

On the canonical connection of a 3-web

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Dedicated to Professor A. Rapcsák on his 70th birthday

1. Introduction

Let M be a smooth manifold of dimension $2r$. We say that a 3-web is defined on M if there is given three foliations $\lambda_1, \lambda_2, \lambda_3$ of r -dimensional leaves in M such that every point of M belongs to exactly one leaf of each foliation and the tangent spaces of the leaves through a point of M are in general position.

The notion of a 3-web on a 2-manifold was introduced by W. BLASCHKE [4]. The generalization to higher dimension was studied by S.-S. CHERN [5]. A systematic investigation of this structure has been made by M. A. AKIVIS and his followers in the last 15 years [1].

We shall call the leaves of the first foliation λ_1 as horizontal leaves and of the second family λ_2 as vertical leaves, their tangent vectors horizontal and vertical tangent vectors.

Let $P(M)$ be the subbundle of the linear frame bundle $L(M)$ over M consisting of the frames $\{\bar{e}_1, \dots, \bar{e}_r, \tilde{e}_1, \dots, \tilde{e}_r\}$, where $\bar{e}_1, \dots, \bar{e}_r$ are horizontal, $\tilde{e}_1, \dots, \tilde{e}_r$ are vertical tangent vectors. We can define a tensor field on M with help of the third foliation by the following way:

If v is a horizontal tangent vector at $x \in M$, let $\mathcal{P}_x v$ be the vertical tangent vector at x such that $v - \mathcal{P}_x v$ is a tangent vector to the leaf of the third foliation λ_3 through $x \in M$. The inverse map of $\mathcal{P}_x$ is denoted by Q_x . The matrix

$$\begin{pmatrix} 0 & \mathcal{P}_x \\ Q_x & 0 \end{pmatrix}$$

with respect to the frame $\{\bar{e}_1, \dots, \bar{e}_r, \tilde{e}_1, \dots, \tilde{e}_r\}$ defines a tensor field $\mathcal{T}_x$ on M having the property $\mathcal{T}_x^2 = \text{id}$.

The adapted frame bundle $Q(M)$ of the three-web is defined as the subbundle of $P(M)$ consisting of the frames $\{\bar{e}_1, \dots, \bar{e}_r, \tilde{e}_1, \dots, \tilde{e}_r\}$ such that $\tilde{e}_i = \mathcal{P} \bar{e}_i$, $i = 1, \dots, r$. S.-S. CHERN defined the canonical connection on the bundle $Q(M)$ uniquely determined by the structure equations [5]

$$\begin{aligned} d\bar{\omega}^i &= -\omega_j^i \wedge \bar{\omega}^j + a_{jk}^i \bar{\omega}^j \wedge \bar{\omega}^k, \\ d\tilde{\omega}^i &= -\omega_j^i \wedge \tilde{\omega}^j - a_{jk}^i \tilde{\omega}^j \wedge \tilde{\omega}^k, \end{aligned}$$

where ω_j^i are the components of the connection form, $\bar{\omega}^1, \dots, \bar{\omega}^r, \tilde{\omega}^1, \dots, \tilde{\omega}^r$ are the components of the basic form over $Q(M)$, and the tensor field a_j^i is called as the torsion tensor of the three-web. The invariants of this connection are the invariants of the 3-web.

In the following we shall investigate the canonical connection, particularly we shall give its description expressed by the Lie-loop structure on the horizontal leaves induced by the 3-web.

2. The calculation of the connection form

The foliations of horizontal and vertical leaves of a 3-web give a local direct product decomposition of the manifold $M=U \times V$. If $x^1, \dots, x^r$ and $y^1, \dots, y^r$ are the local coordinates on U and V , the collection $x^1, \dots, x^r, y^1, \dots, y^r$ will be a local coordinate system on the manifold M . We suppose that the leaves of the third foliation λ_3 are the level surfaces of the functions $f^i(x^1, \dots, x^r, y^1, \dots, y^r)$, $i=1, \dots, r$. It means that the 3-web is given by the equations:

$$\begin{aligned}\lambda_1: \{(x^1, \dots, x^r, y^1, \dots, y^r) \in U \times V, y^i = c^i\}, \\ \lambda_2: \{(x^1, \dots, x^r, y^1, \dots, y^r) \in U \times V, x^i = c^i\}, \\ \lambda_3: \{(x^1, \dots, x^r, y^1, \dots, y^r) \in U \times V, f^i(x, y) = c^i\},\end{aligned}$$

where $c^1, \dots, c^r$ are constant.

We denote $\bar{f}_j^i = \frac{\partial f^i}{\partial x^j}$, $\tilde{f}_j^i = \frac{\partial f^i}{\partial y^j}$. We can suppose that $\det(\bar{f}_j^i) \neq 0$, $\det(\tilde{f}_j^i) \neq 0$ and we denote by $\bar{g}_j^i$ and $\tilde{g}_j^i$ the inverses of the matrices $\bar{f}_j^i$ and $\tilde{f}_j^i$, respectively. We shall consider the frame field $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^r} \right\}$ on $U \times V$. A vector $X^i \frac{\partial}{\partial x^i} - Y^i \frac{\partial}{\partial y^i}$ is tangent to the leaf $\in \lambda_3$ if and only if

$$(\bar{f}_j^i dx^j + \tilde{f}_j^i dy^j) \left(X^i \frac{\partial}{\partial x^i} - Y^i \frac{\partial}{\partial y^i} \right) = 0.$$

It follows that the map $P_{(x,y)}: T_x U \rightarrow T_y V$ can be written in the form

$$(1) \quad \mathcal{P}_j^i dx^j \otimes \frac{\partial}{\partial y^i} = \tilde{g}_k^i \bar{f}_j^k dx^j \otimes \frac{\partial}{\partial y^i}$$

and its inverse is

$$(2) \quad \mathcal{Q}_j^i dy^j \otimes \frac{\partial}{\partial x^i} = \bar{g}_k^i \tilde{f}_j^k dy^j \otimes \frac{\partial}{\partial x^i}.$$

The frame $\left\{ \bar{g}_j^i \frac{\partial}{\partial x^i}, \tilde{g}_j^i \frac{\partial}{\partial y^i}, j=1, \dots, r \right\}$ is an adapted one, its dual coframe is

$$\bar{\omega}^i = \bar{f}_j^i dx^j, \quad \tilde{\omega}^i = \tilde{f}_j^i dy^j.$$

We extend the connection ω_j^i from the adapted frame bundle $Q(M)$ to the larger bundle $P(M)$ containing $Q(M)$ as subbundle. We denote by $\bar{\omega}_j^i, \tilde{\omega}_j^i$ the components of the extended connection form, and we compute their expressions in the frame

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^r}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^r} \right\},$$

which is a section of the bundle $P(M)$. Our calculation is a modified version of the calculations in [2].

By standard formulas we have

$$(3) \quad \bar{\omega}_j^i = \bar{g}_l^i (d\tilde{f}_j^l + \omega_k^l \tilde{f}_j^k), \quad \tilde{\omega}_j^i = \tilde{g}_l^i (d\tilde{f}_j^l + \omega_k^l \tilde{f}_j^k).$$

These forms satisfy the structure equations

$$\begin{aligned} 0 &= d(dx^i) = -\bar{\omega}_j^i \wedge dx^j + \bar{g}_l^i a_j^l \tilde{f}_h^j \tilde{f}_m^k dx^h \wedge dx^m, \\ 0 &= d(dy^i) = -\tilde{\omega}_j^i \wedge dy^j - \tilde{g}_l^i a_j^l \tilde{f}_h^j \tilde{f}_m^k dy^h \wedge dy^m. \end{aligned}$$

Using Cartan's lemma we get

$$(4) \quad \begin{aligned} \bar{\omega}_j^i &= \bar{g}_l^i \left(\frac{\partial^2 f^l}{\partial x^h \partial x^j} dx^h + \frac{\partial^2 f^l}{\partial y^h \partial x^j} dy^h + \omega_k^l \tilde{f}_j^k \right) = \\ &= \bar{g}_l^i a_m^l \tilde{f}_h^m \tilde{f}_j^k dx^h + \bar{\lambda}_{j^i h} dx^h, \end{aligned}$$

$$(5) \quad \begin{aligned} \tilde{\omega}_j^i &= \tilde{g}_l^i \left(\frac{\partial^2 f^l}{\partial x^h \partial y^j} dx^h + \frac{\partial^2 f^l}{\partial y^h \partial y^j} dy^h + \omega_k^l \tilde{f}_j^k \right) = \\ &= -\tilde{g}_l^i a_m^l \tilde{f}_h^m \tilde{f}_j^k dy^h + \tilde{\lambda}_{j^i h} dy^h, \end{aligned}$$

where $\bar{\lambda}_{j^i h}$ and $\tilde{\lambda}_{j^i h}$ are symmetric in the indices j and h . Excluding the forms ω_k^l from the equations (4) and (5) we get

$$-\frac{\partial^2 f^l}{\partial x^h \partial x^j} \bar{g}_k^j + \frac{\partial^2 f^l}{\partial x^h \partial y^j} \tilde{g}_k^j + a_m^l \tilde{f}_h^m + \tilde{f}_i^l \bar{\lambda}_{j^i h} \bar{g}_k^j = 0$$

and

$$-\frac{\partial^2 f^l}{\partial y^h \partial x^j} \bar{g}_k^j + \frac{\partial^2 f^l}{\partial y^h \partial y^j} \tilde{g}_k^j + a_m^l \tilde{f}_h^m - \tilde{f}_i^l \tilde{\lambda}_{j^i h} \tilde{g}_k^j = 0.$$

Using $a_m^l = -a_k^l{}_m$ we get from these equations

$$\begin{aligned} a_m^l &= -\frac{\partial^2 f^l}{\partial x^h \partial y^j} \tilde{g}_{[k}^j \bar{g}_{m]}^h, \\ \bar{\lambda}_{j^i h} &= \bar{g}_l^i \left(\frac{\partial^2 f^l}{\partial x^h \partial x^j} - \tilde{g}_k^m \tilde{f}_j^k \frac{\partial^2 f^l}{\partial x^h \partial y^m} \right), \\ \tilde{\lambda}_{j^i h} &= \tilde{g}_l^i \left(\frac{\partial^2 f^l}{\partial y^h \partial y^j} - \tilde{g}_k^m \tilde{f}_j^k \frac{\partial^2 f^l}{\partial y^h \partial x^m} \right), \end{aligned}$$

where $[..]$ and $(..)$ denote the antisymmetric part and the symmetric part of an

expression. We substitute these into (4) and (5), then we get

$$(6) \quad \bar{\omega}_j^i = \bar{g}_l^i \left(\frac{\partial^2 f^l}{\partial x^j \partial x^h} - \frac{\partial^2 f^l}{\partial x^h \partial y^m} \tilde{g}_k^m \tilde{f}_j^k \right) dx^h,$$

$$(7) \quad \tilde{\omega}_j^i = \tilde{g}_l^i \left(\frac{\partial^2 f^l}{\partial y^j \partial y^h} - \frac{\partial^2 f^l}{\partial y^h \partial x^m} \bar{g}_k^m \tilde{f}_j^k \right) dy^h.$$

Proposition 1. *The components $\bar{\omega}_j^i$ and $\tilde{\omega}_j^i$ of the extended canonical connection form can be written in the form*

$$\bar{\omega}_j^i = Q_m^i \frac{\partial \mathcal{P}_j^m}{\partial x^h} dx^h, \quad \tilde{\omega}_j^i = \mathcal{P}_m^i \frac{\partial Q_j^m}{\partial y^h} dy^h.$$

PROOF. Using the relations $\tilde{f}_m^l \tilde{g}_k^m = \delta_k^l$ and $\bar{f}_m^l \bar{g}_k^m = \delta_k^l$ we have

$$\frac{\partial \tilde{f}_m^l}{\partial x^h} \tilde{g}_k^m = -\tilde{f}_m^l \frac{\partial \tilde{g}_k^m}{\partial x^h}, \quad \frac{\partial \bar{f}_m^l}{\partial y^h} \bar{g}_k^m = -\bar{f}_m^l \frac{\partial \bar{g}_k^m}{\partial y^h}.$$

Thus from the equations (6) and (7) follows the Proposition.

3. Parallel vector fields along horizontal and vertical surfaces

Let π and ϱ denote the projection maps $\pi: M = U \times V \rightarrow U$, $\varrho: M = U \times V \rightarrow V$. Let be given a horizontal curve $(x(t), y_0)$ and a vectorfield $X(t)$ along the curve $(x(t), y_0)$. The vectorfield $X(t) = \bar{X}^i \frac{\partial}{\partial x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$ is parallel along $(x(t), y_0)$ if and only if it satisfies the equations

$$\frac{d\bar{X}^i}{dt} + \bar{\omega}_j^i \bar{X}^j = 0, \quad \frac{d\tilde{X}^i}{dt} = 0.$$

Using Proposition 1 we get

$$(9) \quad \frac{d\bar{X}^i}{dt} + Q_m^i \frac{\partial \mathcal{P}_j^m}{\partial x^h} \bar{X}^j \dot{x}^h = Q_m^i \left(\frac{\partial (\mathcal{P}_j^m \bar{X}^j)}{\partial x^h} \dot{x}^h \right) = 0.$$

Similarly a vector field $X(t) = \bar{X}^i \frac{\partial}{\partial x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$ is parallel along a vertical curve $(x_0, y(t))$ if and only if

$$(10) \quad \frac{d\bar{X}^i}{dt} = 0 \quad \text{and} \quad \frac{d\tilde{X}^i}{dt} + \tilde{\omega}_j^i \tilde{X}^j = \mathcal{P}_m^i \left(\frac{\partial (Q_j^m \tilde{X}^j)}{\partial y^h} \dot{y}^h \right) = 0.$$

From the relations (9) and (10) we get the

Theorem 1. *The vector field $X(t) = \bar{X}(t) + \tilde{X}(t) = \bar{X}^i \frac{\partial}{\partial x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$ is parallel along a horizontal curve $(x(t), y_0)$ in M if and only if*

(i) its vertical part $\tilde{X} = \tilde{X}^i \frac{\partial}{\partial y^i}$ is a constant vector, that is

$$\frac{d}{dt} (\varrho_* \tilde{X}) = 0,$$

(ii) the vertical image $\mathcal{P}\tilde{X} = \mathcal{P}_j^i \tilde{X}^j \frac{\partial}{\partial y^i}$ of its horizontal part $\bar{X} = \bar{X}^i \frac{\partial}{\partial x^i}$ is a constant vector, that is

$$\frac{d}{dt} (\varrho_* \mathcal{P}\tilde{X}) = 0.$$

The vector field $X = \bar{X} + \tilde{X} = \bar{X}^i \frac{\partial}{\partial x^i} + \tilde{X}^i \frac{\partial}{\partial y^i}$ is parallel along a vertical curve $(x_0, y(t))$ in M if and only if

(i) its horizontal part $\bar{X} = \bar{X}^i \frac{\partial}{\partial x^i}$ is a constant vector, that is

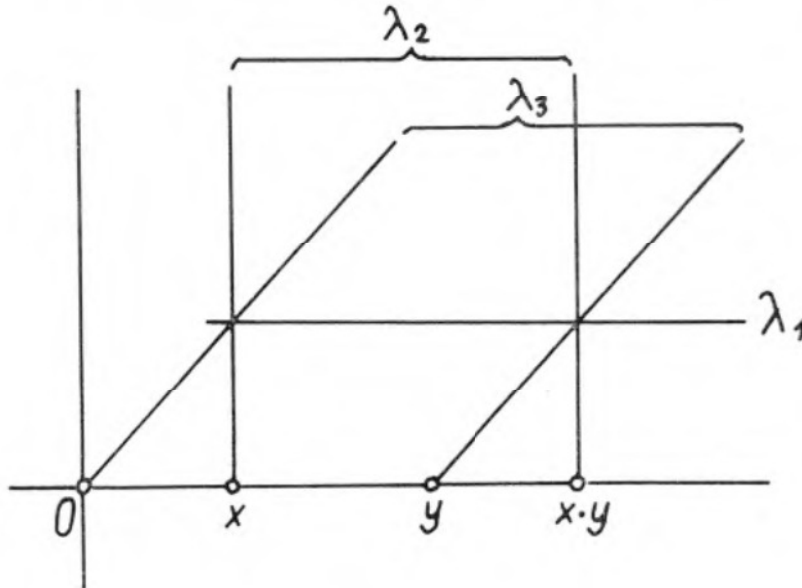
$$\frac{d}{dt} (\pi_* \bar{X}) = 0,$$

(ii) the horizontal image $\mathcal{Q}\tilde{X} = \mathcal{Q}_j^i \tilde{X}^j \frac{\partial}{\partial x^i}$ of its vertical part $\tilde{X} = \tilde{X}^i \frac{\partial}{\partial y^i}$ is a constant vector, that is

$$\frac{d}{dt} (\pi_* \mathcal{Q}\tilde{X}) = 0.$$

The proof is an immediate consequence of the equations (9) and (10).

Now we fix an origin $0 \in M$ and we suppose that its coordinates are zero. On the horizontal leaf U through 0 we can introduce a loop operation (non-associative but invertible multiplication with unit) by the following schema



It is well known that if we consider on U the introduced loop structure, than we can define a 3-web on the direct product $U \times U$ such that the horizontal leaves ($\in \lambda_1$) are $U \times \{x_0\}$, $x_0 \in U$, the vertical leaves ($\in \lambda_2$) are $\{x_0\} \times U$, and the leaves of λ_3 are

$$\{(x, y) \in U \times U : x \cdot y = z_0, \text{ where } z_0 \in U \text{ is constant}\}.$$

This 3-web is isomorphic to the original one [6].

Theorem 2. Let $\mathcal{L}_g: U \rightarrow U$ and $\mathcal{R}_g: U \rightarrow U$, $g \in U$ denote the left and the right translations on the loop U . A horizontal vector field $\bar{X}$ is parallel along the horizontal leaf $U \times \{e\}$ where e is the unit in U , if and only if

$$\bar{X}(g, e) = (\mathcal{L}_g)_* \bar{X}(e, e).$$

A vertical vector field $\tilde{X}$ is parallel along the vertical leaf $\{e\} \times U$ if and only if

$$\tilde{X}(e, g) = (\mathcal{R}_g)_* \tilde{X}(e, e).$$

Epecially if U is a Lie group then the connection induced on the horizontal leaf $U \times \{e\}$ is defined by the left invariant vector fields, and the connection induced on the vertical leaf $\{e\} \times U$ is defined by the right invariant vector field on the Lie group U .

PROOF. We know by Theorem 1. that the horizontal vector field $\bar{X}^i \frac{\partial}{\partial x^i}$ is parallel along a horizontal leaf if and only if $\mathcal{P}_j^i \bar{X}^j = c^i$ (constant). We get by (2) $\bar{X}^j = Q_h^j c^h = \tilde{g}_k^j \tilde{f}_h^k c^h$, where the function $f^i(x^1, \dots, x^r, y^1, \dots, y^r)$ can be interpreted as the coordinate functions of the multiplication, that is

$$(x \cdot y)^i = f^i(x^1, \dots, x^r, y^1, \dots, y^r).$$

We suppose that the coordinates of the unit $e \in U$ are zero. We consider the horizontal leaf $U \times \{e\}$ through the origin $0 = (e, e)$. The parallel horizontal vector field $\bar{X}$ has to satisfy

$$\bar{X}^j(x, 0) = Q_h^j(x, 0) c^h = \tilde{g}_k^j(x, 0) \tilde{f}_h^k(x, 0) c^h.$$

Since $x \cdot e = x$ we have $f^i(x, 0) = x^i$, $\tilde{f}_k^i(x, 0) = \delta_k^i$ and $\tilde{g}_k^i(x, 0) = \delta_k^i$. Thus we get

$$Q_h^i(x, 0) = \tilde{f}_h^i(x, 0).$$

Since $(\mathcal{L}_x y)^i = f^i(x, y)$, the tangent map of $\mathcal{L}_g: U \rightarrow U$ can be written in the form

$$(\mathcal{L}_g)_* \left(\bar{X}_0^i \frac{\partial}{\partial x^i} \right) = \bar{X}_0^i \frac{\partial f^k}{\partial y^i} \frac{\partial}{\partial x^k} = \tilde{f}_h^k \bar{X}_0^h \frac{\partial}{\partial x^k} = Q_h^k \bar{X}_0^h \frac{\partial}{\partial x^k},$$

which proves the first assertion.

For vertical vector fields similar arguments prove the statement.

References

- [1] М. А. Акивис, О три-тканях многомерных поверхностей, Труды Геометрического Семинара, ВИНТИ Москва, **2** (1961), 7—31.
- [2] М. А. Акивис—А. М. Шелехов, О вычислении тензоров кривизны и кручения многомерной три-ткани, Сибирский Мат. Журнал **12** (1971), 953—960.
- [3] М. А. Акивис—А. М. Шелехов, Основы теории тканей, Калининский Государственный Университет, Калинин (1981).
- [4] W. BLASCHKE—G. BOL, Geometrie der Gewebe, Springer, Berlin (1938).
- [5] S.-S. CHERN, Eine Invariantentheorie der 3-Gewebe aus r -dimensionalen Mannigfaltigkeiten in R_{2n} , *Abhandl. Math. Semin. Hamburg. Univ.*, **11** (1936), 335—358.
- [6] G. PICKERT, Projektive Ebene, Springer, Berlin—Göttingen—Heidelberg (1955).

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