4 matrices for which $\sup_n \|x_n\|$ is finite, with coordinatewise operations and

$$\|x\| = \sup_n \|x_n\|$$

Let B be the C^* -subalgebra of C consisting of all $x = (x_n)$ such that x_n converges in norm to a matrix of the form

$$\begin{pmatrix} a(x) & b(x) & 0 & 0 \\ c(x) & d(x) & 0 & 0 \\ 0 & 0 & a(x) & b(x) \\ 0 & 0 & c(x) & d(x) \end{pmatrix} \quad \text{as, } n \rightarrow \infty$$

for some complex numbers $a(x)$, $b(x)$, $c(x)$ and $d(x)$. We write

$$M(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}.$$

$\hat{B}$ is homeomorphic to $N \cup \{\infty\}$, each $n \in N$ corresponding to a 4-dimensional representation π_n given by $\pi_n(x) = x_n$ and ∞ to the 2-dimensional representation π_M given by $\pi_M(x) = M(x)$. In this example, we show that $\overline{P(B)}$ cannot be written as a union of closed simplicial faces of $S(B)$. Consider

$$e_1, e_2 \in C^2 \quad \text{and} \quad \xi_1, \xi_4 \in C^4$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since ξ_1 and ξ_4 are orthogonal unit vectors in C^4 , then they are linearly independent. Finally, using the definition of B , we can prove that.

$$(1/2)(w_{e_1} \circ \pi_M + w_{e_2} \circ \pi_M) = w^* - \lim_{\frac{\xi_1 + \xi_4}{\sqrt{2}}} w_{\frac{\xi_1 + \xi_4}{\sqrt{2}}} \circ \pi_n$$

where

$$w_{e_i}(A) = \langle Ae_i, e_i \rangle \quad \text{for all } A \in M_2(C), \quad i = 1, 2.$$

Thus, there exist two distinct equivalent pure states of B , $\phi_1 = w_{e_1} \circ \pi_M$, $\phi_2 = w_{e_2} \circ \pi_M$ such that

$$(1/2)(\phi_1 + \phi_2) \in \overline{P(B)}.$$

So (*) fails in this example and by proposition 3.2, $\overline{P(B)}$ is not a union of closed simplicial faces of $S(B)$.

We note that in [11. proposition 3.3.5], $\overline{P(B)}$ is explicitly determined:

$$\begin{aligned} \overline{P(B)} = & \{w_\xi \circ \pi_n : n = 1, 2, \dots \quad \text{and} \quad \xi \text{ is a unit vector in } C^4\} \\ & \cup \{\psi \circ \pi_M : \psi \text{ is any state of } M_2(C)\} \end{aligned}$$

It is also shown in [11, Proposition 3.3.11] that if a C^* -algebra C is defined by changing the definition of B to allow the limit matrix to be

$$\begin{pmatrix} M(x) & 0 \\ 0 & N(x) \end{pmatrix}$$

where $M(x), N(x) \in M_2(C)$, then

$$\begin{aligned} \overline{P(C)} = & \{w_\xi \circ \pi_n : n = 1, 2, \dots \text{ and } \xi \text{ is a unit vector in } C^4\} \\ & \cup \{a(w_\xi \circ \pi_M) + (1 - \alpha)(w_\eta \circ \pi_N) : 0 \leq \alpha \leq 1 \text{ and } \xi, \eta \text{ are} \\ & \text{unit vectors in } C^2\} \end{aligned}$$

and hence $\overline{P(C)}$ is a union of simplicial faces of $S(A)$ (singletons and line segments).

Finally, we note that if A is a C^* -algebra such that $LC(H_\pi) \subseteq A \subseteq L(H_\pi)$ and $A/LC(H_\pi)$ is abelian then

$$\overline{P(A)} = \cup \{F_\xi : \xi \text{ is a unit vector in } H_\pi\}$$

where

$$F_\xi = \{\alpha w_\xi + (1 - \alpha)g : 0 \leq \alpha \leq 1, g \in S(A)/LC(H_\pi)\}$$

a simplicial closed face of $S(A)$ [11, §3]. In this connection, see proposition 3.4 (ii).

5. Simplicial faces in factorial state spaces of a C^* -algebra

In this section, we find a necessary and sufficient condition for the factorial state space of a C^* -algebra A to be a union of closed simplicial faces.

Let $F(A)$ be the set of all ϕ in $S(A)$ such that $\pi_\phi(A)'$ is a factor. We define the factorial state space of A as the w^* -closed of $F(A)$ and we denote it by $\overline{F(A)}$.

Proposition 5.1. *Let A be a unital C^* -algebra. Then A is abelian if, and only if, $\overline{F(A)}$ is a union of closed simplicial faces of $S(A)$.*

PROOF. ($\longrightarrow$) if A is abelian, then

$$\overline{F(A)} = \overline{P(A)} = P(A).$$

Hence $\overline{F(A)} = \bigcup_{\phi \in P(A)} \{\phi\}$, a union of closed simplicial faces of $S(A)$.

($\leftarrow$) Suppose A is not abelian. Then there exists an irreducible representation π with $\dim H_\pi > 1$. Choose $\xi_1, \xi_2 \in H_\pi$ so that they are linearly independent unit vectors. Let

$$\begin{aligned}\psi_1(a) &= \langle \pi(a)\xi_1, \xi_1 \rangle \quad \text{and} \\ \psi_2(a) &= \langle \pi(a)\xi_2, \xi_2 \rangle \quad \text{for all } a \in A.\end{aligned}$$

It is easy to check that ψ_1, ψ_2 are distinct equivalent pure states of A . Let $\phi = (1/2)(\psi_1 + \psi_2)$. By [4, Th 2.1], we get

$$\phi \in F_I(A) (\subseteq \overline{F(A)})$$

Finally, ϕ does not belong to any closed simplicial faces of $S(A)$. For, suppose F is a face of $S(A)$ such that $\phi \in F$. Therefore, $\psi_1, \psi_2 \in F$ and F is not a simplex.

In the non-unital case, consider the restriction map r given by $r : S(\tilde{A}) \longrightarrow Q(A)$, where $\tilde{A}$ is the C^* -algebra obtained from A by the adjoining of an identity. Now since

$$r(F(\tilde{A})) = F(A) \cup \{0\}$$

and

$$\begin{aligned}0 \in \overline{P(A)} \subset \overline{F(A)}, \quad [5; 2.12.13], \text{ we obtain} \\ r(\overline{F(\tilde{A})}) = \overline{F(A)}.\end{aligned}$$

Proposition 5.2. *Let A be a non-unital C^* -algebra. Then the following are equivalent:*

- (i) A is abelian
- (ii) $\overline{F(A)}$ is a union of closed simplicial faces of $Q(A)$

PROOF. It is clear that A is abelian if, and only if, $\tilde{A}$ is abelian. Moreover, since r is an affine homeomorphism, $\overline{F(A)}$ is a union of closed simplicial faces of $Q(A)$ if, and only if, $\overline{F(\tilde{A})}$ is a union of closed simplicial faces of $S(\tilde{A})$. The result then follows from proposition 5.1.

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- [11] A. M. ZAKI Ph. D., Thesis, *Alderdeen University*, 1987.

A. M. ZAKI
ABHA TEACH. JUN. TRAIN. COLLEGE
ABHA, P.O. BOX 249
SAUDI ARABIA

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