

The gammaoperators in L^p spaces

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§ 1. Introduction

The gammaoperators [3]

$$G_n(f; x) = \frac{x^{n+1}}{n!} \int_0^\infty e^{-xu} u^n f\left(\frac{n}{u}\right) du \stackrel{\text{def}}{=} \int_0^\infty g_n(x, u) f\left(\frac{n}{u}\right) du \quad (x > 0)$$

were introduced by M. MÜLLER and A. LUPAS and several papers have been devoted to the investigation of their approximation-theoretical properties. Concerning global uniform approximation we have proved in [7].

Theorem A. Let f be a continuous and bounded function defined on $(0, \infty)$. For every $0 < \alpha \leq 1$ the statements

$$|G_n(f) - f| \leq K_1 n^{-\alpha}, \quad n = 1, 2, \dots$$

and

$$e^x |A_h^2(g; x)| \leq K_2 h^{2\alpha}, \quad h > 0, x \in (-\infty, \infty), g(x) = f(e^x)$$

are equivalent.

Here

$$A_h^2(g; x) = g(x-h) - 2g(x) + g(x+h)$$

is the usual symmetric second difference of g .

The aim of the present article is to prove the analogous result in the L^p -metric. It is easy to see that for every function $f \in L^p(0, \infty)$ $G_n(f; x)$ is defined for every $x > 0$ and $n \geq 2$. Now the analogue of Theorem A in L^p is

Theorem 1. Let $\varphi(x) = x$, $1 \leq p < \infty$, $f \in L^p(0, \infty)$ and $0 < \alpha \leq 1$. Then the relations

$$(1.1) \quad \|G_n(f) - f\|_{L^p(0, \infty)} = O(n^{-\alpha}) \quad (n \rightarrow \infty)$$

and

$$(1.2) \quad \|\Delta_{h\varphi}^2(f)\|_{L^p(0,\infty)} = O(h^{2\alpha}) \quad (h \rightarrow 0+0)$$

are equivalent.

Corollary 1. For any $0 < \alpha \leq 1$ (1.1) is equivalent to

$$(1.3) \quad \left\{ \int_{-\infty}^{\infty} e^x |\Delta_h^2(g; x)|^p dx \right\}^{1/p} \leq Kh^{2\alpha}, \quad g(x) = f(e^x), \quad (h > 0).$$

Corollary 2. If $\alpha = 1$ then for $p > 1$ (1.1) is equivalent to the fact that f has a locally absolutely continuous derivative f' with $x^2 f'' \in L^p(0, \infty)$, and for $p = 1$ (1.1) holds if and only if f is locally absolutely continuous and $x^2 f'(x)$ is of bounded variation on $(0, \infty)$.

Naturally, it is understood in Corollary 2 that f coincides a.e. with a function having the stated properties.

We made the assumption $0 < \alpha \leq 1$ because a simple modification of our proof shows, that $\{G_n\}$ is saturated of order $\{n^{-1}\}$, i.e. $f \in L^p(0, \infty)$ and

$$\liminf_{n \rightarrow \infty} n \|G_n(f) - f\|_{L^p(0,\infty)} = 0$$

implies that f is zero a.e.

Now let us introduce the following modified L^p -modulus of smoothness:

$$\omega(f; \delta)_p = \sup_{0 \leq h \leq \delta} \|\Delta_{h\varphi}^2(f)\|_{L^p(0,\infty)}, \quad \varphi(x) = x, \quad \delta > 0.$$

This ω has the usual properties of the ordinary L^p -modulus of smoothness (i.e. when $\varphi \equiv 1$) e.g. there is a K independent of f , $\delta > 0$ and $\lambda \geq 1$ such that

$$(1.4) \quad \omega(f; \lambda\delta)_p \leq K\lambda^2 \omega(f; \delta)_p$$

(see [5, Theorem 1]).

For $0 < \alpha < 1$ Theorem 1 is contained in

Theorem 2. Let $1 \leq p < \infty$, $f \in L^p(0, \infty)$ and

$$E_n(f) = \|G_n(f) - f\|_{L^p(0,\infty)}.$$

Then there is a constant K_p depending only on p such that the estimates

$$(1.5) \quad E_n(f) \leq K_p \omega\left(f; \frac{1}{\sqrt{n}}\right)_p \quad (n \geq 2)$$

and

$$(1.6) \quad \omega\left(f; \frac{1}{\sqrt{n}}\right)_p \leq K_p \frac{\|f\| + E_2(f) + \dots + E_n(f)}{n} \quad (n \geq 2)$$

hold.

We shall prove Theorem 2 in the next paragraph while the proof of Theorem 1 will be given in § 3.

§ 2. Proof of Theorem 2

In what follows $\varphi(x)=x$ and K is a positive constant depending only on p and not necessarily the same at each occurrence.

I. PROOF OF (1.5). Let $L^p = L^p(0, \infty)$, $\|\cdot\|_p = \|\cdot\|_{L^p(0, \infty)}$ and

$$(2.1) \quad K(f; t)_p = \inf_{g \in D_p} (\|f - g\|_p + t^2 \|\varphi^2 g''\|_p),$$

$$(2.2) \quad D_p = \{g \in L^p | g' \text{ loc. abs. cont., } \varphi^2 g'' \in L^p\}$$

the so called (Peetre-) K -functional. In [6] we proved the estimate

$$(2.3) \quad K(f; t)_p \leq K\omega(f; t)_p \quad (0 < t < 1).$$

Therefore, if we show that for $n \geq 2$

$$(2.4) \quad \|G_n(f)\|_p \leq K \|f\|_p \quad (f \in L^p)$$

and

$$(2.5) \quad \|G_n(f) - f\|_p \leq \frac{K}{n} \|\varphi^2 f''\|_p \quad (f \in D_p)$$

are satisfied then choosing a $g_n \in D_p$ with the property

$$\|f - g_n\|_p + \frac{1}{n} \|\varphi^2 g_n''\|_p \leq 2K \left(f; \frac{1}{\sqrt{n}}\right)_p$$

(see (2.1)) we obtain by (2.3) the required estimate:

$$\begin{aligned} \|G_n(f) - f\|_p &\leq \|G_n(f - g_n)\|_p + \|f - g_n\|_p + \|G_n(g_n) - g_n\|_p \leq \\ &\leq K \left(\|f - g_n\|_p + \frac{1}{n} \|\varphi^2 g_n''\|_p\right) \leq KK \left(f; \frac{1}{\sqrt{n}}\right)_p \leq K\omega \left(f; \frac{1}{\sqrt{n}}\right)_p. \end{aligned}$$

For later application we shall prove somewhat more than (2.4), namely

$$(2.6) \quad \|G_n(f)\|_p \leq \left(1 + \frac{1}{n}\right)^{1/p} \|f\|_p \quad (f \in L^p, n \geq 2).$$

A change of variable yields

$$(2.7) \quad G_n(f; x) = \int_0^\infty \frac{e^{-\tau} \tau^n}{n!} f\left(\frac{nx}{\tau}\right) d\tau.$$

Now let us use the fact that the norm of an integral is not greater than the integral of the corresponding norms (cf. [4, p. 271]) by which

$$\begin{aligned} \left\{ \int_0^\infty \left| \int_0^\infty \frac{e^{-\tau} \tau^n}{n!} f\left(\frac{nx}{\tau}\right) d\tau \right|^p dx \right\}^{1/p} &\leq \int_0^\infty \frac{e^{-\tau} \tau^n}{n!} \left\{ \int_0^\infty \left| f\left(\frac{nx}{\tau}\right) \right|^p dx \right\}^{1/p} d\tau = \\ &= \|f\|_p \int_0^\infty \frac{e^{-\tau} \tau^{n+1/p}}{n! n^{1/p}} d\tau = \|f\|_p \frac{\Gamma(n+1/p+1)}{n! n^{1/p}} \leq \\ &\leq \|f\|_p \frac{(\Gamma(n+1))^{1-1/p} (\Gamma(n+2))^{1/p}}{n! n^{1/p}} = \|f\|_p \frac{(n+1)^{1/p}}{n^{1/p}} \end{aligned}$$

where, at the last but one step we also applied the convexity of $\log \Gamma(x)$ (see [1].)

In the proof of (2.5) we need the following lemma

Lemma 1. *If $x > 0$, $n \geq 2$ and*

$$h_x(t) = \frac{(t-x)^2}{x^2} \max\left(1; \frac{x}{t}\right)$$

then

$$G_n(h_x; x) \leq \frac{10}{n}.$$

Indeed, a simple calculation gives (see e.g. [3]) that

$$G_n((t-x)^2; x) = \frac{x^2}{n-1} \quad (n \geq 2)$$

and so

$$\begin{aligned} G_n(h_x; x) &= \int_0^\infty g_n(x, u) x^{-2} \left(\frac{n}{u} - x\right)^2 \max\left(1, \frac{ux}{n}\right) du = \\ &= \int_0^{2n/x} + \int_{2n/x}^\infty \leq 2x^{-2} G_n((t-x)^2; x) + 4x^{-2} \frac{n+1}{n} G_{n+1}((t-x)^2; x) \leq \frac{10}{n}, \end{aligned}$$

where we used the identity $(ux/n)g_n(x, u) = ((n+1)/n)g_{n+1}(x, u)$ and the fact that for $u \geq 2n/x$, $n \geq 2$ the inequality

$$\left(\frac{n}{u} - x\right)^2 \leq 4 \left(\frac{n+1}{u} - x\right)^2$$

holds.

Returning to the proof of (2.5), for an $f \in D_p$ we can write by Taylor's formula

$$\begin{aligned} (2.8) \quad f(t) - f(x) &= f'(x)(t-x) + \int_x^t (t-u) f''(u) du = \\ &= f'(x)(t-x) + \int_0^{t-x} \frac{t-x-u}{(x+u)^2} ((x+u)^2 f''(x+u)) du. \end{aligned}$$

Here

$$\frac{|t-x-u|}{(x+u)^2} \leq \frac{|t-x|}{x^2} \max\left(1; \frac{x}{t}\right) \quad (u \in (0, t-x))$$

by which

$$\left| \int_0^{t-x} \frac{t-x-u}{(x+u)^2} ((x+u)^2 f''(x+u)) du \right| \leq \frac{(t-x)^2}{x^2} \max\left(1; \frac{x}{t}\right) M(\varphi^2 f''; x)$$

where $M(g; x)$ denotes the Hardy—Littlewood-maximal function of g (see [4, p. 5]). Now for $p > 1$ the maximal-inequality ([4, p. 5]), (2.8) and Lemma 1 together with the easily verifiable facts

$$G_n(t; x) \equiv x, \quad G_n(1; x) \equiv 1, \quad |G_n(g_1; x)| \leq G_n(g_2; x) \quad \text{for} \quad |g_1| \leq g_2$$

yield

$$\begin{aligned}\|G_n(f) - f\|_p &\equiv \|G_n(h, \cdot) M(\varphi^2 f''; \cdot)\|_p \equiv \\ &\equiv \frac{10}{n} \|M(\varphi^2 f'')\|_p \equiv \frac{K}{n} \|\varphi^2 f''\|_p\end{aligned}$$

which is (2.5).

For $p=1$ we choose a more direct argument. All the transformations below are permitted because of the nonnegativity of the occurring functions. Using (2.8) and $G_n(t-x; x)=0$ we obtain

$$\begin{aligned}\|G_n(f) - f\|_1 &= \int_0^\infty \left| \int_0^\infty \left(\int_x^{nx/\tau} \left(\frac{nx}{\tau} - u \right) f''(u) du \right) \frac{e^{-\tau} \tau^n}{n!} d\tau \right| dx \equiv \\ &\equiv \int_0^\infty \int_0^n \left(\int_x^{nx/\tau} \left(\frac{nx}{\tau} - u \right) |f''(u)| du \right) \frac{e^{-\tau} \tau^n}{n!} d\tau dx + \\ &+ \int_0^\infty \int_n^\infty \left(\int_{nx/\tau}^x \left(u - \frac{nx}{\tau} \right) |f''(u)| du \right) \frac{e^{-\tau} \tau^n}{n!} d\tau dx = \\ &= \int_0^n \int_0^\infty \int_0^{nx/\tau} () du dx d\tau + \int_n^\infty \int_0^\infty \int_{nx/\tau}^x () du dx d\tau = \\ &= \int_0^n \left(\int_0^\infty \left(\int_{\tau u/n}^u \left(\frac{nx}{\tau} - u \right) dx \right) |f''(u)| du \right) \frac{e^{-\tau} \tau^n}{n!} d\tau + \\ &+ \int_n^\infty \left(\int_0^\infty \left(\int_u^{\tau u/n} \left(u - \frac{nx}{\tau} \right) dx \right) |f''(u)| du \right) \frac{e^{-\tau} \tau^n}{n!} d\tau = \\ &= \frac{1}{2} \left(\int_0^n + \int_n^\infty \right) \int_0^\infty \frac{\tau}{n} \left(\frac{nu}{\tau} - u \right)^2 |f''(u)| \frac{e^{-\tau} \tau^n}{n!} du d\tau = \\ &= \frac{n+1}{2n} \int_0^\infty |f''(u)| \left(\int_0^\infty \left(\frac{nu}{\tau} - u \right)^2 \frac{e^{-\tau} \tau^{n+1}}{(n+1)!} d\tau \right) du = \\ &= \frac{1}{2n} \int_0^\infty u^2 |f''(u)| du\end{aligned}$$

since an easy computation gives

$$\begin{aligned}&\int_0^\infty \left(\frac{nu}{\tau} - u \right)^2 \frac{e^{-\tau} \tau^{n+1}}{(n+1)!} d\tau = \\ &= u^2 \left(\frac{n}{(n+1)} \frac{\Gamma(n)}{(n-1)!} - 2 \frac{n}{(n+1)} \frac{\Gamma(n+1)}{n!} + \frac{\Gamma(n+2)}{(n+1)!} \right) = \frac{u^2}{n+1}.\end{aligned}$$

The proof of (1.5) is thus complete.

II. PROOF OF (1.6). We shall use the following lemma which is a modification of a result of A. GRUNDMANN [2].

Lemma 2. Let $\mathcal{B}$ be a Banach space, $D \subseteq \mathcal{B}$ a linear subspace, $S: D \rightarrow \mathcal{B}$, $T_n: \mathcal{B} \rightarrow D$ ($n=2, 3, \dots$) linear operators with the properties

$$(2.9) \quad \|ST_n f\| \leq K_1 n \|f\| \quad (f \in \mathcal{B})$$

and

$$(2.10) \quad \|ST_n f\| \leq \left(1 + \frac{K_1}{n}\right) \|Sf\| \quad (f \in D)$$

where K_1 is a positive constant. Then

$$(2.11) \quad \inf_{g \in D} \left(\|f - g\| + \frac{1}{n} \|Sg\| \right) \leq \frac{K_2}{n} \left(\|f\| + \sum_{i=2}^n \|T_i f - f\| \right) \quad (n = 2, 3, \dots)$$

where the constant K_2 depends only on K_1 .

PROOF. Let $2^m \leq n < 2^{m+1}$, $E_i = \|T_i f - f\|$, and for $0 \leq v < m$ let k_v be defined by

$$2^v < k_v \leq 2^{v+1}, \quad E_{k_v} = \min_{2^v < i \leq 2^{v+1}} E_i.$$

Clearly, if $K\left(\frac{1}{n}\right)$ is the infimum on the left of (2.11) then

$$(2.12) \quad K\left(\frac{1}{n}\right) \leq \|f - T_{k_{m-1}} f\| + \frac{1}{2^m} \|ST_{k_{m-1}} f\|$$

and we can see by (2.9) and (2.10) that here

$$\begin{aligned} \|ST_{k_{m-1}} f\| &\leq \|ST_{k_{m-1}}(f - T_{k_{m-2}} f)\| + \|ST_{k_{m-1}} T_{k_{m-2}} f\| \leq \\ &\leq K_1 k_{m-1} E_{k_{m-2}} + \left(1 + \frac{K_1}{k_{m-1}}\right) \|ST_{k_{m-2}} f\|. \end{aligned}$$

Iterating this it follows by $2^v < k_v \leq 2^{v+1}$ that

$$\begin{aligned} \|ST_{k_{m-1}} f\| &\leq K_1 \left(k_{m-1} E_{k_{m-2}} + \left(1 + \frac{K_1}{k_{m-1}}\right) k_{m-2} E_{k_{m-3}} + \dots \right. \\ &\quad \left. \dots + \left(1 + \frac{K_1}{k_{m-1}}\right) \dots \left(1 + \frac{K_1}{k_2}\right) k_1 E_{k_0} \right) + \left(1 + \frac{K_1}{k_{m-1}}\right) \dots \left(1 + \frac{K_1}{k_1}\right) \|ST_{k_0} f\| \leq \\ &\leq K_1 \prod_{v=0}^{m-1} \left(1 + \frac{K_1}{2^v}\right) \left(2 \sum_{v=0}^{m-1} 2^v E_{k_v} + K_1 \|f\|\right) \leq K_2 \left(\sum_{k=2}^{2^m} E_k + \|f\| \right) \end{aligned}$$

which, together with (2.12), proves the lemma.

Now let us apply Lemma 2 with $\mathcal{B} = L^p(0, \infty)$, D as in (2.2), $Sf = \varphi^2 f''$, $T_n = G_n$. If we show (2.9) and (2.10) then we obtain from the lemma that (see (2.1))

$$(2.13) \quad K\left(f; \frac{1}{\sqrt{n}}\right)_p \leq \frac{K}{n} \left(\|f\|_p + \sum_{i=2}^n \|G_i(f) - f\|_p \right).$$

Since the inequality $\omega\left(f; \frac{1}{\sqrt{n}}\right)_p \leq KK\left(f; \frac{1}{\sqrt{n}}\right)_p$ was proved in [6, Theorem 1] the estimate (1.6) follows from (2.13).

Thus, there remains to be shown that

$$(2.14) \quad \|\varphi^2 G_n''(f)\|_p \leq Kn \|f\|_p \quad (f \in L^p, n \geq 2)$$

and

$$(2.15) \quad \|\varphi^2 G_n''(f)\|_p \leq \left(1 + \frac{K}{n}\right) \|\varphi^2 f''\|_p \quad (f \in D_p, n \geq 2)$$

are satisfied with a K independent of n and f .

Let us begin with (2.14). Since for fixed $x > 0$ and $(h) < x/2$

$$\begin{aligned} \frac{1}{h} \left| e^{-u(x+h)} u^n f\left(\frac{n}{u}\right) - e^{-ux} u^n f\left(\frac{n}{u}\right) \right| &\leq \left| \frac{e^{-uh} - 1}{-uh} \right| e^{-ux} u^{n+1} \left| f\left(\frac{n}{u}\right) \right| \leq \\ &\leq K e^{|h|u} e^{-ux} u^{n+1} \left| f\left(\frac{n}{u}\right) \right| \leq K e^{-ux/2} u^{n+1} \left| f\left(\frac{n}{u}\right) \right| \end{aligned}$$

it follows from the dominated convergence theorem that

$$\frac{d \left(\int_0^\infty e^{-ux} u^n f\left(\frac{n}{u}\right) du \right)}{dx} = - \int_0^\infty e^{-ux} u^{n+1} f\left(\frac{n}{u}\right) du$$

and applying this once more we obtain

$$\begin{aligned} (2.16) \quad x^2 G_n''(f; x) &= \int_0^\infty \frac{d^2 g_n(x, u)}{dx^2} f\left(\frac{n}{u}\right) du = \int_0^\infty ((n-xu)^2 + (n-2ux)) \times \\ &\times g_n(x, u) f\left(\frac{n}{u}\right) du = n \int_0^\infty g_n^*(x, u) f\left(\frac{n}{u}\right) du + n G_n(f; x) - \\ &- 2(n+1) G_{n+1} \left(f\left(\frac{n}{n+1} \cdot\right); x \right) \end{aligned}$$

where

$$g_n^*(x, u) = \frac{1}{n} (n-ux)^2 g_n(x, u) = \frac{x^{n+1}}{n \cdot n!} (n-ux)^2 u^n e^{-ux}.$$

Simple computations give

$$\begin{aligned} \int_0^\infty g_n^*(x, u) du &= \frac{1}{n} \left(\int_0^\infty [n^2 g_n(x, u) - 2n(n+1) g_{n+1}(x, u) + \right. \\ &\left. + (n+1)(n+2) g_{n+2}(x, u)] du \right) = 1 + \frac{2}{n} \end{aligned}$$

and (use that $\Gamma(x+1)=x\Gamma(x)$)

$$\int_0^\infty g_n^*(1, u) \frac{u^{1/p}}{n^{1/p}} du = \frac{1}{n^{1+1/p}n!} \left[n^2 \Gamma\left(n+1+\frac{1}{p}\right) - 2n \Gamma\left(n+2+\frac{1}{p}\right) + \Gamma\left(n+3+\frac{1}{p}\right) \right] = \frac{\Gamma(n+1+1/p)}{n!n^{1/p}} \frac{n^2 - 2n(n+1+1/p) + (n+1+1/p)(n+2+1/p)}{n} \equiv K$$

and so (2.6) and its proof give (2.14) (see (2.16)).

It is easy to give a formal proof for (2.15). We obtain from (2.7) that for $f \in D_p$

$$(2.17) \quad G_n''(f; x) = \int_0^\infty \frac{e^{-\tau} \tau^n}{n!} \frac{n^2}{\tau^2} f''\left(\frac{nx}{\tau}\right) d\tau$$

and so, by (2.6),

$$\|\varphi^2 G_n''(f)\|_p \leq \left(1 + \frac{1}{n}\right)^{1/p} \|\varphi^2 f\|_p \leq \left(1 + \frac{K}{n}\right) \|\varphi^2 f''\|_p.$$

To justify this formal approach all we have to prove is that the derivation after the integral sign in (2.17) is permitted. But

$$\frac{1}{2} \lim_{h \rightarrow 0} h^{-2} \Delta_h^2(G_n(f); x) = G_n''(f; x)$$

a.e.

$$\frac{1}{2} \lim_{h \rightarrow 0} h^{-2} \left[f\left(\frac{n(x-h)}{\tau}\right) - 2f\left(\frac{nx}{\tau}\right) + f\left(\frac{n(x+h)}{\tau}\right) \right] = \frac{n^2}{\tau^2} f''\left(\frac{nx}{\tau}\right)$$

and

$$\begin{aligned} h^{-2} \Delta_h^2(G_n(f); x) &= \int_0^\infty \frac{e^{-\tau} \tau^n}{n!} h^{-2} \left[f\left(\frac{n(x-h)}{\tau}\right) - 2f\left(\frac{nx}{\tau}\right) + f\left(\frac{n(x+h)}{\tau}\right) \right] d\tau = \\ &= \frac{1}{n!} \int_0^\infty h^{-2} \left(e^{-\tau(x-h)/x} \tau^n \left(\frac{x-h}{x}\right)^{n+1} - 2e^{-\tau} \tau^n + e^{-\tau(x+h)/x} \tau^n \left(\frac{x+h}{x}\right)^{n+1} \right) f\left(\frac{nx}{\tau}\right) d\tau = \\ &\stackrel{\text{def}}{=} \frac{1}{n!} \int_0^\infty e^{-\tau} \tau^n h^{-2} h_{x,n,h}^*(\tau) f\left(\frac{nx}{\tau}\right) d\tau. \end{aligned}$$

An easy consideration gives for $|h| < \frac{x}{2}$ that we have

$$h^{-2} |h_{x,n,h}^*(\tau)| \leq K_{x,n} (1 + \tau^2 + e^{\tau|h|/x}) \leq K_{x,n} (1 + \tau^2 + e^{\tau/2})$$

and so, by the dominated convergence theorem, (2.17) follows from the relations above.

The proof of Theorem 2 is complete.

§ 3. Proof of Theorem 1

That (1.2) implies (1.1) for all p and α follows from Theorem 2, (1.5). Also, the implication (1.1) $\Rightarrow$ (1.2) for $0 < \alpha < 1$ follows from Theorem 2, (1.6) (see also (1.4)).

Thus, there has remained to show that (1.1) implies (1.2) for $\alpha=1$ and all $p \geq 1$. To this end we first verify that if $g \in C^2(0, \infty)$ has compact support in $(0, \infty)$ and if

$$A_n(f, g) = n \int_0^\infty (G_n(f) - f) g \quad (f \in L^p)$$

then

$$|A_n(f, g)| \leq K_g \|f\|_p$$

with a constant K_g independent of $f \in L^p$ and $n \geq 2$.

Indeed, with the notation

$$h(u) = g\left(\frac{1}{u}\right)$$

we obtain by a change of variable

$$\begin{aligned} |A_n(f, g)| &= n \left| \int_0^\infty \int_0^\infty \frac{e^{-\tau} \tau^n}{n!} \left(f\left(\frac{n\tau}{x}\right) - f(x) \right) g(x) d\tau dx \right| = \\ &= n \left| \int_0^\infty \int_0^\infty \frac{e^{-\tau} \tau^n}{n!} f(x) \left(\frac{\tau}{n} g\left(\frac{\tau}{n} x\right) - g(x) \right) d\tau dx \right| = \\ &= \left| \int_0^\infty f(x) \left\{ (n+1) \left[G_{n+1}\left(h; \frac{n}{(n+1)x}\right) - h\left(\frac{n}{(n+1)x}\right) \right] + \right. \right. \\ &\quad \left. \left. + \left[(n+1) h\left(\frac{n}{(n+1)x}\right) - nh\left(\frac{1}{x}\right) \right] \right\} dx \right|. \end{aligned}$$

Let $g(x)=0$ outside the interval (a, b) , $a>0$, $b<\infty$. By Theorem A

$$(n+1) \left| G_{n+1}\left(h; \frac{n}{(n+1)x}\right) - h\left(\frac{n}{(n+1)x}\right) \right| \leq K_g$$

and for $x \geq a/2$

$$\left| (n+1) h\left(\frac{n}{(n+1)x}\right) - nh\left(\frac{1}{x}\right) \right| \leq \|h\|_{\sup} + n \|h'\|_{\sup} \frac{1}{(n+1)x} \leq \frac{K_h}{a} \leq K_g$$

(for $x < a/2$ the expression on the left hand side is zero), and these together with the preceding equality prove our statement for $p=1$.

If $p>1$ then we argue as follows. Let

$$p_n(x) = (n+1) \left| G_{n+1}\left(h; \frac{n}{(n+1)x}\right) - h\left(\frac{n}{(n+1)x}\right) \right| + \left| (n+1) h\left(\frac{n}{(n+1)x}\right) - nh\left(\frac{1}{x}\right) \right|.$$

We have just seen that $p_n(x)$ is bounded, $|p_n(x)| \leq Kg$, which gives

$$\int_0^{2b} p_n(x) dx \leq Kg.$$

For $x > 2b$ we have $h\left(\frac{n}{(n+1)x}\right) = h\left(\frac{1}{x}\right) = 0$ and so

$$\begin{aligned} \int_{2b}^{\infty} p_n(x) dx &= \int_{2b}^{\infty} (n+1) \left| \int_0^{\infty} \frac{e^{-\tau} \tau^{n+1}}{(n+1)!} g\left(\frac{x\tau}{n}\right) d\tau \right| dx \leq \\ &\leq \int_{2b}^{\infty} \int_{an/x}^{bn/x} \frac{e^{-\tau} \tau^{n+1}}{n!} \left| g\left(\frac{x\tau}{n}\right) \right| d\tau dx = \int_0^{n/2} \frac{e^{-\tau} \tau^{n+1}}{n!} \int_{an/\tau}^{bn/\tau} \left| g\left(\frac{x\tau}{n}\right) \right| dx d\tau \leq \\ &\leq Kg \int_0^{n/2} \frac{e^{-\tau} \tau^{n+1}}{n!} d\tau \leq Kg n \frac{e^{-n/2} (n/2)^{n+1}}{n!} = o(1) \end{aligned}$$

where, at the last step we used the Stirling formula. Thus,

$$\int_0^{\infty} p_n(x) dx \leq Kg, \quad |p_n(x)| \leq Kg$$

which imply by the Hölder-inequality

$$\begin{aligned} |A_n(f, g)| &\leq \int_0^{\infty} |f(x)| p_n(x) dx \leq \|f\|_p \|p_n\|_q \leq \\ &\leq \|f\|_p \|p_n\|_{\sup}^{1-1/q} \|p_n\|_1^{1/q} \leq Kg \|f\|_p \left(\frac{1}{p} + \frac{1}{q} = 1 \right) \end{aligned}$$

what was to be proved.

An easy consideration gives (cf. [8, Lemma 5.5] and [3]) that if $f \in C^2(0, \infty) \cap L^p(0, \infty)$ then

$$\lim_{n \rightarrow \infty} n(G_n(f; x) - f(x)) = \frac{1}{2} x^2 f''(x)$$

uniformly on compact subsets of $(0, \infty)$. Thus, if $f \in C^2(0, \infty) \cap L^p(0, \infty)$ and $g \in C^2(0, \infty)$ has compact support in $(0, \infty)$ then

$$A(f, g) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} A_n(f, g) = \frac{1}{2} \int_0^{\infty} x^2 f''(x) g(x) dx$$

exists and integrating by parts twice we can see that

$$A(f, g) = \frac{1}{2} \int_0^{\infty} (x^2 g(x))'' f(x) dx.$$

Since we verified that $|A_n(f, g)| \leq Kg \|f\|_p$ and $C^2(0, \infty) \cap L^p(0, \infty)$ is dense in $L^p(0, \infty)$, it follows that $A(f, g)$ exists for every $f \in L^p(0, \infty)$ and $g \in C^2(0, \infty)$,

support $g \subseteq (0, \infty)$, furthermore,

$$A(f, g) = \frac{1}{2} \int_0^\infty (x^2 g(x))'' f(x) dx.$$

Now let us suppose that $\|G_n(f) - f\|_p = O(n^{-1})$. By weak-compactness there is a subsequence $\{n_k\}$ of the natural numbers such that $\{n_k(G_{n_k}(f) - f)\}$ converges weakly to a $h \in L^p$ for $1 < p < \infty$ and to a bounded signed Borel measure μ for $p = 1$.

Let us consider first the case $p > 1$. We have

$$(3.1) \quad \frac{1}{2} \int_0^\infty f(x)(x^2 g(x))'' dx = A(f, g) = \lim_{k \rightarrow \infty} A_{n_k}(f, g) = \int_0^\infty h(x) g(x) dx$$

for every $g \in C^2(0, \infty)$, support $g \subseteq (0, \infty)$. One solution of (3.1) is given by

$$f_1(x) = \int_x^\infty \int_\tau^\infty \frac{2h(u)}{u^2} du d\tau$$

and, as a standard argument gives, the associated homogeneous equation $\frac{1}{2} \int_0^\infty f(x)(x^2 g(x))'' dx = 0$ ($g \in C^2(0, \infty)$, support $g \subseteq (0, \infty)$) has only solutions that are linear (a.e.) (sketch of this last assertion: putting

$$(x^2 g(x))'' = \begin{cases} 1 & \text{if } y \leq x \leq y+a \text{ or } y+3a \leq x \leq y+4a \\ -1 & \text{if } y+a < x < y+4a \\ 0 & \text{otherwise} \end{cases}$$

into the preceding equality and differentiating with respect to a it follows that $f(y+a) - 3f(x+3a) + 2f(y+4a) = 0$ for $a \geq 0$ and almost every $y > 0$. Then the same holds everywhere for the continuous function $F_n(x) = \int_x^{x+1/n} f(u) du$. If F_n is non-linear then subtracting a linear function from it we can suppose that for some $c, d > 0$ $F_n(c) = F_n(d) = 0$ but $F_n(x) > 0$ for $x \in (c, d)$. If $y+3a \in (c, d)$ is a maximum point of F_n and a is small then $y+a$ and $y+4a$ must again be maximum points — contradiction. Thus, each F_n is linear and so $f(x) = \lim_{n \rightarrow \infty} n \int_x^{x+1/n} f(u) du$ is also a linear function (a.e.). Thus,

$$f(x) = ax + b + \int_x^\infty \int_\tau^\infty \frac{2h(u)}{u^2} du d\tau \quad (\text{a.e.})$$

by which $x^2 f''(x) = 2h(x) \in L^p(0, \infty)$. Now (1.2) follows from this by [5, Theorem 2].

Similarly, in the case $p = 1$ we have

$$f(x) = ax + b + \int_x^\infty \int_\tau^\infty \frac{2 d\mu(u)}{u^2} d\tau.$$

Here $a=0$ since the last term on the right tends to zero as $x \rightarrow \infty$ and $f \in L^1(0, \infty)$; and so

$$\frac{1}{2} x^2 f'(x) = x^2 \int_x^\infty \frac{d\mu(u)}{u^2}.$$

We shall prove that $\frac{1}{2} x^2 f'(x)$ is of bounded variation on $(0, \infty)$ and then [5, Theorem 2] yields (1.2). If $|\mu|$ denotes the total variation of μ then for all points $0 < x_1 < \dots < x_{n+1} < \infty$ we have

$$\begin{aligned} \sum_{i=1}^n \left| \frac{1}{2} x_i^2 f'(x_i) - \frac{1}{2} x_{i+1}^2 f'(x_{i+1}) \right| &\leq \sum_{i=1}^n x_i^2 \int_{x_i}^{x_{i+1}} \frac{d|\mu|(u)}{u^2} + \\ &+ \sum_{i=1}^n (x_{i+1}^2 - x_i^2) \int_{x_{i+1}}^\infty \frac{d|\mu|(u)}{u^2} \leq \sum_{i=1}^n \int_{x_i}^{x_{i+1}} d|\mu|(u) + \\ &+ \sum_{i=2}^n x_i^2 \int_{x_i}^{x_{i+1}} \frac{d|\mu|(u)}{u^2} + x_{n+1}^2 \int_{x_{n+1}}^\infty \frac{d|\mu|(u)}{u^2} \leq 2|\mu|(0, \infty) < \infty \end{aligned}$$

which proves that $\frac{1}{2} x^2 f''(x)$ has finite total variation on $(0, \infty)$.

The proof of Theorem 1 is complete.

Finally, we prove the corollaries. Corollary 2 is an immediate consequence of Theorem 1 and [5, Theorem 2]. To prove Corollary 1 let $g(x) = f(e^x)$ ($x \in (-\infty, \infty)$). Making the change of variable $\log x = u$ we obtain that for $0 < h < 1$

$$\begin{aligned} \left\{ \int_0^\infty |A_{hx}^2(f; x)|^p dx \right\}^{1/p} &\leq \left\{ \int_{-\infty}^\infty e^u |A_{\log(1+h)}^2(g; u)|^p du \right\}^{1/p} + \\ &+ \left\{ \int_{-\infty}^\infty e^u |g(u - \log(1-h)) - g(u + \log(1+h))|^p du \right\}^{1/p} = I_1(h) + I_2(h). \end{aligned}$$

If we assume (1.3) then $I_1(h) \leq Kh^{2\alpha}$. Let $0 < \tau < 1$ and let m be chosen according to $\frac{1}{2} \leq 2^m \tau \leq 1$. Since

$$g(x+2\tau) - g(x) - 2(g(x+\tau) - g(x)) = \Delta_\tau^*(g; x)$$

where

$$\Delta_\tau^*(g; x) = g(x) - 2g(x+\tau) + g(x+2\tau)$$

is the second order forward difference of g , we obtain

$$\begin{aligned} \left\{ \int_{-\infty}^\infty e^u |g(u+\tau) - g(u)|^p du \right\}^{1/p} &\leq \frac{1}{2} \left\{ \int_{-\infty}^\infty e^u |g(u+2\tau) - g(u)|^p du \right\}^{1/p} + \\ &+ \frac{1}{2} \left\{ \int_{-\infty}^\infty e^u |\Delta_\tau^*(g; u)|^p du \right\}^{1/p} \end{aligned}$$

the iteration of which yields

$$\begin{aligned} \left\{ \int_{-\infty}^{\infty} e^u |g(u+\tau) - g(u)|^p du \right\}^{1/p} &\leq \sum_{i=1}^{m-1} \frac{1}{2^i} \left\{ \int_{-\infty}^{\infty} e^u |\Delta_{2^{i-1}\tau}^*(g; u)|^p du \right\}^{1/p} + \\ &+ \frac{1}{2^m} \left\{ \int_{-\infty}^{\infty} e^u |\Delta_{2^{m-1}\tau}^*(g; u)|^p du \right\}^{1/p} \leq \\ &\leq K \sum_{i=1}^{m-1} 2^{-i} I_1(2^{i-1}\tau) + K 2^{-m} \|f\|_p \leq K \sum_{i=1}^m 2^{-i} (2^i \tau)^{2\alpha} \leq \\ &\leq \begin{cases} \tau^{2\alpha} & \text{if } 0 < 2\alpha < 1 \\ \left(\log \frac{1}{\tau} \right) \tau & \text{if } 2\alpha = 1 \\ \tau & \text{if } 1 < 2\alpha \leq 2 \end{cases} \leq K\tau^{2\alpha}. \end{aligned}$$

Putting here $\tau = -\log(1-h) - \log(1+h) \sim h^2$ we get

$$I_2(h) \leq K \left\{ \int_{-\infty}^{\infty} e^u |g(u+\tau) - g(u)|^p du \right\}^{1/p} \leq K\tau^{2\alpha} \leq Kh^{2\alpha}.$$

Collecting our estimates we can see that (1.3) implies (1.2). That (1.2) also implies (1.3) can be proved by the same method (using the transformation $e^u = x$) and the proof is complete.

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