Latent additivity and a differential equation

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1. Statement of the problem

In psychology one sometimes studies two-parameter families of distribution functions $V(z/x, y)$ where x and y are the parameters. These distribution functions often depend on x and y only through a real-valued function F such that $V(z/x, y)= =V_1(z/F(x, y))$. A very important class of models related to Rasch's theory of specific objectivity (see [2], [3], [4] e.g.) are models where F is of the form $F(x, y)= =f(g(x)+h(y))$. Functions of this form will be called latent additive in the following. (In some references these functions also are called nomographic.) It will be shown that the differential equation

$$F_{xy}(F_{xx}F_y^2-F_{yy}F_x^2) = F_xF_y(F_{xxy}F_y-F_{xyy}F_x)$$

nearly characterizes the class of latent additive functions. (The full result is given in corrollary 1 at the end of the paper.)

2. Theorems about latent additive functions

Let A be some subset of $\mathbf{R}^k$. $C^n(A)$ is the set of all functions defined on A such that all partial derivatives up to order n exist and are continuous.

Theorem 1. *Let I_1 and I_2 be (possibly infinite) intervals in $\mathbf{R}_1$ and let $g\in C^3(I_1)$ $h\in C^3(I_2)$ and $f\in C^3(g(I_1)+h(I_2))$. Then $F(x, y)=f(g(x)+h(y))$ fulfills the equation*

$$F_{xy}(F_{xx}F_y^2-F_{yy}F_x^2) = F_xF_y(F_{xxy}F_y-F_{xyy}F_x).$$

PROOF. Trivial.

This result is essentially known (see [1] e.g.) and it is also known that there exist functions fulfilling our differential equation without being latent additive. (An example can be found in [1].)

Therefore we will try to impose further conditions on F such that fulfillment of our differential equation also yields latent additivity.

For technical reasons we first prove

Lemma 1. *Let I_1 and I_2 be intervals (possibly infinite) in $\mathbf{R}_1$ and let $R=I_1\times I_2$. For $n\geqq 1$ let $F\in C^n(R)$ fulfill the partial differential equation $F_x=F_y$. Then there exists $f\in C(I_1+I_2)$ with $F(x,y)=f(x+y)$.*

PROOF. Let $T(x,y)=(x+y,x-y)$. Then $T^{-1}=\dfrac{1}{2}T$ and the projection of $T^{-1}(R)$ onto the x-axis is

$$S_1=\frac{1}{2}(I_1+I_2).$$

Let $S=T^{-1}(R)$ and $G(x,y)=F\big(T(x,y)\big)$. (G is defined on S). Since T is a linear isomorphism between S and R $G\in C^n(S)$ and we have

$$G_y(x,y)=F_x(x,y)-F_y(x,y)=0.$$

Therefore there exists $g\in C^n(S_1)$ with $G(x,y)=g(x)$ and therefore

$$F(x,y)=G\left(\frac{1}{2}(x+y,x-y)\right)=g\left(\frac{1}{2}(x+y)\right).$$

Defining $f(x)=g\left(\dfrac{1}{2}x\right)$ for x I_1+I_2 we have

$$F(x,y)=f(x+y)\quad\text{and}\quad f\in C^n(I_1+I_2).$$

With the help of this lemma we can state sufficient conditions for latent additivity.

Theorem 2. *Let I_1 and I_2 be (possibly infinite) intervals in $\mathbf{R}_1$. Let $R=I_1\times I_2$ and $F\in C^3(R)$. Let furthermore $F_x>0$ and $F_y>0$ on R and F fulfill the third-order partial differential equation $F_{xy}(F_{xx}F_y^2-F_{yy}F_x^2)=F_xF_y(F_{xxy}F_y-F_{xyy}F_x)$. Then there exist $f\in C^3(I_1)$, $g\in C^3(I_2)$ and $h\in C^3\big(f(I_1)+h(I_2)\big)$ with $f'>0$, $g'>0$ and $h'>0$ on their domains such that $F(x,y)=f\big(g(x)+h(y)\big)$ on R.*

PROOF. Define $H(x,y)=(F_yF_{xx}-F_xF_{xy})/F_xF_y$. Then one easily sees $H_y=0$ and $H\in C^1(R)$. Therefore there exists $r\in C^1(I_1)$ with $H(x,y)=r(x)$. Let $s(x)$ be an indefinite integral of $r(x)$ and let $g(x)$ be an indefinite integral of $\exp\big(s(x)\big)$. Then $g\in C^3(I_1)$ and $g'>0$ on I_1. Furthermore we have

$$\frac{g''}{g'}=(\ln g')'=(F_yF_{xx}-F_xF_{xy})/F_xF_y.$$

Therefore defining $T(x,y)=g'(x)\cdot F_y(x,y)/F_x(x,y)$ we have

$$T_x=g'(F_{xy}F_x-F_{xx}F_y)/F_x^2+g''F_yF_x$$

and we easily see $T_x=0$.

We also have $T\in C^2(R)$ and therefore there exists $t\in C^2(I_2)$ with $T(x,y)=t(y)$. Let $h(y)$ be an indefinite integral of $t(y)$. Since $T(x,y)=t(y)>0$ h is strictly increasing. Since g also is strictly increasing the inverse functions g^{-1} and h^{-1} are defined on the intervals $J_1=g(I_1)$ and $J_2=h(I_2)$. We have

$$g'(x)F_y(x,y)/F_x(x,y)=h'(y)$$

or equivalently

$$F_y(x, y)/h'(y) = F_x(x, y)/g'(x) \quad \text{for all} \quad (x, y) \in K.$$

Now we define

$$G(x, y) = F\big(g^{-1}(x), h^{-1}(y)\big).$$

G is defined on $J_1 \times J_2$ and we have

$$G_x(x, y) = F_x\big(g^{-1}(x), h^{-1}(y)\big)/g'\big(g^{-1}(x)\big)$$
$$G_y(x, y) = F_y\big(g^{-1}(x), h^{-1}(y)\big)/h'\big(h^{-1}(y)\big).$$

Therefore we have $G_x = G_y$ on $J_1 \times J_2$. Furthermore we have $1/(g' o g^{-1}) \in C^2(J_1)$ and $1/(h' o h^{-1}) \in C^2(J_2)$. Therefore $G \in C^3(J_1 \times J_2)$ and according to Lemma 1 there exists $f \in C^3(J_1 + J_2)$ with $G(x, y) = f(x+y)$. Since $F(x, y) = G\big(g(x), h(y)\big)$ we have $F(x+y) = f\big(g(x)+h(y)\big)$ and our theorem is proved.

Combining both theorems we get the following characterization result for latent additive functions:

Corollary 1. *Let I_1 and I_2 be (possibly infinite) intervals in $\mathbf{R}_1$, let $F \in C^3(I_1 \times I_2)$ and $F_x > 0$ and $F_y > 0$ on $I_1 \times I_2$. Then $F(x, y)$ is of the form $F(x, y) = f\big(g(x)+h(y)\big)$ with $g \in C^3(I_1)$, $h \in C^3(I_2)$ and $f \in C^3\big(g(I_1)+h(I_2)\big)$ if and only if $F_{xy}(F_{xx}F_y^2 - F_{yy}F_x^2) = F_x F_y (F_{xxy} F_y - F_{xyy} F_x)$.*

with $g \in C^3(I_1)$, $h \in C^3(I_2)$ and $f \in C^3\big(g(I_1)+h(I_2)\big)$ if and only if $F_{xy}(F_{xx}F_y^2 - F_{yy}F_x^2) = F_x F_y (F_{xxy} F_y - F_{xyy} F_x)$.

[1] R. C. Buck, Approximate Complexity and Functional Representation. *Journal of Mathematical Analysis and Applications* **70** (1979), 280—298.

[2] G. Fischer, Einführung in die Theorie psychologischer Tests. Verlag Hans Huber, Bern 1974, S. 407ff.

[3] G. Rasch, On general laws and the meaning of measurement in psychology. *Berkeley symposium on mathematical statistics and probability. Univ. of California Press* 1961.

[4] G. Rasch, An informal report on a theory of objectivity in comparisons. *Proc of the NUFFIC international summer session, The Hague, July 14—28*, 1966.

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