

On the error terms of $\sum_{n \leq x} \varphi_{f,t}^{(k)}(n)$ and $\sum_{n \leq x} \psi_{f,t}^{u,v}(n)$

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ABSTRACT

Let $f(n)$ be a nonconstant polynomial with integer coefficients and k, t, u and v positive integers. The first author introduced earlier the generalized Euler totient function $\varphi_{f,t}^{(k)}(n)$ and the generalized Dedekind ψ -functions $\Psi_{f,t}^{(k)}(n)$, $\psi_{f,t}^{(k)}(n)$, and $\psi_{f,t}^{u,v}(n)$ and obtained asymptotic formulae for their summatory functions subject to the condition $N_f(n) = O(n^\varepsilon)$ for some ε , $0 < \varepsilon < 1$, $N_f(n)$ being the number of incongruent solutions mod n of $f(x) \equiv 0 \pmod{n}$. In this paper we obtain these asymptotic formulae without any such condition and with better estimates for the error functions. Key words and phrases: Möbius function, Euler's totient function, Dedekind's ψ -function.

1. Introduction. Let $f=f(x)$ be a nonconstant polynomial with integer coefficients and let k and t be positive integers. In [1] the first author studied the generalized Euler totient function $\varphi_{f,t}^{(k)}(n)$ defined to be the number of incongruent t -tuples of integers $(a_1, a_2, \dots, a_t) \pmod{n}$ such that $((f(a_1), f(a_2), \dots, f(a_t)), n)_k = 1$, it being understood that the t -tuples $(a_1, a_2, \dots, a_t)$ and $(b_1, b_2, \dots, b_t)$ are congruent mod n iff $a_i \equiv b_i \pmod{n}$ for $1 \leq i \leq t$ and the symbol $(a, b, c, \dots, e)_k$ stands for the largest k th power common divisor of $a, b, c, \dots, e$ and $(a, b, c, \dots, e)_1 = (a, b, c, \dots, e)$ with the convention $(0, 0, \dots, 0)_k = 0$.

This function $\varphi_{f,t}^{(k)}(n)$ reduces when $f(x)=x$, $t=1$ and $k=1$ to Euler's totient function $\varphi(n)$ and for special choices of t and k , the various extensions of $\varphi(n)$ studied by E. COHEN, E. K. HAVILAND, JORDAN, V. L. KLEE, P. J. MCCARTHY, and SCHEMMELE. $\varphi_{f,t}(n)$ has been studied by P. K. MENON. (For details we refer to [1])

Among other things, in [1] it has been proved that as $x \rightarrow \infty$

$$(1.1) \quad \sum_{n \leq x} \varphi_{f,t}^{(k)}(n) = \alpha x^{t+1} + O(x^{te + \frac{1}{k}}),$$

where

$$(1.2) \quad \alpha = \frac{1}{t+1} \sum_{n=1}^{\infty} \frac{\mu(n) N_f^t(n^k)}{n^{k(t+1)}} = \frac{1}{t+1} \prod_p \left\{ 1 - \frac{N_f^t(p^k)}{p^{t(k+1)}} \right\}$$

under the condition

$$(1.3) \quad N_f(n) = O(n^\varepsilon) \text{ for some } 0 < \varepsilon < 1;$$

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here $N_f(n)$ denotes the number of incongruent solutions mod n of $f(x) \equiv 0 \pmod{n}$, $N_f^t(n) = (N_f(n))^t$, and $\mu(n)$ the Möbius function.

In the case of $\varphi(n)$ we have the well known result due to F. MERTENS, namely

$$(1.4) \quad \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x);$$

thus in this special case the main terms of (1.1) and (1.4) agree but the error term in (1.1) is not as good as that in (1.4).

Similarly, in [2] and [3], the first author introduced and studied the generalized Dedekind ψ -functions $\Psi_{f,t}^{(k)}(n)$, $\psi_{f,t}^{(k)}(n)$ and $\psi_{f,t}^{u,v}(n)$. He proved that under the condition (1.3)

$$(1.5) \quad \sum_{n \leq x} \Psi_{f,t}^{(k)}(n) = \frac{x^{t+1}}{t+1} \sum_{n=1}^{\infty} \frac{\mu^2(n) N_f^t(n^k)}{n^{k(t+1)}} + O(x^{\delta t + \frac{1}{k}}),$$

$$(1.6) \quad \sum_{n \leq x} \psi_{f,t}^{(k)}(n) = \frac{x^{kt+1}}{kt+1} \sum_{n=1}^{\infty} \frac{\mu^2(n) N_f^t(n^k)}{n^{kt+1}} + O(x^{\delta kt+1})$$

for every δ such that

$$(1.7) \quad 1 > \delta > \max \left(\varepsilon, 1 - \frac{1}{kt} \right);$$

and with the condition that ε of (1.3) is $< \frac{1}{u}$,

$$(1.8) \quad \sum_{n \leq x} \psi_{f,t}^{u,v}(n) = \frac{x^{vt+1}}{vt+1} \sum_{n=1}^{\infty} \frac{\varrho_{f,t}^{u,v}(n)}{n^{vt+1}} + E(x)$$

where

$$(1.9) \quad E(x) = \begin{cases} O(x^{vt}) & \text{if } vt(1-u\varepsilon) > 1 \\ O(x^{1+u\theta+uvte}) & \text{for every } \theta < \frac{vt(1-u\varepsilon)}{u} \end{cases}$$

if $vt(1-u\varepsilon) \leq 1$.

Here $\varrho_{f,t}^{u,v}(n)$ is the multiplicative arithmetical function defined by

$$(1.10) \quad \varrho_{f,t}^{u,v}(n) = \prod_{p^\alpha \parallel n} \binom{u}{\alpha} N_f^t(p^v),$$

where $p^\alpha \parallel n$ means that $p^\alpha | n$ and $p^{\alpha+1} \nmid n$ and

$$\binom{u}{\alpha} = \frac{u(u-1) \dots (u-\alpha+1)}{1 \cdot 2 \dots \alpha}, \quad \binom{u}{0} = 1 \quad \text{for nonnegative integers}$$

u and α .

In the case when $f(x)=x$, the functions $\Psi_{f,1}^{(k)}(n)$, $\psi_{f,1}^{(k)}(n)$, and $\psi_{f,1}^{k,1}(n)$ reduce respectively to $\Psi_k(n)$, $\psi_k(n)$, and $\psi_{(k)}(n)$ which were studied earlier by D. SURYANARAYANA [6] as extensions of the Dedekind's ψ -function $\psi(n)$ which has the arithmetical form

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p} \right),$$

the product being extended over all prime divisors of n . In fact, $\Psi_1(n) = \psi_1(n) = \psi_{(1)}(n) = \psi(n)$. The estimates for error terms in his asymptotic formulae for $\sum_{n \leq x} \Psi_k(n)$, $\sum_{n \leq x} \psi_k(n)$, and $\sum_{n \leq x} \psi_{(k)}(n)$ turned out to be a little better than those of (1.5), (1.6) and (1.8) in these special cases.

Also, it is known now that (1.3) always holds, ε of course depending upon the polynomial f . In fact, the second author and P. V. KRISHNAIAH [5] recently proved that

$$N_f(n) \leq A^{\omega(n)} n^{1-\frac{1}{h}},$$

h being the degree of f and A an absolute positive constant and $\omega(n)$ the number of distinct prime factors of n . Since $\omega(n) = O\left(\frac{\log n}{\log \log n}\right)$, (1.3) follows and thus the asymptotic formulae given in (1.1), (1.5), (1.6), and (1.8) hold for every nonconstant polynomial f .

In this paper, using entirely different arguments, we give proofs of the above formulae with better estimates for the error terms.

2. Preliminaries. As before, f is a nonconstant polynomial with integer coefficients and h the degree of f . Let D be the g.c.d. of the coefficients of f and $\gamma(n)$ the largest square free divisor of n . It is well known that $N_f(n)$ is a multiplicative function of n , i.e., $N_f(mn) = N_f(m)N_f(n)$ if $(m, n) = 1$. Let

$$(2.1) \quad c = \max \{h, u\},$$

u being the largest prime divisor of D .

We shall also need the following:

$$(2.2) \quad \varphi_{f,t}^{(k)}(n) = \sum_{d^k \delta = n} \mu(d) N_f^t(d^k) \delta^t,$$

$$(2.3) \quad \Psi_{f,t}^{(k)}(n) = \sum_{d^k \delta = n} \mu^2(d) N_f^t(d^k) \delta^t,$$

$$(2.4) \quad \psi_{f,t}^{(k)}(n) = \sum_{d \delta = n} \mu^2(d) N_f^t(d^k) \delta^{kt},$$

$$(2.5) \quad \psi_{f,t}^{u,v}(n) = \sum_{d \delta = n} \varrho_{f,t}^{u,v}(d) \delta^{vt},$$

$$(2.6) \quad \varrho_{f,t}^{1,v}(n) = \mu^2(n) N_f^t(n^v)$$

and

$$(2.7) \quad \varrho_{f,t}^{u,v}(n) = \sum_{d \delta = n} \varrho_{f,t}^{1,v}(d) \varrho_{f,t}^{u-1,v}(\delta).$$

Of these (2.2) is Corollary 1 of Theorem 2 in [1], (2.3) is an easy consequence of 2.15 of [2], (2.4) is (3) of 2.18 of [2], (2.6) is (2.9) of [3] and (2.5) and (2.7) are respectively Theorems 2.2 and Lemma 2.1 of [3].

Lemma 2.1. For all n ,

$$N_f(n) \leq C^{\omega(n)} \frac{n}{\gamma(n)}.$$

PROOF. By (2.1) we have $N_f(p) \leq c$ for all primes p . Also it is known that $N_f(p^\alpha) \leq p^{\alpha-1} N_f(p)$ for all primes p and all positive integers α . The multiplicativity of $N_f(n)$ gives the lemma.

Lemma 2.2. For positive integral k ,

$$\sum_{n \leq x} k^{\omega(n)} = O(x(\log 2x)^{k-1}).$$

PROOF. We use induction on k . First, we note

$$(2.8) \quad (k+1)^{\omega(n)} \leq \sum_{d|n} k^{\omega(d)}.$$

Clearly the functions appearing on both sides of (2.8) are positive and it is easily seen that they are multiplicative and (2.8) holds when $n=p^\alpha$, p a prime and α a positive integer; thus (2.8) holds for all n .

The lemma is clearly true for $k=1$. Assuming its truth for the positive integer k , we have, by (2.8),

$$\begin{aligned} \sum_{n \leq x} (k+1)^{\omega(n)} &\leq \sum_{n \leq x} \sum_{d \leq n} k^{\omega(d)} = \sum_{\delta \leq x} \sum_{d \leq x/\delta} k^{\omega(d)} = \\ &= O\left(\sum_{\delta \leq x} \left(\frac{x}{\delta}\right) \left(\log \frac{2x}{\delta}\right)^{k-1}\right) = O\left(x(\log 2x)^{k-1} \sum_{\delta \leq x} \frac{1}{\delta}\right) = O(x(\log 2x)^k), \end{aligned}$$

giving the truth of the lemma for $k+1$ and the proof is complete.

Lemma 2.3. As $x \rightarrow \infty$

$$(2.9) \quad \sum_{n \leq x} \mu^2(n) N_f^t(n^k) = O(x^{1+(k-1)t} (\log 2x)^{c^t-1}),$$

$$(2.10) \quad \sum_{n \leq x} \frac{\mu^2(n) N_f^t(n^k)}{n^{kt}} = \begin{cases} O((\log 2x)^c) & \text{if } t = 1 \\ 0(1), & \text{if } t \geq 2 \end{cases}$$

and

$$(2.11) \quad \sum_{n > x} \frac{\mu^2(n) N_f^t(n^k)}{n^{k(t+1)}} = O\left(\frac{(\log 2x)^{c^t-1}}{x^{k+t-1}}\right).$$

PROOF. By Lemmas 2.1 and 2.2, we have

$$\sum_{n \leq x} \mu^2(n) N_f^t(n^k) \leq \sum_{n \leq x} \mu^2(n) n^{(k-1)t} C^{t\omega(n)} \leq x^{(k-1)t} \sum_{n \leq x} C^{t\omega(n)} = O(x^{1+(k-1)t} (\log 2x)^{c^t-1})$$

and this proves (2.9). Now (2.10) and (2.11) follow from (2.9) by the partial summation theorem (cf. [4], Theorem 421).

Lemma 2.4. As $x \rightarrow \infty$

$$(2.12) \quad \sum_{n \leq x} \varrho_{f,t}^{u,v}(n) = O(x^{1+(v-1)t} (\log 2x)^{uc^t-1}),$$

$$(2.13) \quad \sum_{n \leq x} \frac{\varrho_{f,t}^{u,v}(n)}{n^{vt}} = \begin{cases} O((\log 2x)^{uc}), & \text{if } t = 1 \\ O(1), & \text{if } t \geq 2 \end{cases}$$

and

$$(2.14) \quad \sum_{n > x} \frac{\varrho_{f,t}^{u,v}(n)}{n^{v(t+1)}} = O\left(\frac{(\log 2x)^{uc^t-1}}{x^t}\right).$$

PROOF. We use induction on u for proving (2.12). For $u=1$, (2.12) is clear in virtue of (2.6) and (2.9). Now, assume (2.12) for $u-1$ where $u \geq 2$ is an integer. Then, by (2.7) and the induction hypothesis,

$$\begin{aligned}
 \sum_{n \leq x} \varrho_{f,t}^{u,v}(n) &= \sum_{d \leq x} \varrho_{f,t}^{1,v}(d) \varrho_{f,t}^{u-1,v}(\delta) = \sum_{d \leq x} \varrho_{f,t}^{1,v}(d) \sum_{\delta \leq x/d} \varrho_{f,t}^{u-1,v}(\delta) = \\
 (2.15) \quad &= O \left(\sum_{d \leq x} \varrho_{f,t}^{1,v}(d) \left(\frac{x}{d} \right)^{1+(v-1)t} \left(\log \frac{x}{d} \right)^{(u-1)c^t-1} \right) = \\
 &= O \left(x^{1+(v-1)t} (\log x)^{(u-1)c^t-1} \sum_{d \leq x} \frac{\varrho_{f,t}^{1,v}(d)}{d^{1+(v-1)t}} \right).
 \end{aligned}$$

Now, by (2.6) and partial summation, we get

$$\sum_{n \leq x} \frac{\varrho_{f,t}^{1,v}(n)}{n^{1+(v-1)t}} = O((\log x)^{c^t})$$

and hence (2.12) follows because of (2.15).

(2.13) and (2.14) follow from (2.12) by partial summation.

3. Main results.

Theorem 1.

$$(3.1) \quad \sum_{n \leq x} \varphi_{f,t}^{(k)}(n) = \alpha x^{t+1} + \begin{cases} O(x(\log x)^c), & \text{for } t = 1 \\ O(x^t), & \text{for } t \geq 2 \end{cases}$$

where

$$(3.2) \quad \alpha = \frac{1}{t+1} \sum_{n=1}^{\infty} \frac{\mu(n) N_f^t(n^k)}{n^{k(t+1)}} = \frac{1}{t+1} \prod_p \left\{ 1 - \frac{N_f^t(p^k)}{p^{k(t+1)}} \right\}$$

and the product on the right is extended over all primes p .

PROOF. The series in the definition of α converges absolutely by (2.11) and the general term of the series is a multiplicative function of n . Hence by the Euler's infinite product factorization theorem (cf. [4], Theorem 285) follows the equality of the sum and product in (3.2).

Now, by (2.2), we obtain

$$\begin{aligned}
 \sum_{n \leq x} \varphi_{f,t}^{(k)}(n) &= \sum_{d^k \delta \leq x} \delta^t \mu(d) N_f^t(d^k) = \sum_{d \leq x^{1/k}} \mu(d) N_f^t(d^k) \sum_{\delta \leq x/d^k} \delta^t = \\
 &= \sum_{d \leq x^{1/k}} \mu(d) N_f^t(d^k) \left\{ \left(\frac{x}{d^k} \right)^{t+1} \frac{1}{t+1} + O \left(\frac{x}{d^k} \right)^t \right\} = \\
 &= \frac{x^{t+1}}{t+1} \sum_{d=1}^{\infty} \frac{\mu(d) N_f^t(d^k)}{d^{k(t+1)}} + O \left(x^{t+1} \sum_{d > x^{1/k}} \frac{\mu^2(d) N_f^t(d^k)}{d^{k(t+1)}} \right) + O \left(x^t \sum_{d \leq x^{1/k}} \frac{\mu^2(d) N_f^t(d^k)}{d^{kt}} \right)
 \end{aligned}$$

and the result follows now by an application of (2.10) and (2.11).

Theorem 2.

$$(3.3) \quad \sum_{n \leq x} \Psi_{f,t}^{(k)}(n) = \beta x^{t+1} + \begin{cases} O(x(\log x)^c), & \text{for } t = 1 \\ O(x^t), & \text{for } t \geq 2 \end{cases}$$

where

$$(3.4) \quad \beta := \frac{1}{t+1} \sum_{n=1}^{\infty} \frac{\mu^2(n) N_f^t(n^k)}{n^{k(t+1)}} = \frac{1}{t+1} \prod_p \left\{ 1 + \frac{N_f^t(p^k)}{p^{k(t+1)}} \right\}.$$

PROOF. The proof is the same as that of Theorem 1 except that we use (2.3) in place of (2.2).

Theorem 3.

$$(3.5) \quad \sum_{n \leq x} \psi_{f,t}^{(k)}(n) = \gamma x^{kt+1} + \begin{cases} O(x^k(\log x)^c), & \text{for } t = 1 \\ O(x^{tk}), & \text{for } t \geq 2 \end{cases}$$

where

$$(3.6) \quad \gamma := \frac{1}{kt+1} \sum_{n=1}^{\infty} \frac{\mu^2(n) N_f^t(n^k)}{n^{k(t+1)}} = \frac{1}{kt+1} \prod_p \left\{ 1 + \frac{N_f^t(p^k)}{p^{k(t+1)}} \right\}.$$

PROOF. We use (2.4) in place of (2.2) and the proof runs similar to that of Theorem 1. We omit the details.

Theorem 4.

$$(3.7) \quad \sum_{n \leq x} \psi_{f,t}^{u,v}(n) = \delta x^{vt+1} + \begin{cases} O(x^v(\log x)^{uc}), & \text{for } t = 1 \\ O(x^{vt}), & \text{for } t > 1 \end{cases}$$

where

$$\delta = \frac{1}{vt+1} \sum_{n=1}^{\infty} \frac{q_{f,t}^{u,v}(n)}{n^{vt+1}}.$$

PROOF. We use (2.5), (2.13) and (2.14) in place of (2.2), (2.10) and (2.11) and proceed as in the proof of Theorem 1. We omit the details.

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