

Hypoelliptic convolution equations in the space $\mathcal{H}'\{M_p\}$

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0. Introduction

In the papers [5] and [6] we introduced and analysed the space of distributions $\mathcal{H}'\{M_p\}$ and gave necessary and sufficient conditions for the solvability of the convolution equation

$$(1) \quad S * U = V, \quad S \in \mathcal{O}'_c(\mathcal{H}'\{M_p\})$$

in the space $\mathcal{H}'\{M_p\}$. $\mathcal{O}'_c(\mathcal{H}'\{M_p\})$ denotes the space of convolutors on $\mathcal{H}'\{M_p\}$.

For the convenience of the reader we repeat some definitions from [5] and [6]. The space $\mathcal{H}\{M_p\}$ is the space of smooth functions $\varphi(x)$ on $\mathbf{R}$ such that for every $p \in \mathbf{N}$

$$\gamma_p(\varphi) := \sup \{ |\varphi^{(j)}(x)| \exp(M_p(x)); x \in \mathbf{R}, 0 \leq j \leq n \} < \infty.$$

The space $\mathcal{H}\{M_p\}$ equipped with the sequence of norms becomes a Fréchet space. The dual of $\mathcal{H}\{M_p\}$, denoted by $\mathcal{H}'\{M_p\}$, can be regarded as a subspace of $\mathcal{D}'$, the space of finite-order distributions. The space $\mathcal{O}'_c(\mathcal{H}'\{M_p\})$ is the space of convolutors on $\mathcal{H}'\{M_p\}$. A distribution S is in $\mathcal{O}'_c(\mathcal{H}'\{M_p\})$ iff for every $\varphi \in \mathcal{H}\{M_p\}$ $S * \varphi$ is again in $\mathcal{H}\{M_p\}$; one proves then that this implies that for every $T \in \mathcal{H}'\{M_p\}$ $S * T \in \mathcal{H}'\{M_p\}$.

In this paper we shall give necessary and sufficient conditions for the hypoellipticity of (1) and of the convolutor S . These conditions will be given on the Fourier transform of S , which by [6] is an entire analytic function.

Throughout the paper we suppose that a sequence of even, convex continuously differentiable functions $\{M_p(x)\}$ is given such that, for every $p \in \mathbf{N}$, the function $M_p(x)$ is increasing for $x > 0$; $M_p(0) = 0$, $\lim_{|x| \rightarrow \infty} M_p(x)/|x| = \infty$. We assume that the following condition is satisfied:

(A) For every $p \in \mathbf{N}$ there exist $p' \in \mathbf{N}$ and X_p , $p' \equiv X_p > p$, so that $M_p(px) \equiv M_{p'}(x)$ for $|x| \equiv X_p$.

It was shown in [5] that this condition implies the condition (N) ([1], p. 111), in general for a greater $p' \in \mathbf{N}$. We denote by $r(p)$ the smallest natural number p'

for which the inequalities in (A) hold, and the function $\exp(M_p(x) - M_{p'}(x))$ is summable on the set $\mathbf{R}$.

It is important to note that we do not assume the smoothness of the functions $M_p(x)$, $p \in \mathbf{N}$. However, since smoothness is necessary in Section 5., we have to define smooth functions which behave at infinity as (for instance) $M_p(x)$ (see Section 4.).

We are now ready to define the space $\mathcal{E}\mathcal{H}'\{M_p\}$.

1. The Space $\mathcal{E}\mathcal{H}'\{M_p\}$

Definition 1. $\mathcal{E}\mathcal{H}'\{M_p\}$ is the space of smooth functions $U(x)$ defined on $\mathbf{R}$ which satisfy the inequalities

$$(2) \quad |U^{(j)}(x)| \leq C_j \cdot \exp(M_{p_0}(x)), \quad j = 0, 1, 2, \dots$$

for some $C_j > 0$ and $p_0 \in \mathbf{N}$ (which depends only on $U(x)$).

The space $\mathcal{E}\mathcal{H}'\{M_p\}$ can be obtained as the dual space of the space of convolutors $\mathcal{O}'_c(\mathcal{H}'\{M_p\})$, but this is not essential for us. Rather, let us return to the convolution equation (1). If $S \in \mathcal{O}'_c(\mathcal{H}'\{M_p\})$ and $U \in \mathcal{E}\mathcal{H}'\{M_p\}$, then the following Lemma shows that $V := S * U$ must be in $\mathcal{E}\mathcal{H}'\{M_p\}$.

Lemma 1. Let $S \in \mathcal{O}'_c(\mathcal{H}'\{M_p\})$ and $U \in \mathcal{E}\mathcal{H}'\{M_p\}$. Then $V := S * U$ is in $\mathcal{E}\mathcal{H}'\{M_p\}$.

PROOF. Let us suppose that $U^{(j)}(x)$, $j = 0, 1, 2, \dots$, satisfy (2), and let us put $p_1 := r(r(p_0))$. By Theorem 3 from [5] for given $p_1 \in \mathbf{N}$ there exists $m \in \mathbf{N}_0$ and a continuous function $F(x)$ with the property $|F(x)| \leq C \exp(-M_{p_1}(x))$ for some $C > 0$, such that $S = D^m F$. (D denotes the distributional derivative.) Hence

$$\begin{aligned} |(S * U)^{(j)}(x)| &= \left| \int_{\mathbf{R}} F(y) U^{(m+j)}(x-y) dy \right| \leq \\ &\leq \int_{\mathbf{R}} C \exp(-M_{p_1}(y)) C_{m+j} \exp(M_{p_0}(x-y)) dy \leq \\ &\leq \int_{\mathbf{R}} C_j' \exp(-M_{p_1}(x-y) + M_{p_0}(y)) dy. \end{aligned}$$

From the inequality

$$M_{p_0}(y) \leq M_{r(p_0)}(y/2) + K \leq M_{r(p_0)}(x) + M_{r(p_0)}(x-y) + K$$

for $x, y \in \mathbf{R}$ and some $K = K(p_0) > 0$ we obtain at last

$$|(S * U)^{(j)}(x)| \leq C_j'' \exp(M_{r(p_0)}(x)). \quad \square$$

The converse statement is not true in general; if, however, every solution of (1) is in $\mathcal{E}\mathcal{H}'\{M_p\}$ when V is, then the equation (1) and the convolutor S are called hypoelliptic in $\mathcal{H}'\{M_p\}$.

We shall introduce two new conditions for the functions $M_p(x)$, $p \in \mathbf{N}$; in the following, we suppose that they are satisfied:

(B) For every pair $(p, q) \in \mathbb{N}^2$ there exist $k \in \mathbb{N}$ and $X(p, q) > 0$ such that

$$M_p(qx) \leq M_p(kx)/k \quad \text{for } |x| \geq X(p, q).$$

(Observe that this condition is not satisfied for an arbitrary convex function which increases in infinity faster than any linear function. On the other hand, every function from the sequence $\{M_p(x)\}$, where $M_p(x) := p \cdot |x|^s$, $s > 1$ fixed natural number, satisfies this condition. We know that this sequence leads to the space of exponential distributions $\mathcal{H}'_s$. If $M_p(x) := M(px)$, where $M(x)$ is a convex function which grows faster than any linear function when $|x| \rightarrow \infty$ ([8], [3]) then this condition becomes:

For $q \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and $X(q) > 0$ such that

$$(3) \quad M(qx) \leq M(kx)/k \quad \text{for } |x| \geq X(q).$$

The other condition is:

(C) For every $p \in \mathbb{N}$ there exist $p' \in \mathbb{N}$, $k \in \mathbb{N}$ and $K_p > 0$ such that

$$M_p(x+y) \leq kM_p(x) + M_{p'}(y) + K_p \quad \text{for } x, y \in \mathbb{R}.$$

(This condition is a "new" one only for $0 \leq |y| < |x|$, otherwise it follows from (A).)

The main result of this paper is the following

Theorem. Suppose $S \in \mathcal{O}'_c(\mathcal{H}'\{M_p\})$ and let $\hat{S}(w)$, $w = u + iv \in \mathbb{C}$ be its Fourier transform. The following conditions are equivalent:

(H₁) S is hypoelliptic in $\mathcal{H}'\{M_p\}$;

(H₂) a) For each $p \in \mathbb{N}$ there exist $B, M > 0$ such that

$$|\hat{S}(u)| \leq |u|^{-B} \quad \text{for } |u| \geq M;$$

(b) $\lim (\tilde{M}_p(v)/\log |w|) = \infty$ when $|w| \rightarrow \infty$ staying on the surface $\hat{S}(w) = 0$;

(H₃) For every $p \in \mathbb{N}$ and $d > 0$ there exists a constant $\bar{B} > 0$ so that for every $m \in \mathbb{N}$ there exists $C_m > 0$ with the property

$$|1/\hat{S}(w)| \leq |w|^{\bar{B}} \exp(d \cdot \tilde{M}_p(v))$$

for

$$|\tilde{M}_p(v)| \leq m \log |w| \quad \text{and} \quad |w| \geq C_m.$$

$\tilde{M}_p(v)$ denotes the function dual in the sense of Young to the function $M_p(x)$ ([2] p. 18).

2. The Necessity of (H₂)

The implication (H₁) $\Rightarrow$ (H₂a) can be proved in the same way as in [7], because the proof in [7] runs for the space of tempered distributions $\mathcal{S}'$, and obviously $\mathcal{S}' \subset \mathcal{H}'\{M_p\}$. In order to prove (H₁) $\Rightarrow$ (H₂b) we follow the lines of the proof of Theorem 7 in [7], though in our case this proof needs certain modifications. The following statement implies at once the necessity of the condition (H₂b).

Proposition 1. If each continuous solution of the homogenous equation

$$(4) \quad S * U = 0$$

which satisfies the estimate

$$(5) \quad U(x) = O(\exp(M_p(x))) \quad \text{when } |x| \rightarrow \infty$$

for some $p \in \mathbf{N}$, is in fact a continuously differentiable function in some neighbourhood of zero, then the condition (H_2b) holds.

PROOF. Let us fix p and let $\mathcal{L} = \{x \in \mathbf{R}; |x| \leq 1\}$ be the mentioned neighbourhood of zero. Let H_p denote the space of continuous solutions of (4) which satisfy (5) and let $H_p^* = H_p \cap C^1(\mathcal{L})$, where $C^1(\mathcal{L})$ denotes the space of continuously differentiable functions on the interval $\mathcal{L}$. The spaces H_p and H_p^* endowed with the norms

$$\|U\|_p := \sup \{|U(x)| \exp(-M_p(x)); x \in \mathbf{R}\} \quad \text{and}$$

$$\|U\|_p^* := \sup \{|U'(x)|; x \in \mathcal{L}\} + \|U\|_p$$

respectively are Banach spaces. Since these spaces have the same elements, from the closed graph theorem it follows that

$$(6) \quad \|U\|_p^* \leq C_p \|U\|_p \quad \text{for every } U \in H_p \text{ and some } C_p > 1.$$

If $w = u + iv$ is such that $\hat{S}(w) = 0$, then $U_0(x) := \exp(iwx) \in H_p$ and $\|U_0\|_p = \sup \{\exp(-vx - M_p(x)); x \in \mathbf{R}\} = \exp(\tilde{M}_p(v))$. The relation (6) implies now

$$(C_p - 1) \exp(\tilde{M}_p(v)) \geq |w| \quad \text{or}$$

$$(7) \quad \tilde{M}_p(v)/\log |w| \geq 1 \quad \text{for sufficiently large } |w|.$$

Let us take $p < p' < p''$ such that $M_p(p'x) \leq M_{p''}(x)$ for sufficiently large $|x|$ (condition (A) implies the existence of such natural numbers p' and p'' for given $p \in \mathbf{N}$). So, we obtain

$$(8) \quad \tilde{M}_p(x/p') \leq \tilde{M}_{p''}(x) \quad \text{or} \quad \tilde{M}_p(x) \leq \tilde{M}_{p''}(p'x)$$

for sufficiently large $|x|$. From (7) (p replaced with p'') and (8) we get

$$\tilde{M}_p(v)/\log |w| \leq \tilde{M}_p(v)/\tilde{M}_{p''}(v) \leq \tilde{M}_{p''}(p'v)/\tilde{M}_{p''}(v) \leq p'$$

for sufficiently large $|w|$ and $\hat{S}(w) = 0$. This implies $\lim \tilde{M}_p(v)/\log |w| = \infty$ when $|w| \rightarrow \infty$ on the surface $\hat{S}(w) = 0$. $\square$

3. The Implication $(H_2) \Rightarrow (H_3)$

In order to prove this implication, we shall state two Lemmas. Let $\{d_p\}$ be a sequence of positive numbers which monotonically tends to zero. Let us denote by $\{\underline{m}_p(x)\}$ the sequence of continuous functions such that $\underline{m}_p(x) := M'_p(x)$ for $x > d_p > 0$ and $\underline{m}_p(x) := M'_p(d_p)x/d_p$ for $0 \leq x \leq d_p$. By $\{\underline{M}_p(x)\}$ we denote the corresponding sequence of convex functions, i.e.

$$\underline{M}_p(x) := \int_0^{|x|} \underline{m}_p(t) dt, \quad x \in \mathbf{R}, \quad p = 1, 2, \dots$$

It is clear that the functions $M_p(x)$ (resp. $\tilde{M}_p(x)$) and $\underline{M}_p(x)$ (resp. $\underline{\tilde{M}}_p(x)$) have the same behaviour in infinity. Obviously, condition (A) holds for the sequence $\{\tilde{M}_p(x)\}$ provided that it holds for $\{M_p(x)\}$. The following lemma is obvious, so we omit the proof.

Lemma 2. $\lim_{y \rightarrow 0+} (\tilde{M}_p(xy)/\tilde{M}_p(y)) = x^2$ for every $x \in \mathbb{R}$.

The next lemma is a version of Lemma 1 from [4].

Lemma 3. For given $A, \bar{B}, b > 0$ and $p \in \mathbb{N}$ there exists a positive constant N such that if $U = U(x, y)$ is a harmonic function for $x^2 + y^2 < T^2$ which satisfies the inequalities

$$(9) \quad U(x, 0) \leq 0, \quad U(x, y) \leq -a \underline{\tilde{M}}_p(y) - \bar{B} \underline{\tilde{M}}_p(R)$$

for $x^2 + y^2 < T^2$ then

$$(10) \quad U(x, y) \leq a \underline{\tilde{M}}_p(y) + (\bar{B} + b) \underline{\tilde{M}}_p(R)$$

for $x^2 + y^2 < R^2$ provided that $0 < a < A$, $0 < R < T/N$.

PROOF. If we suppose that this lemma were false, then in the same way as in [4] we can show that there exist sequences $\{a_n\}$, $\{T_n\}$ and $\{R_n\}$ such that $0 < a_n < A$, $T_n > n \cdot R_n$ and a sequence of harmonic functions $\{U_n(x, y)\}$ such that

$$U_n(x, 0) \leq 0, \quad U_n(x, y) \leq -a_n \underline{\tilde{M}}_p(y) - \bar{B} \underline{\tilde{M}}_p(R_n) \quad \text{for } x^2 + y^2 < T_n^2$$

and $U_n(x_n, y_n) \geq a_n \cdot \underline{\tilde{M}}_p(y_n) + (\bar{B} + b) \cdot \underline{\tilde{M}}_p(R_n)$ for some (x_n, y_n) which satisfies $x_n^2 + y_n^2 \leq R_n^2$.

Let us put

$$V_n(x, y) := U_n(R_n x, R_n y) / \underline{\tilde{M}}_p(R_n), \quad x'_n = x_n / R_n, \quad y'_n = y_n / R_n.$$

We obtain

$$V_n(x, 0) \leq 0, \quad V_n(x, y) \leq -a_n \underline{\tilde{M}}_p(R_n y) / \underline{\tilde{M}}_p(R_n) - \bar{B}$$

for $x_n'^2 + y_n'^2 \leq n^2$ and

$$V_n(x'_n, y'_n) \geq a_n \underline{\tilde{M}}_p(R_n y'_n) / \underline{\tilde{M}}_p(R_n) + (\bar{B} + b)$$

for $x_n'^2 + y_n'^2 < 1$.

As in [4], using Lemma 2, we can show that there exists a harmonic function $V(x, y)$ and real numbers x_0, y_0 and a_0 such that

$$(11) \quad V(x, 0) \leq 0, \quad V(x, y) \leq -a_0 y^2 / 2 - \bar{B}, \quad V(x_0, y_0) \geq a_0 y_0^2 / 2 + (\bar{B} + b).$$

From Harnack's inequality it follows that $V(x, y)$ is of the form $V(x, y) = cy + d$; (11) implies $d \leq 0$. Putting $y := -y_0$ in the second inequality in (11) we obtain $-cy_0 + d \geq a_0 y_0^2 / 2 - \bar{B}$, and this with the last inequality in (11) yields $2d \geq b > 0$, contradicting $d \leq 0$. $\square$

The proof of the implication $(H_2) \Rightarrow (H_3)$

Let $p, m \in \mathbb{N}$ and $b > 0$ be given. Using the method from [6] one can prove the following

Proposition 2. For given $p \in \mathbf{N}$ and $d > 0$ there exists a positive constant n with the property that the Fourier transform $\hat{S}(w)$ of the convolutor S satisfies the inequality

$$(12) \quad |\hat{S}(w)| \leq |u|^n \exp(d \tilde{M}_p(v))$$

for sufficiently large $|w|$.

Let $w = u + iv$ be a complex number such that $0 \leq \tilde{M}_p(v) < m \cdot \log |w|$ and let L be a positive constant which we fix later. The function

$$F_w(z) := \hat{S}\left(u + z \frac{v}{|v|}\right), \quad z = x + iy \in \mathbf{C}$$

is an entire analytic function of the complex variable z . Condition (H_2) implies

$$(13) \quad |F_w(x)| \leq (2|u|)^{-B} \quad \text{for} \quad \tilde{M}_p(x) < L \log |u|$$

and using (12)

$$(14) \quad |F_w(z)| \leq (2|u|)^n \exp(d \tilde{M}_p(y))$$

for sufficiently large $|w|$ and $\tilde{M}_p(|z|) \leq L \cdot \log |u|$.

Let us analyse the function

$$(U_w(z) := -\log(2|u|)^B |F_w(z)|).$$

It is harmonic in z , if $\tilde{M}_p(|z|) < L \cdot \log |u|$, for sufficiently large $|w|$. Let us put $A := 1 + d$, $a := d$, $\bar{B} := (B + n + 1)/(m + 1)$, $b := 1/(m + 1)$, $R := \tilde{M}_p^{-1}((m + 1) \cdot \log |u|)$, where “ -1 ” stands for the inverse function. Relations (13) and (14) show that we can apply Lemma 3 for these A , $\bar{B}$, b and the given $p \in \mathbf{N}$. Now, Lemma 3 implies the existence of a natural number N , so that for $|z| < \tilde{M}_p^{-1}(L \cdot \log |u|)/N$ we have

$$(15) \quad U_w(z) \leq d \tilde{M}_p(y) + (B + n + 2) \log |u|.$$

The moment has come to use condition (B). For a given $p \in \mathbf{N}$ and given $q := N$ (N from Lemma 3) there exists $k \in \mathbf{N}$ such that (B) holds for sufficiently large values of the variable. Putting $L := k(m + 1)$ we see that $R \cdot N < T := k(m + 1) \cdot \log |u|$. The monotonicity of $\tilde{M}_p(v)$ implies $|v| < R$, since $\tilde{M}_p(v) < m \cdot \log |w| \leq (m + 1) \times \log |u| = \tilde{M}_p(R)$. So, we can put $z = i|v|$ in (15) which gives at last

$$(1/|\hat{S}(u + iv)|) \leq |u|^{2B + 2n + 3} \exp(d \tilde{M}_p(v))$$

for sufficiently large $|w|$, and this is the inequality in condition (H_3) with $\tilde{M}_p$ instead of $\tilde{M}_p$. But $\tilde{M}_p(v) = \tilde{M}_p(v) + C'_p$ for sufficiently large $|v|$, and this proves $(H_2) \Rightarrow (H_3)$.

4. The functions $N_p(x)$ and $Q_p(x)$, $p \in \mathbf{N}$

In order to prove the sufficiency of condition (H_3) , we shall introduce some sequences of smooth functions.

Let $\omega: \mathbf{R} \rightarrow \mathbf{R}$ be a smooth nonnegative function which satisfies the following conditions: $\text{supp } \omega \subset [0, 1]$ and $\int_{\mathbf{R}} \omega(x) dx = 1$. For $x > 1$ and $p \in \mathbf{N}$ we define

$$\bar{N}_p(x) := \int_{\mathbf{R}} \tilde{M}_p(t) \omega(x - t) dt.$$

By $\underline{N}_p(x)$ we denote the smooth even function on $\mathbf{R}$ which equals $\bar{N}_p(x)$ for $x > 1$, and is increasing for $x > 0$. The next Lemma shows that the sequences $\{N_p(x)\}$ and $\{\tilde{M}_p(x)\}$ are equivalent in the sense that for every function from the first sequence there exists a greater one (at least, for $|x|$ sufficiently large) from the second and vice versa.

Now, we write down the following condition, which is by Lemma 4 equivalent to (H_3) from the Theorem.

(H'_3) For every $p \in \mathbf{N}$ and $d > 0$ there exists a constant $\bar{B} > 0$ so that for every $m \in \mathbf{N}$ there exists $C_m > 0$ such that

$$|1/\hat{S}(w)| \leq |w|^{\bar{B}} \exp(d \underline{N}_p(v)) \quad \text{for} \quad \underline{N}_p(v) < m \log |w|$$

and $|w| \geq C_m$.

Remark. One checks easily that this condition implies that there exists an $\bar{L}_1 > \bar{B}$ which fits for any d_1 , $0 < d_1 < d$, ($\bar{B}$ and d as in (H'_3)), though in general for greater C_m .

Lemma 4. Let $p \in \mathbf{N}$, $p > r(2)$ be given. Then:

- $\underline{N}_p(x) \leq \tilde{M}_p(x)$ for $|x| \geq 1$;
- There exists $p' \in \mathbf{N}$, $1 < p' < p$ such that $\tilde{M}_p(x) \leq \underline{N}_{p'}(x) + K_{pp'}$ for each $x \in \mathbf{R}$ and some $K_{pp'} > 0$.
- The condition (H'_3) is equivalent to the condition (H_3) .

PROOF. a) Let $x > 1$. Then

$$\underline{N}_p(x) = \int_{\mathbf{R}} \tilde{M}_p(x-t) \omega(t) dt = \int_0^1 \tilde{M}_p(x-t) \omega(t) dt \leq \tilde{M}_p(x) \int_0^1 \omega(t) dt = \tilde{M}_p(x).$$

b) Let p' be the largest natural number such that $r(p') \leq p$. Let us show first

$$(16) \quad \tilde{M}_p(x) \leq \tilde{M}_{p'}(x-1)$$

for sufficiently large $|x|$.

In fact, condition (A) implies $M_{p'}(p'x) \leq M_p(x)$, hence $\tilde{M}_{p'}(x/p') \leq \tilde{M}_p(x)$ for sufficiently large $|x|$. Since $\tilde{M}_{p'}(x)$ increases with $|x|$, the relation (16) follows. This implies

$$\tilde{M}_p(x) \leq \tilde{M}_{p'}(x-1) \int_0^1 \omega(t) dt \leq \int_{\mathbf{R}} \tilde{M}_{p'}(x-t) \omega(t) dt = \underline{N}_{p'}(x)$$

for sufficiently large $|x|$, and since $\tilde{M}_p(x)$ and $\underline{N}_p(x)$ are even b) follows.

c) Let us suppose that (H_3) holds. From part b) it follows that for every p' there exists $p_0 > p'_0$ such that for sufficiently large $|x|$ the inequality:

$$\tilde{M}_{p_0}(x) < \underline{N}_{p'_0}(x) \quad \text{holds. If} \quad \underline{N}_{p'_0}(v) < m \log |w| \quad \text{and}$$

$|w| \geq C_m$ then $\tilde{M}_{p_0}(v) < m \cdot \log |w|$. This implies that for $p = p_0$ and sufficiently large $|w|$ (H'_3) holds, and hence (H_3) holds for $p = p'_0$. The opposite implication follows in a similar way from part a) of the Lemma. $\square$

We prove now an important property of the functions $\underline{N}_p(x)$, namely: for given $(p, q) \in \mathbb{N}^2$ there exists $k = k(p, q) \geq q$ such that

$$(17) \quad \underline{N}_p(qx) \leq k \underline{N}_p(x) \quad \text{for } |x| \text{ sufficiently large.}$$

In fact, from (B) follows that for given $(p, q_1) \in \mathbb{N}^2$ there exists $k = k(p, q_1) \geq q_1$, such that $\tilde{M}_p(q_1 y) \leq k \cdot \tilde{M}_p(y)$ for $|y|$ sufficiently large. We choose $q_1 \geq q$ such that $q_1 |y| \geq q |y| + q - 1$ (for large $|y|$), so $\tilde{M}_p(qy + q - 1) \leq k \cdot \tilde{M}_p(y)$. This implies

$$\tilde{M}_p(qy + qt - t) \leq k \cdot \tilde{M}_p(y) \quad \text{for } t \in [0, 1] \text{ and } |y| \text{ large.}$$

Putting $y = x - t$ in the last inequality, multiplying it with $\omega(t)$ and integrating by t on $[0, 1]$ we obtain (17).

Next, we shall need the sequence of smooth functions on $\mathbb{R}$, $\{Q_p(x)\}_{p \in \mathbb{N}}$ which are defined in the following way:

$$(18) \quad Q_p(x) = \int_0^1 \underline{N}_p^{-1}(x-t) \omega(t) dt \quad \text{for } x \geq 1;$$

($\underline{N}_p^{-1}(x)$ is the inverse function of $\underline{N}_p(x)$, $x > 0$). We define $Q_p(x)$ on $(-\infty, 1]$ so that each $Q_p(x)$ becomes smooth and even on $\mathbb{R}$ and increasing on $(0, \infty)$. We need later the obvious inequality

$$(19) \quad Q_p(x) \leq \underline{N}_p^{-1}(x), \quad |x| \geq 1, p \in \mathbb{N}.$$

5. The (p, q) -parametrix for S

Definition 2. Let $S \in \mathcal{O}'_c(\mathcal{H}'\{M_p\})$ and $p, q \in \mathbb{N}$. The distribution $P \in \mathcal{H}'\{M_p\}$ is called the (p, q) — parametrix for S if it satisfies the following two conditions:

(P_1) There exists $n \in \mathbb{N}$ and a continuous function $F(x)$ on $\mathbb{R}$ with the property

$$(20) \quad F(x) = O(\exp(-N_p(x))) \quad \text{when } |x| \rightarrow \infty \text{ such that } P(x) = D^n F(x),$$

where $N_p(x)$ is an even, smooth nonnegative function such that for $x > 1$

$$(21) \quad N_p(x) = \int_{\mathbb{R}} M_p(t) \omega(x-t) dt$$

and the sequence $\{N_p(x)\}$ is monotonically increasing (see [5]).

(P_2) If $W(x) := \delta(x) - (S * P)(x)$, then $W \in C^q(\mathbb{R})$ and for every j , $0 \leq j \leq q$,

$$W^{(j)}(x) = O(\exp(-N_p(x))) \quad \text{when } |x| \rightarrow \infty.$$

We prove now

Proposition 3. Let $S \in \mathcal{O}'_c(\mathcal{H}'\{M_p\})$ and $(p, q) \in \mathbb{N}^2$ be given. There exists a (p, q) — parametrix P for S if $\hat{S}(w)$ satisfies the condition (H'_3) .

PROOF. In the proof of this proposition we use the idea of an analogous statement from [7]. However, we give the complete proof since many details on the functions $M_p(x)$ occur.

Let b be an even number such that

$$(22) \quad b > (\bar{B} + 1)/2;$$

$\bar{B}$ is the constant from (H'_3) for given $p \in \mathbf{N}$ and $d > 0$. By the assumption, for given $m \in \mathbf{N}$ (which we fix later), there exists $C_m > 0$ such that the function

$$(23) \quad F(x, w) := \frac{1}{2\pi} \frac{\exp(ixw)}{\hat{S}(w)w^{2b}}, \quad w = u + iv \in \mathbf{C},$$

is analytic for $N_p(v) \leq m \cdot \log |w|$ and $|w| \geq C_m > 1$. Condition (22) implies that the function $F(x, u)$ is integrable for each $x \in \mathbf{R}$ on the set $I_m := \{u \in \mathbf{R}; |u| \geq C_m\}$. If we put

$$h(x) := \int_{I_m} F(x, u) du \quad \text{and} \quad g(x) := D^{2b}h(x) \quad \text{then}$$

$$S * g = \delta - \frac{1}{2\pi} \int_{-C_m}^{C_m} \exp(ixu) du.$$

Let $a > 0$, $p \in \mathbf{N}$ and $c \in \mathbf{N}$ be given; take $q = 2c$ and the constant $k > 2c$ from (17) such that

$$(24) \quad c < k < ma/2$$

(as we see later, this can be achieved by letting m increase. Let $s(t)$ be an odd smooth function on $\mathbf{R}$ such that $s(t) = C_m$ for $0 < t \leq C_m$ and $s(t) = C_m \cdot \exp(aN_p(t))$ for $|t| \geq 2C_m$. We suppose also that $s(t)$ increases for $t > C_m$. With $s^*(t)$ we denote the increasing smooth function on $\mathbf{R}$, which equals $s(t)$ for $|t| \geq C_m$, with the property $s^{*(j)}(0) = 0$ for $j = 0, 1, 2, \dots$. Let $z(t)$ be an even smooth function on $\mathbf{R}$ such that $z(t) = 0$ for $|t| \leq C_m$ and $z(t) = c \cdot Q_p(N_p(t))$ for $|t| \geq 2C_m$ (see (18) and (21)); we suppose $z(t)$ to be increasing for $t > C_m$. From (17), (19) and (24) follows that

$$(25) \quad N_p(2z(t)) \leq m \log |s(t)| \quad \text{for} \quad |t| \geq C_m.$$

We take the curve $L(x) \equiv w(t) := s(t) + i \cdot z(t) \cdot \operatorname{sgn} x$, $t \in I_m$. Condition (24) shows that the function $h(x)$ can be written as

$$h(x) = \int_{L(x)} F(x, u) du = \int_{I_m} F(x, w(t)) w'(t) dt$$

provided that $2b > \bar{B} + dm + 2$.

Let us put

$$(26) \quad h_1(x) := \frac{1}{2\pi} \int_{L_1(x)} \frac{\exp(ixw) dw}{\hat{S}(w)w^{2b}}, \quad |x| \geq C_m,$$

$$(27) \quad h_2(x) := \frac{1}{2\pi} \int_{L_2(x)} \frac{\exp(ixw) dw}{\hat{S}(w)w^{2b}}, \quad |x| \geq C_m$$

where $L_1(x)$ and $L_2(x)$ are parts of the curve $L(x)$ for the values of the parameter t such that $|t| \geq |x|$, respectively $C_m \leq |t| \leq |x|$.

We put now

$$(28) \quad W(x) := (S * D^{2b} h_2)(x) + \frac{1}{2\pi} \int_{-c_m}^{c_m} \exp(ixu) du$$

and finally $P(x) := D^{2b} h_1(x)$.

The distribution P is the (p, q) -parametrix for S , and $W(x)$ from (28) is just the function from the condition (P_2) . In fact, from the following two lemmas conditions (P_1) and (P_2) follow.

Lemma 5. *The function $h_1(x)$ from (26) is $O(\exp(-c \cdot N_p(x)))$ when $|x| \rightarrow \infty$.*

PROOF. We write $h_1(x)$ as

$$h_1(x) = \frac{1}{2\pi} \int_{\{t: |t| \leq |x|\}} \frac{\exp(ixw(t)) w'(t) dt}{\hat{S}(w(t)) w^{2b}(t)}.$$

First of all let us prove

$$(29) \quad |\exp(ixw(t))| \leq K_0 \exp(-c N_p(x))$$

for sufficiently large $|x|$, $|t| \leq |x|$ and some $K_0 > 0$. Since $\underline{N}_p^{-1}(x-1) \leq Q_p(x) \leq \underline{N}_p^{-1}(x)$ and (see [3]) $\frac{|x|}{M_p^{-1}(x)} \leq \tilde{M}_p^{-1}(x)$ we have by Lemma 4 and [5]

$$Q_p(x) \leq \underline{N}_p^{-1}(|x|-1) \leq \tilde{M}_p^{-1}(|x|-1) \leq \frac{|x|-1}{N_p^{-1}(|x|-1)} \leq \frac{|x|-1}{N_p^{-1}(x)}.$$

This implies

$$Q_p(N_p(t)) \leq Q_p(N_p(x)) \leq \frac{N_p(x)-1}{N_p^{-1}(N_p(x))} = \frac{N_p(x)-1}{x}, \quad |t| \leq |x|,$$

hence

$$c|x|Q_p(N_p(t)) \leq cN_p(x) - c$$

and so (29) follows.

Let us observe now that

$$(30') \quad |Q_p^{(j)}(x)| \leq K'_j \cdot Q_p(x) \leq K'_j \cdot |x|,$$

$$(30'') \quad |N_p^{(j)}(x)| \leq K''_j \cdot N_p(x) \quad \text{and (see [5])},$$

(30''') $N_p(x) \leq K'''_j \cdot \exp(aN_p(x))$, $|x|$ sufficiently large, for some K'_j , K''_j and K'''_j (which depend also on $p \in \mathbb{N}$). To simplify the notations, we shall use the same letter K_j for these constants; moreover, when some constant depends only on $j \in \mathbb{N}_0$ and a fixed $p \in \mathbb{N}$ we denote it also by K_j .

Let us estimate $|z^{(j)}(t)|$. For $j=1$ the relations (30) imply

$$(31) \quad |z'(t)| \leq K_1 \exp(2adN_p(t))$$

and by induction

$$|z^{(j)}(t)| \leq K_j \exp(a(j+1)dN_p(t)), \quad j \in \mathbb{N}_0.$$

Now, it is easy to prove

$$(32) \quad |w^{(j)}(t)| \leq K_j \exp(a(1+(j+1)d)N_p(t))$$

for $|t|$ sufficiently large and some $K_j > 0$, $j \in \mathbb{N}_0$.

At last, we have

$$(33) \quad |w(t)| \cong K \exp(a N_p(t)) \quad \text{for } |t| \text{ sufficiently large}$$

and some $K > 0$.

Combining (H'_3) , (29), (32) (for $j=1$) and (33) we obtain

$$\begin{aligned} |h_1(x)| &\leq K'_0 \exp(-c N_p(x)) \int_{\mathbb{R}} \exp\left(\left(a\bar{B} + \frac{dam}{2} + a(1+2d) - 2ab\right) N_p(t)\right) dt = \\ &= O(\exp(-c N_p(x))) \quad \text{when } |x| \rightarrow \infty \end{aligned}$$

provided that

$$(34) \quad b > 1 + \left(\bar{B} + \frac{dm}{2} + 1 + 2d\right)/2,$$

where b is an even natural number. So we obtain Lemma 5. $\square$

Lemma 6. For given $(p, q) \in \mathbb{N}^2$ there exist sufficiently small constants d, a and sufficiently large constants m, c such that

$$W^{(j)}(x) = O(\exp(-N_p(x))) \quad \text{when } |x| \rightarrow \infty$$

for every $j, 0 \leq j \leq q$. ($W(x)$ from (28)).

PROOF. It is clear that the function $h_2(x)$ defined by (27) is smooth and equals zero for $|x| \leq C_m$. We suppose that $x > 0$ (the case $x < 0$ is analogous).

Let us prove now

$$(35) \quad |h_2^{(j)}(x)| \leq K_j \exp(ak_1 N_p(x))$$

for some $k_1 = k_1(j, d) \geq 1$ (one can compute it exactly) and some K_j . In fact we have

$$(35') \quad 2\pi h_2^{(j)}(x) = \int_{\{t: C_m \leq |t| \leq x\}} \frac{\exp(iw(t)x) w'(t)}{\hat{S}(w(t)) w(t)^{2b}} dt + R_j(x)$$

where

$$\begin{aligned} R_0(x) &= 0, \quad R_1(x) = (y_1(x) \exp(ixw(x)) - y_1(-x) \exp(ixw(-x)))w'(x), \quad R_{j+1}(x) = \\ &= R'_j(x) + (y_1(x) \exp(ixw(x)) (iw(x))^{j+1} - y_1(-x) \exp(ixw(-x)) (iw(-x))^{j+1})w'(x), \\ y_1(x) &= y_2(w(x)) = 1/(\hat{S}(w(x))w(x)^{2b}), \quad j=0, 1, \dots \end{aligned}$$

Our next goal is to prove that the function $y_1(w) = 1/(\hat{S}(w) \cdot w^{2b})$ is bounded together with its derivatives on the set $\Omega_1 = \{w = u + iv \in \mathbb{C}, \frac{N_p(2v)}{2} \leq m \cdot \log |u|, |w| \geq T > C_m\}$ for sufficiently large T . In order to prove this, we shall use a version of Theorem 1.2.4 from [9], p. 17; also, we use the notations from [9].

First of all, the integral

$$\iint_{\Omega} |y_1(w)| dw \wedge d\bar{w}$$

converges in view of the condition $2b > \bar{B} + dm + 2$, where $\Omega = \{w = u + iv \in \mathbb{C}, \frac{N_p(v)}{2} \leq m \cdot \log |u|, |w| \geq C_m\}$ (see (25)).

Let $\psi(w) \in C_0^\infty(\mathbb{C})$ be such that $\psi(w) = 1$ on $B(0, 1)$ and $\psi(w) = 0$ for $w \notin B(0, 2)$, where $B(w_0, R) = \{w \in \mathbb{C}, |w - w_0| \leq R\}$ $w_0 \in \mathbb{C}, R > 0$.

If $w \in \Omega_1$, then for T sufficiently large there exists $w_0 \in \Omega$ such that $w \in B(w_0, 1)$ and $B(w_0, 2) \subset \Omega$. The function $y_1(w)$ is analytic on Ω_1 , so $\frac{\partial}{\partial \bar{w}} (y_1(w) \cdot \psi(w - w_0)) = y_1(w) \cdot \frac{\partial}{\partial \bar{w}} \psi(w - w_0)$, $w \in \Omega_1$. By the Cauchy formula we get

$$y_1(w) \psi(w - w_0) = \frac{1}{2\pi i} \left(\int_{\{\zeta: |\zeta - w_0| = 2\}} \frac{y_1(\zeta) \psi(\zeta - w_0)}{\zeta - w} d\zeta + \iint_{B(w_0, 2)} y_1(\zeta) \frac{\frac{\partial}{\partial \bar{\zeta}} \psi(\zeta - w_0)}{\zeta - w} d\zeta \wedge d\bar{\zeta} \right).$$

Observing that the first integral is zero, differentiating this equality j — times and using the following facts: $\frac{\partial}{\partial \bar{\zeta}} \psi(\zeta - w_0)$ is zero on $B(w_0, 1)$ and $\sup \{1/|\zeta - w|; w \in B(w_0, 1), \zeta \in \text{supp } \frac{\partial}{\partial \bar{\zeta}} \psi(\zeta - w_0)\} < \infty$, we get the following inequality:

$$|y_1^{(j)}(w)| \leq K_j' \iint_{\Omega} |y_1(\zeta)| d\zeta \wedge d\bar{\zeta} =: K_j, \quad j = 0, 1, \dots$$

for any $w \in \Omega_1$ and some $K_j > 0$. Observe that this K_j does not depend on w or w_0 .

In view of (32) this implies that the last integral in (35') can be estimated with

$$2x \cdot K_j \exp(ak_1' N_p(x)) \leq K_j \exp(a(k_1' + d) N_p(x))$$

and the remainders $R_j(x)$ with $K_j \cdot (\exp(ak_1'' \cdot N_p(x)))$ for some $k_1', k_1'' \geq 1$ which depend on $j \in \mathbb{N}_0$ and $d > 0$ ($|x|$ sufficiently large). So we obtain (35) for $k_1 = \max(k_1' + d, k_1'')$.

Let us observe that Lemma 6. implicitly states that $W(x)$ is in fact a function from the class C^q ; from the smoothness of $h_2(x)$ and (35) this follows at once. We have

$$W^{(j)}(x) = (S * h_2^{(j+2b)})(x) + \left(\frac{1}{2\pi} \int_{-C_m}^{C_m} \exp(ixu) du \right)_x^{(j)}.$$

Since S is a convolutor on $\mathcal{H}'\{M_p\}$, for given p_2 (which we fix later) there exists a continuous bounded function on $\mathbb{R}$ and a non-negative integer n such that $S = D^n(G(x) \exp(-N_{p_2}(x)))$. This implies

$$\begin{aligned} (36) \quad W^{(j)}(x) &= \\ &= \int_{-\infty}^{\infty} G(y) \exp(-N_{p_2}(y)) \left(\int_{\{t: {}_m C \leq |t| \leq |x-y|\}} F(x-y, w(t)) w'(t) dt \right)_y^{(j+2b+n)} dy + \\ &\quad + \frac{1}{2\pi} \left(\int_{-C_m}^{C_m} \exp(iux) du \right)_x^{(j)} \end{aligned}$$

($F(x, w)$ from (23) and $w(t) = s(t) + iz(t) \cdot \text{sgn}(x - y)$).

We suppose that $x > C_m$, write the inner integral in (36) as

$$\int_{\{t: C_m \leq |t| \leq |x-y|\}} = \int_{t: -m \leq |t| \leq x} + \int_{\{t: x \leq |t| \leq |x-y|\}}$$

and estimate the obtained integrals separately. We shall show that they are both $O(\exp(-N_p(x)))$ when $x \rightarrow \infty$. The first integral can be written as

$$(37) \quad \int_{-\infty}^{\infty} G(y) \exp(-N_{p_2}(y)) \left(\int_{\{t: C_m \leq |t| \leq x\}} (F(x-y, w(t)) w'(t) dt)_y^{(j+2b+n)} \right) dy = \\ = \int_{-\infty}^x + \int_x^{\infty} \pm \int_x^{\infty} (G(y) \exp(-N_{p_2}(y)) \left(\int_{\{t: C_m \leq |t| \leq x\}} \frac{\exp(i(x-y)w(t))(-iw(t))^{j+n}}{\hat{S}(w(t))} dt \right) dy$$

where in the second integral $w(t) = s(t) - iz(t)$, and in the others $w(t) = s(t) + iz(t)$.

Now the first integral and the third integral with the "plus" sign in (37) give

$$(-1)^{j+n} \frac{2\pi}{2\pi} \int_{\{t: C_m \leq |t| \leq x\}} \frac{\exp(ixw(t)) w^{j+n}(t) w'(t) dt}{\hat{S}(w(t))} \times \\ \times \int_{-\infty}^{\infty} G(y) \exp(-N_{p_2}(y)) \exp(-yiw(t)) dy = \left(\frac{1}{2\pi} \int_{L_1(x)}^{\infty} \exp(ixw) dw \right)_x^{(j)}.$$

Take now $p_1 \geq r(p)$ such that the integral $\int_{-\infty}^{\infty} \exp(-N_{p_1}(y) + c|y|Q_p(N_p(y))) dy$ converges and (for a given k_2) $p_2 \geq r(p_1)$ such that $N_{p_2}(y) \geq N_{p_1}(y) + k_2 N_p(y)$, $y \in \mathbf{R}$. (p_1 and p_2 with such a property exist by (A)). Then the fourth integral in (37) (with the "-" sign) can be estimated with

$$\text{const } 2x |w^{j+2b+n}(x)| |w'(x)| |y_2(w(x))| \exp(-k_2 N_{p_1}(x)) \times \\ \times \int_x^{\infty} \exp(-N_{p_1}(y) + cyQ_p(N_p(y))) dy = O(\exp(k_3 a - k_2 N_p(x)))$$

when $x \rightarrow \infty$, where $k_3 > 0$ can be computed exactly; it depends on j, n, b and d . We take $k_2 > k_3 + 1 > k_3 a + 1$ if $0 < a < 1$.

At last, we can estimate the second integral in (37) as $O(\exp(k_3 a - k_2 N_p(x)))$ when $x \rightarrow \infty$, k_3 and k_2 as before.

The integral

$$\int_{-\infty}^{\infty} G(y) \exp(-N_{p_2}(y)) \left(\int_{\{t: x \leq |t| \leq |x-y|\}} F(x-y, w(t)) w'(t) dt \right)_y^{(j+n+2b)} dy$$

can be written as

$$(38) \quad \int_{-\infty}^{\infty} G(y) \exp(-N_{p_2}(y)) \left(\int_{\{t: x \leq |t| \leq |x-y|\}} \frac{\exp(i(x-y)w(t))(-iw(t))^{j+n}}{\hat{S}(w(t))} w'(t) dt \right) dy + \\ + S_j(x)$$

where the S_j are the "remainders" (we estimate them at the end).

The integral in (38) we write as

$$(39) \quad \int_{-\infty}^{\infty} = \int_{-\infty}^0 + \int_0^x + \int_x^{2x} + \int_{2x}^{\infty} = I_1 + I_2 + I_3 + I_4.$$

Now, if necessary, we increase the previous p_1 so that the inequality in the condition (C) holds for some $k=k_4$; this implies:

$$N_p(x-y) \leq k_4 N_p(x) + N_{p_1}(y)$$

for x sufficiently large and $y < 0$. If necessary, we increase $p_2 \geq r(p_1)$ so that the integral

$$\int_{-\infty}^{\infty} |yG(y)| \exp(k_5 a N_{p_1}(y) - N_{p_2}(y)) dy$$

converges and obtain

$$|I_1| \leq K_1 \exp(-c N_p(x)) \int_{-\infty}^0 yG(y) \exp(k_4 k_5 a N_p(x) \times$$

$$\times \exp(k_5 a N_{p_1}(y) - N_{p_2}(y)) dy = O(\exp(k_4 k_5 a - c) N_p(x)) \quad \text{when } x \rightarrow \infty;$$

here $k_5 = (1+d)(j+2b+n) + (1+2d)$.

Next, since for $0 < y < x$ we have $N_p(x-y) \leq N_p(x) - N_{p_1}(y)$ (for given p there exists $p_1 \geq r(p)$) and if p_2 is so large that

$$\int_{\pm\infty}^{\infty} |yG(y)| \exp(c N_{p_1}(y) - N_{p_2}(y)) dy < \infty$$

we have $|I_2| \leq K_2 \cdot \exp((k_5 a - c) N_p(x))$.

In a similar way we get (without condition (C)!):

$$I_3 = O(\exp((k_5 a - c) N_p(x)))$$

and

$$I_4 = O(\exp(-c N_p(x))) \quad \text{when } x \rightarrow \infty$$

hence the integral in (38) is $O(\exp(k_4 k_5 a - c) N_p(x))$ when $x \rightarrow \infty$.

The remainders $S_j(x)$ are linear combinations of the integrals

$$\int_{-\infty}^{\infty} G(y) \exp(-N_{p_2}(y)) (y_1^{(n_1)}(x-y))^{n_2} ((xw(x-y))_{(y)}^{n_3})^{n_4} \times \\ \times ((iw(x-y))_y^{(n_5)})^{n_6} w^{(n_7)}(x-y) \exp(i(x-y)w(x-y)) dy, \quad n_i \in \mathbb{N}_0$$

so by the previously used inequalities we get

$$S_j(x) = O(\exp((k_6 a - c) N_p(x))) \quad \text{when } x \rightarrow \infty,$$

for some k_6 which depends on j, n, b and d .

Now we can write

$$W^{(j)}(x) = O(\exp(k_7 a - c) N_p(x)) + \frac{1}{2\pi} \left(\int_{L^*(x)} \exp(ixw) dw \right)_x^{(j)}$$

where $L^*(x)$ is the contour obtained by joining the segment $[-C_m, C_m]$ on the real axis with the curve $w^*(t) = s^*(t) + iz(t)$ for $C_m \leq |t| \leq x$, where $k_7 = \max(k_4 k_5, k_5, k_6)$. Let us show

$$(40) \quad \frac{1}{2\pi} \left(\int_{L^*(x)} \exp(ixw) dw \right)_x^{(j)} = O \left(\exp \left(k_8 - \frac{c}{2} \right) N_p(x) \right)$$

when $x \rightarrow \infty$ for some $k_8 > 1$ which depends on j, n, b and d .

From the Cauchy theorem it follows that we can replace $L^*(x)$ with the contour $L_1^*(x)$, which is the complementary contour to $L^*(x)$, i.e. its equation is $w^*(t) = s^*(t) + iz(t)$ for $0 < x < |t|$. So we get that the integral in (40) is equal to

$$(41) \quad \frac{1}{2\pi} \int_{L_1^*(x)} (iw)^j \exp(ixw) dw + S_j^*(x)$$

where $S_j^*(x)$ is the remainder. For sufficiently large x we can estimate the last integral with

$$\begin{aligned} & \exp \left(-\frac{c}{2} N_p(x) \right) \int_{|t| \geq x} \exp \left(-\frac{c}{2} N_p(t) \right) \exp(k'_8 a N_p(t)) dt = \\ & = O \left(\exp \left(-\frac{c}{2} N_p(x) \right) \right) \quad \text{when } x \rightarrow \infty, \quad k'_8 = k'_8(j, b, d) \end{aligned}$$

provided that $k'_8 a < \frac{c}{2} + 1$. The remainders $S_j^*(x)$ are estimated easily as $O \left(\exp \left(\left(k''_8 a - \frac{c}{2} \right) N_p(x) \right) \right)$ when $x \rightarrow \infty$ for some $k''_8 > 0$. Taking $k_8 = \max(k'_8, k''_8)$ we obtain (40). So, for $k_9 := \max_{0 \leq j \leq q} \max(k_7, k_8)$ we get

$$(42) \quad W^{(j)}(x) = O \left(\exp \left(\left(k_9 a - \frac{c}{2} \right) N_p(x) \right) \right) \quad \text{when } x \rightarrow \infty \text{ and } 0 \leq j \leq q.$$

Take now $c = 2p + 2$, choose p_1, p_2 and n as indicated before. Taking now $b > \bar{B} + 3$ ($\bar{B}$ from (H'_3)), a so small that $k_9 a - c/2 < -1$, m sufficiently large that $m > 2c/a$ and observing that k_9 does not grow by decreasing d , take $d < 2/m$, (see the Remark after condition (H'_3)) we obtain the Lemma. $\square$

As we said before, proving Lemmas 5 and 6, we finish the proof of Proposition 3. We prove in the sixth section that the existence of a (p, q) -parametrix for the convolutor S implies its hypoellipticity.

6. Sufficiency of (H_3) for hypoellipticity

Proposition 4. Let us suppose that for given $(p, q) \in \mathbb{N}^2$ there exists a (p, q) -parametrix $P \in \mathcal{H}'\{M_p\}$ for the convolutor $S \in \mathcal{O}'_c(\mathcal{H}'\{M_p\})$. Then the equation

$$(1) \quad S * U = V$$

and S are hypoelliptic in $\mathcal{H}'\{M_p\}$.

PROOF. Let $V \in \mathcal{E}\mathcal{H}'\{M_p\}$ be given and let us suppose that there exists a solution $U \in \mathcal{H}'\{M_p\}$ of (1). We know that there exists $p_1 \in \mathbb{N}$ such that

$$|V^{(j)}(x)| \leq C_j \exp(M_{p_1}(x)) \quad \text{for every } j = 0, 1, 2, \dots$$

Theorem 1 in [5] implies the existence of numbers $k_1 \in \mathbb{N}_0$ and $p_2 \in \mathbb{N}$ and a continuous function $K_1(x)$ on $\mathbb{R}$ such that

$$U(x) = D^{k_1} K_1(x) \quad \text{and} \quad |K_1(x)| \leq C \exp(M_{p_2}(x)).$$

Let $j \in \mathbb{N}_0$ be given and put $p := r(r(p_1 + p_2))$, $q := j + k_1$. For this pair (p, q) we take the parametrix P for the convolutor S , so that the conditions (P_1) and (P_2) from Definition 2. hold. This implies

$$(43) \quad U = U * \delta = U * (S * P + W) = U * (S * P) + U * W = (U * S) * P + U * W \Rightarrow$$

$$(43') \quad U = V * P + U * W.$$

The existence and the associativity of the convolutions in (43) follows from the assumptions on U, V, P and S .

We have yet to show the smoothness of the two summands in (43'), and that they are $O(\exp(M_{p_0}(x)))$ when $|x| \rightarrow \infty$ for some $p_0 \in \mathbb{N}$, i.e.

$$(44) \quad U^{(j)}(x) = O(\exp(M_{p_0}(x))) \quad \text{when } |x| \rightarrow \infty$$

and p_0 should not depend on $j \in \mathbb{N}_0$.

The function $V * P$ is smooth, since

$$(V * P)(x) = (V * D^n F)(x) = (V^{(n)} * F)(x) = \int_{\mathbb{R}} V^{(n)}(x - y) F(y) dy$$

and generally

$$(V * P)^{(j)}(x) = \int_{\mathbb{R}} V^{(n+j)}(x - y) F(y) dy.$$

Let us estimate this integral:

$$\left| \int_{\mathbb{R}} V^{(n+j)}(x - y) F(y) dy \right| \leq c_{n,j} \int_{\mathbb{R}} \exp(M_{p_1}(x - y)) \exp(-N_p(y)) dy.$$

Using the inequality

$$(45) \quad \begin{aligned} M_{p_1}(x - y) &\leq M_{r(p_1)}(x) + M_{r(p_1)}(y) \leq M_{r(p_1)}(x) + N_{r(r(p_1))}(y) + \\ &+ K' \leq M_{r(p_1)}(x) + N_{r(r(p_1 + p_2))}(y) + K' \end{aligned}$$

for some $K' = K(p_2)$ we obtain

$$(46) \quad |V * P)^{(j)}(x)| \leq c \exp(M_{r(p_1)}(x)), \quad c > 0.$$

Since W is a C^q function, the choice of p shows that this is true also for $U * W$. So, we have

$$(U * V)^{(j)}(x) = (K_1 * W)^{(j+k_1)}(x)$$

and

$$|U * W)^{(j)}(x)| \leq C' \int_{\mathbb{R}} \exp(M_{p_2}(y)) \exp(-N_p(x - y)) dy$$

for some $C' > 0$. Hence from a relation like (45) we obtain

$$(47) \quad |(U * W)^{(j)}(x)| \leq C'' \exp(M_{r(p_2)}(x)).$$

From (43'), (46) and (47) we see that putting $p_0 := \max(r(p_1), r(p_2))$ we get (44). Observe that p_0 does not depend on j . $\square$

References

- [1] I. M. GEL'FAND, G. E. ŠILOV, Generalized Functions, Vol. II, *Academic Press, New York, London* 1968.
- [2] I. M. GEL'FAND, G. E. ŠILOV, Generalized Function, Vol. III, *Academic Press, New York, London* 1967.
- [3] O. V. GRUDZINSKI, Convolutions-Gleichungen in Räumen von Beurling Distributionen endlicher Ordnung, *Habilitationsschrift, Kiel*, 1980.
- [4] L. HÖRMANDER, On the Range of Convolution Operators, *Ann. Math.* **76** (1962), pp. 148—169.
- [5] S. PILIPOVIĆ, A. TAKAČI, The Space $\mathcal{H}'\{M_p\}$ and Convolutors, Proceedings of the Moscow Conference on Generalized Functions, *Moscow* 1981, pp. 415—426.
- [6] S. PILIPOVIĆ, A. TAKAČI, Solvability of Convolution Equations in the Space $\mathcal{H}'\{M_p\}$, to appear in *Revue Roum. Math. Pur. Appl.*
- [7] G. SAMPSON, Z. ZIELEZNY, Hypoelliptic Convolution Equations in $\mathcal{H}'_p$, $p > 1$, *Trans. Amer. Math. Soc.* **223** (1976), pp. 133—154.
- [8] J. WŁOKA, Über die Hurewič — Hörmanderschen Distributionsräume, *Math. Ann.*, **160** (1965), pp. 321—362.
- [9] L. HÖRMANDER, An Introduction to Complex Analysis in Several Complex Variables, (In Russian), *Mir, Moskva*, 1968.

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