

A common fixed point theorem

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We prove the following theorem.

Theorem. *Let S and T be commuting mappings of the complete metric space (X, d) into itself satisfying the inequality*

$$(1) \quad d(Sx, Ty) \leq c \cdot \max \{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}$$

for all x, y in X , where $0 < c < 1$. Suppose that

$$(2) \quad \begin{aligned} &\max \{d(S^i x, S^j x), d(T^i x, T^j x), d(x, S^i T^j x): 0 \leq i, j \leq n+1\} \leq \\ &\leq b \cdot \max \{d(S^i x, S^j x), d(T^i x, T^j x), d(x, S^i T^j x): 0 \leq i, j \leq n\} \end{aligned}$$

for some x in X and $n=1, 2, \dots$, where $0 < bc < 1$. Then S and T have a unique common fixed point z . Further z is the unique fixed point of S and T .

PROOF. With an x satisfying inequality (2) let

$$E_n = \{d(S^i x, S^j x), d(T^i x, T^j x), d(x, S^i T^j x): 0 \leq i, j \leq n\},$$

$$F_n = \{d(S^i T^{i'} x, T^j S^{j'} x): 1 \leq i, j \leq n; 0 \leq i', j' \leq n\}$$

and let

$$K_n = \max E_n, \quad L_n = \max F_n$$

for $n=1, 2, \dots$. It follows that inequality (2) can be written in the form

$$(3) \quad K_{n+1} \leq bK_n.$$

We now note that inequality (1) can be applied to every term in F_n to give terms in either E_n or F_n and it follows that

$$L_n \leq c \cdot \max \{K_n, L_n\}.$$

Since $c < 1$, we have

$$(4) \quad L_n \leq cK_n$$

for $n=1, 2, \dots$.

Inequality (1) can also be applied n times to the term $d((ST)^n x, T(ST)^{n-1} x)$ and then to the resulting terms, before terms in E_n appear. It follows that

$$d((ST)^n x, T(ST)^{n-1} x) \leq c^n \cdot \max \{K_n, L_n\} \leq c^n \cdot \max \{K_n, cK_n\}$$

on using inequality (4) and so since $c < 1$

$$d((ST)^n x, T(ST)^{n-1} x) \leq c^n K_n.$$

Now using inequality (3) it follows that

$$d((ST)^n x, T(ST)^{n-1} x) \leq c^n b^{n-1} K_1$$

for $n=1, 2, \dots$.

Similarly we can prove that

$$d(T(ST)^n x, (ST)^n x) \leq c^n b^{n-1} K_1$$

for $n=1, 2, \dots$. Since $bc < 1$, it follows that the sequence

$$\{x, Tx, STx, \dots, (ST)^n x, T(ST)^n x, \dots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X .

Using inequality (1) we have

$$\begin{aligned} & d(Sz, T(ST)^n x) \leq \\ & \leq c \cdot \max \{d(z, (ST)^n x), d(z, Sz), d((ST)^n x, T(ST)^n x), d(z, T(ST)^n x), d((ST)^n x, Sz)\} \end{aligned}$$

and on letting n tend to infinity we have

$$d(Sz, z) \leq cd(Sz, z).$$

It follows that z is a fixed point of S .

Similarly we can prove that z is a fixed point of T .

Now suppose that S has a second fixed point z' . Then

$$\begin{aligned} & d(z', z) = d(Sz', Tz) \leq \\ & \leq c \cdot \max \{d(z', z), d(z', Sz'), d(z, Tz), d(z', Tz), d(z, Sz')\} = cd(z', z) \end{aligned}$$

and so z is the unique fixed point of S . The proof that z is the unique fixed point of T follows similarly. This completes the proof the theorem.

We note that the condition that inequality (2) holds for some x in X is not necessary if X is bounded, see [2]. It is not known whether or not this condition is necessary if X is unbounded.

The condition that S and T commute is necessary even if X is bounded, see [1].

References

- [1] B. FISHER, On a conjecture on common fixed points, *Math. Sem. Notes, Kobe Univ.*, **6** (1978), 153—6.
- [2] B. FISHER, Results on common fixed points on bounded metric spaces, *Math. Sem. Notes, Kobe Univ.*, **7** (1979), 73—80.

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