U-lifters of single morphisms in the category of F-algebras

By JENŐ SZIGETI (Budapest)

1. Introduction

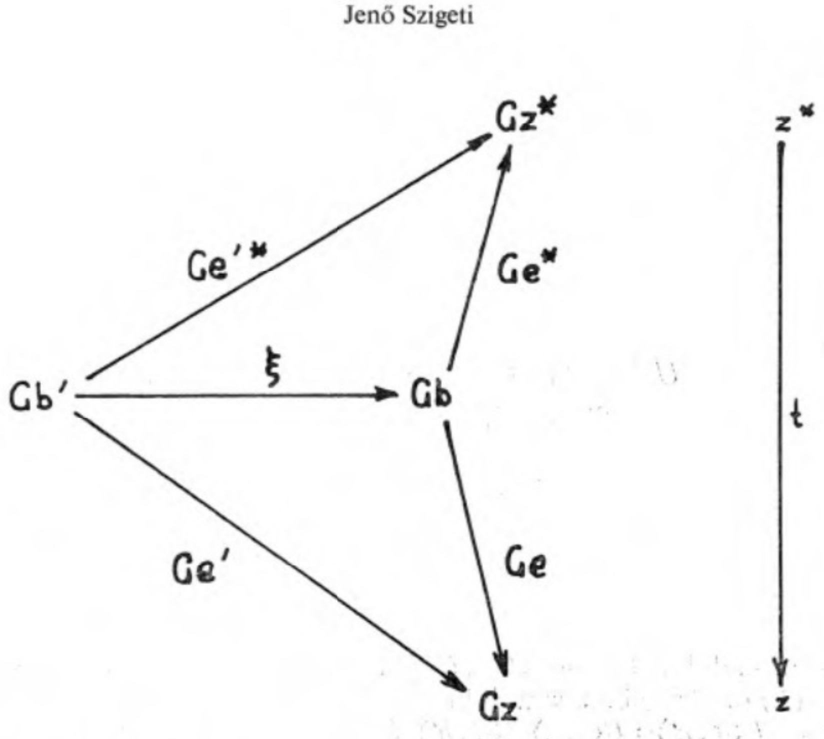
Given an endofunctor $F: \mathscr{A} \to \mathscr{A}$ on the category $\mathscr{A}$ the construction of the category $\mathscr{A}(F)$ of F-algebras is well known (see e.g. in [1—5, 7, 11, 14—15]). For a morphism ξ: $U\langle a', h'\rangle \to U\langle a, h\rangle$ ($\langle a', h'\rangle, \langle a; h\rangle$ are F-algebras and $U: \mathscr{A}(F) \to \mathscr{A}$ is the canonical forgetful functor) one can define the U-lifter of ξ as in [10]. If U is presemitopological then the existence of this U-lifter is clear since presemitopologicity was defined by requiring U-lifters for all $U \circ D \to U\langle a, h\rangle$ cones, where D ranges on functors into $\mathscr{A}(F)$ (cf. [9—10]). Originally the more general notion of a G-lifter ($G: \mathscr{B} \to \mathscr{A}$ is a functor) for a single morphism first appeared in [8] under the name $(1_{\mathscr{B}}, G)$-quotient.

The main aim of the present paper is to give sufficient conditions for the existence of certain U-lifters. The first result (2.1) concerns the general case of G-lifters. In (2.4) an image factorization system $(\mathscr{E}, \mathscr{M})$ will be used in order to obtain U-lifters of morphisms $\xi \in \mathscr{E}$. Finally the condition in (2.6) that F preserves certain colimits will give the existence of all U-lifters (of single morphisms).

2. Constructions providing lifters

Let $G: \mathscr{B} \to \mathscr{A}$ be a functor and $\xi: Gb' \to Gb$ be a morphism in $\mathscr{A}$; then we say that ξ has a G-lifter if there is a triple $\langle e'^*, z^*, e^*\rangle$ with $z^* \in |\mathscr{B}|$, $e'^*: b' \to z^*$, $e^*: b \to z^*$ and $(Ge^*)\circ\xi = Ge'^*$ such that for each triple $\langle e', z, e\rangle$ with $z \in |\mathscr{B}|$, $e': b' \to z$, $e: b \to z$ and $(Ge)\circ\xi = Ge'$ there exists a unique morphism $t: z^* \to z$ with $t \circ e^* = e$ and $t \circ e'^* = e'$ (cf. [10]).

(2./1)



2.1. Proposition. *Let $\mathscr{B}$ have pushouts and $V: \mathscr{A} \to \mathscr{B}$ be a left adjoint to the functor $G: \mathscr{B} \to \mathscr{A}$ with counit $\delta: V \circ G \to 1_{\mathscr{B}}$. Then each $\xi: Gb' \to Gb$ has a G-lifter.*

Proof. Take the following pushout in $\mathscr{B}$. ‖

(2./2)

$$\begin{tikzcd} VGb' \arrow[r, "V\xi"] \arrow[d, "\delta_{b'}"'] & VGb \arrow[r, "\delta_b"] & b \arrow[d, "e^*"] \\ b' \arrow[rr, "e'^*"'] & & z^* \end{tikzcd}$$

2.2. *Remark.* Clearly $\dashv G$ in the above proposition can be replaced by the following weaker condition: $\underset{G}{\dashv} G$. The latter relative adjointness (for relative adjoints see [16]) is equivalent to the existence of initial objects in the commacategories $(Gb{\downarrow}G)$ for $b \in |\mathscr{B}|$. ‖

2.3. Corollary. *Suppose that $\mathscr{A}(F)$ has pushouts and free algebras (i.e. there is a left adjoint to $U: \mathscr{A}(F) \to \mathscr{A}$), then there exists a U-lifter for all $\xi: U\langle a', h'\rangle \to U\langle a, h\rangle$.* ‖

2.4. Theorem. *Let $\mathscr{A}$ be a cocomplete $\mathscr{E}$-cowell powered category with an image factorization system $(\mathscr{E}, \mathscr{M})$. If the functor $F: \mathscr{A} \to \mathscr{A}$ preserves $\mathscr{E}$ then each morphism $\xi: U\langle a', h'\rangle \to U\langle a, h\rangle$ with $\xi \in \mathscr{E}$ admits a U-lifter.*

Proof. Let (2./3) represent the $\mathscr{A}$-colimit of the noncommutative triangle (NC) with the colimit morphisms ε and π.

(2./3)
$$\begin{tikzcd}
a \arrow[rr, "\varepsilon"] & & x \\
a' \arrow[u, "\xi"] & & \\
Fa' \arrow[u, "h'"] \arrow[rr] \arrow[r, phantom, "(NC)"] & & Fa \arrow[uu, "\pi"'] \arrow[uull, "h"']
\end{tikzcd}$$

At first we claim that $\varepsilon \in \mathscr{E}$. If $\varepsilon = m \circ e$ is a factorization of ε with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ then there is a unique morphism $p: Fa \to y$ making (2./4) commute.

(2./4)
$$\begin{tikzcd}
& Y \arrow[dr, "m \in \mathcal{M}"] & \\
a \arrow[ur, "e"] & & x \\
a' \arrow[u, "\xi"] & & \\
Fa' \arrow[u, "h'"] \arrow[rr, "F\xi \in \varepsilon"'] & & Fa \arrow[uu, "\pi"'] \arrow[uuul, "p"]
\end{tikzcd}$$

Since the morphisms e and p create a cocone for (NC) there exists a unique $k: x \to y$ with $k \circ \varepsilon = e$ and $k \circ \pi = p$. $k \circ m \circ e = k \circ \varepsilon = 1_y \circ e$ implies $k \circ m = 1_y$ and the equations $m \circ k \circ \varepsilon = m \circ e = \varepsilon$, $m \circ k \circ \pi = m \circ p = \pi$ imply $m \circ k = 1_x$. Thus m is proved to be an isomorphism, consequently $\varepsilon \in \mathscr{E}$ by $m \in \mathscr{E}$.

It follows from $F\varepsilon \in \mathscr{E}$ that the free completion (2./5) of the partial F-algebra $x \xleftarrow{\pi} Fa \xrightarrow{F\varepsilon} Fx$ exi st: ,t he pushout construction will stop).

(2./5)
$$\begin{tikzcd}
{} \arrow[d, "f^*"'] & Fa \arrow[l, "\pi"'] \arrow[r, "F\varepsilon"] & Fx \arrow[d, "Ff^*"] \\
\omega^* & & F\omega^* \arrow[ll, "u^*"]
\end{tikzcd}$$

It can easily be seen that $\langle f^* \circ \varepsilon \circ \xi, \langle \omega^*, u^* \rangle, f^* \circ \varepsilon \rangle$ will be the U-lifter of ξ. ‖

2.5. *Remark.* For the basic properties of image factorization systems in categories see [3, 5—6, 11, 13, 15]. The result on free completions of partial F-algebras used in the proof of (2.4) can be found e.g. in [5, 11]. ‖

Let $\mathcal{N}$ be the following category. Objects: $|\mathcal{N}|=\{0, 1, 2, \dots, n, \dots\}$. Morphisms: the units, and for each $n \geqq 1$ there are n distinct morphisms $d_i^n: n \to 0$ $(i=0, 1, \dots, n-1)$. Composition: obvious.

2.6. Theorem. *Let $\mathcal{A}$ be $\mathcal{N}$-cocomplete (each functor $\mathcal{N} \to \mathcal{A}$ has a colimit in $\mathcal{A}$) and let F preserve these colimits, then each $\xi: U\langle a', h'\rangle \to U\langle a, h\rangle$ has a U-lifter.*

Proof. Define the functor $N: \mathcal{N} \to \mathcal{A}$ by $n \to 0 \mapsto F^{n-1} a' \to a$ with $Nd_i^n = h \circ Fh \circ \dots \circ F^{i-1} h \circ F^i \xi \circ F^i h' \circ \dots \circ F^{n-2} h'$ ($F^0 = 1_{\mathcal{A}}$ and F^{-1} is irregular). Let $q^*: N \to \omega^*$ be a colimit cocone then $Fq^*: F \circ N \to F\omega^*$ is a colimit cocone too. Define $Q: F \circ N \to \omega^*$ as follows: $Q_0 = q_0^* \circ h$, and for $n \geqq 1$ let $Q_n = Q_0 \circ FNd_0^n$. Clearly, Q is a cocone since $h \circ FNd_i^n = Nd_{i+1}^{n+1}$ $(0 \leqq i \leqq n-1)$. Hence there is a unique $u^*: F\omega^* \to \omega^*$ with $u^* \circ Fq^* = Q$. Now $q_0^*: \langle a, h\rangle \to \langle \omega^*, u^*\rangle$ and $q_0^* \circ \xi: \langle a', h'\rangle \to \langle \omega^*, u^*\rangle$ are $\mathcal{A}(F)$-morphisms because of $q_0^* \circ h = Q_0 = u^* \circ Fq_0^*$ and $(q_0^* \circ \xi) \circ h' = q_0^* \circ Nd_0^2 = q_0^* \circ Nd_1^2 = q_0^* \circ h \circ F\xi = u^* \circ F(q_0^* \circ \xi)$. We prove that $\langle q_0^* \circ \xi, \langle \omega^*, u^*\rangle, q_0^*\rangle$ is the required U-lifter. Let $\langle e', \langle \omega, u\rangle, e\rangle$ be a triple with $\langle \omega, u\rangle \in |\mathcal{A}(F)|$, $e': \langle a', h'\rangle \to \langle \omega, u\rangle$, $e: \langle a, h\rangle \to \langle \omega, u\rangle$ and $(Ue) \circ \xi = Ue'$ then define $q: N \to \omega$ as follows: $q_0 = e$, and for $n \geqq 1$ let $q_n = q_0 \circ Nd_0^n$. q is a cocone: for $0 \leqq i \leqq n-1$ we have $q_0 \circ Nd_i^n = e \circ h \circ Fh \circ \dots \circ F^{i-1} h \circ F^i \xi \circ F^i h' \circ \dots \circ F^{n-2} h' = u \circ Fu \circ \dots \circ F^{i-1} u \circ F^i e' \circ F^i h' \circ \dots \circ F^{n-2} h' = e' \circ h' \circ Fh' \circ \dots \circ F^{i-1} h' \circ F^i h' \circ \dots \circ F^{n-2} h' = e \circ \xi \circ h' \circ Fh' \circ \dots \circ F^{n-2} h' = q_0 \circ Nd_0^n = q_n$ since $e' = e \circ \xi$ and $e \circ h = u \circ Fe$, $e' \circ h' = u \circ Fe'$. Hence there is a unique $t: \omega^* \to \omega$ with $t \circ q^* = q$. Then $t \circ q_0^* = q_0 = e$, $t \circ (q_0^* \circ \xi) = e \circ \xi = e'$ and $t: \langle \omega^*, u^*\rangle \to \langle \omega, u\rangle$ is in $\mathcal{A}(F)$ by $t \circ Q = u \circ Fq$. Moreover t is uniquely determined already by $t \circ q_0^* = e$. ‖

At a first glance theorem (2.6) seems to be weaker than the corollary (2.3). However this is not the case. In the following example a very simple situation will be given for which (2.6) is applicable but (2.3) is not.

2.7. *Example*. Let $\mathrm{Set_{fin}}$ denote the full subcategory of Set consisting of all finite sets and consider $U: \mathrm{Set_{fin}}(1) \to \mathrm{Set_{fin}}$ where $1: \mathrm{Set_{fin}} \to \mathrm{Set_{fin}}$ is identical. For a functor $N: \mathcal{N} \to \mathrm{Set_{fin}}$ let $A_n = Nn$ $(n \geqq 0)$ then the colimit of N is of the form $A_n \to A_0 \xrightarrow{\text{canonical}} A_0/\Theta$, where Θ can be obtained as an intersection of appropriate equivalences.

Suppose that a nonvoid $X \in \mathrm{Set_{fin}}$ generates a free algebra $X \xrightarrow{f^*} A^* \xleftarrow{h^*} A^*$ in $\mathrm{Set_{fin}}(1)$. Let $A = \{0, 1, \dots, n\}$ and $h(i) = i+1$ $(0 \leqq i < n)$, $h(n) = 0$ with $n = |A^*|$. Let $f: X \to A$ be a constant (e.g. $f = 0$) then there exists a unique $k: A^* \to A$ making the following diagram commute.

$$
\begin{tikzcd}
X \arrow[r, bend left, "f^*"] \arrow[dr, "f"'] & A^* \arrow[d, "k"] & A^* \arrow[l, bend right, "h^*"'] \arrow[d, "k"] \\
 & A & A \arrow[l, bend left, "h"]
\end{tikzcd}
\tag{2./5}
$$

This is a contradiction since for an $a \in A^*$ we have $n+1$ different elements $k(a)$, $hk(a), \ldots, h^n k(a)$, consequently $a, h^*(a), \ldots, (h^*)^n(a)$ are different elements in A^*. ‖

References

[1] J. Adámek, Free algebras and automata realizations in the language of categories, *Comment. Math. Univ. Carolinae* **15** (1974).

[2] J. Adámek, Limits and colimits in generalized algebraic categories, *Czech. Math. J.* **26** (1976).

[3] J. Adámek, Colimits of algebras revisited, *Bull. Austral, Math. Soc.* **17** (1977).

[4] J. Adámek and V. Koubek, Functorialalgebras and automata, *Kybernetika* **13** (1977).

[5] J. Adámek and V. Trnková, Varietors and machines, *COINS Technical Report 78-6 Univ. of Massachusetts at Amherst* (1978).

[6] H. Andréka and I. Németi, On the more general notion of factorisation, *Preprint of the Math. Inst. Hung. Acad. Sci.* (1979).

[7] M. Barr, Coequalizers and free triples, *Math. Z.* **116** (1970).

[8] G. Greve, (G, V_0)-Quotienten, *Seminarberichte Nr. 1 Fachbereich Mathematik, Fernuniversität Hagen* (1976).

[9] G. Greve, Lifting closed and monoidal structures along semitopological functors, *Lect. Notes in Math. 719, Springer* (1979).

[10] G. Greve, J. Szigeti and W. Tholen, Lifting tensorproducts along non-adjoint functors, *Cahiers Top. et Géom. Diff.* XXIII—4 (1982).

[11] V. Koubek and J. Reiterman, Categorical constructions of free algebras, colimits, and completions of partial algebras. *J. Pure Appl. Algebra* **14** (1979).

[12] S. MacLane, Categories for the Working Mathematician, GTM6, *Springer* (1971).

[13] E. G. Manes, „Algebraic Theories", GTM26, *Springer* (1976).

[14] J. Reiterman, A left adjoint construction related to free triples, *J. Pure Appl. Algebra* **10** (1977/78).

[15] J. Szigeti, Factorization systems in the category of Arbib-Manes automata, *Acta Cybernetica, to appear*.

[16] F. Ulmer, Properties of Dense and Relative Adjoint Functors, *J. Algebra* **8** (1968).

(Received August 8, 1983.)