

On the Lebesgue function on infinite interval, II.

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1. In this paper we shall continue our previous investigations [1] and obtain a bound for the Lebesgue function on the interval $(-\infty, +\infty)$. Moreover, we shall establish a convergence theorem for the Lagrange interpolation on this interval.

Let a weight function $p(x)$ be given such that

$$(1.1) \quad \int_{-\infty}^{\infty} x^k p(x) dx < \infty \quad (k = 0, 1, 2, \dots),$$

then there exists one and only one system of orthonormal polynomials $\{\omega_n(x)\}_{n=0}^{\infty}$ such that $\omega_n(x)$ has exactly n simple real roots and

$$(1.2) \quad -\infty < x_n < x_{n-1} < \dots < x_{v+1} < x_v < \dots < x_2 < x_1 < +\infty.$$

For the orthonormal Hermite polynomials defining the n -th polynomial as

$$(1.3) \quad \omega_n^*(x) = H_n(x) = \frac{(-1)^n}{\sqrt{2^n n!} \sqrt{\pi}} e^{x^2} \{e^{-x^2}\}^{(n)}$$

the weight function is $p(x) = e^{-x^2}$.

We define the Lebesgue function using the roots (1.2)

$$(1.4) \quad \lambda_n(x) \stackrel{\text{def}}{=} \sum_{v=1}^n |l_v(x)|$$

where

$$(1.5) \quad l_v(x) = \frac{\omega_n(x)}{\omega_n'(x_v)(x - x_v)}.$$

Grünwald and Turán [2] proved that given a system of orthonormal polynomials $\{\bar{\omega}_n(x)\}_{n=0}^{\infty}$ with weight functions $p(x) \geq m > 0$ on the interval $[-1, 1]$, the Lebesgue function using the roots of $\bar{\omega}_n(x)$ can be estimated thus:

$$\lambda_n(x) \leq c_1 \sqrt{n}, \quad x \in (-1, 1)$$

and

$$\lambda_n(x) \leq c_2 n \quad x \in [-1, 1]$$

where c_1 and c_2 are constants independent of n .

In the first of this series [1] we extend our investigation to the interval $[0, \infty)$ and the bound

$$\lambda_n(x) = \begin{cases} c_3 x^{(a/2)-(1/4)} n^{1/4}, & \alpha \cong -\frac{1}{2}, \text{ for fixed } x \in (0, \infty), \\ c_4 n^{1/4}, & -1 < \alpha \cong -\frac{1}{2}, \text{ for fixed } x \in [0, \infty) \end{cases}$$

if the weight function satisfy the condition $p(x)x^{-\alpha}e^x \cong m > 0$. The c_3 and c_4 constants are independent of x and n .

We now state the following

Theorem 1. Let $p(x)e^{x^2} \cong m > 0$ and

$$\int_{-\infty}^{\infty} x^k p(x) dx < \infty \quad (k = 0, 1, 2, \dots),$$

then the Lebesgue function, using the roots of $\omega_n(x)$

$$\lambda_n(x) = O(1)n^{1/4}, \quad x \in [-A, A]$$

where A is an arbitrary fixed real number.

Remark. If $p(x) = e^{-x^2}$, then $m = 1$.

2. The following results, the proofs of which may be found similar to those in [1], are true.

(a) The Cotes numbers

$$(2.1) \quad \int_{-\infty}^{\infty} l_v(x) p(x) dx = \int_{-\infty}^{\infty} l_v(x)^2 p(x) dx = \mu_v$$

$$(v = 1, 2, \dots, n; n = 1, 2, 3, \dots).$$

(b) From the orthogonality of $\{\omega_n(x)\}_{n=0}^{\infty}$

$$(2.2) \quad \int_{-\infty}^{\infty} l_i(x) l_j(x) p(x) dx = \begin{cases} 0, & i \neq j \\ \mu_i, & i = j \end{cases}$$

$$(i, j = 1, 2, \dots, n; n = 1, 2, 3, \dots).$$

Using (2.1), (2.2) and the fact that

$$\sum_{v=1}^n l_v(x) \equiv 1 \quad (n = 1, 2, 3, \dots),$$

we have

$$(2.3) \quad \int_{-\infty}^{\infty} \left\{ \sum_{v=1}^n l_v(x) \right\}^2 p(x) dx = \int_{-\infty}^{\infty} \left\{ \sum_{v=1}^n l_v^2(x) \right\} p(x) dx =$$

$$= \int_{-\infty}^{\infty} \left\{ \sum_{v=1}^n l_v(x) \right\} p(x) dx = \int_{-\infty}^{\infty} p(x) dx < \infty.$$

Let us define $\varepsilon_v \stackrel{\text{def}}{=} \text{sign} \{l_v(x_0)\}$, where $x_0 \geq 0$ is an arbitrary fixed value and let

$$(2.4) \quad \Psi_n(x) = \sum_{v=1}^n \varepsilon_v l_v(x).$$

The polynomial $\Psi_n(x)$ is of degree $n-1$ and can be expanded by the Fourier series of orthonormal Hermite polynomials $\{\omega_n^*(x)\}$. Hence from the inequality Cauchy's we get

$$(2.5) \quad [\Psi_n(x)]^2 = \left[\sum_{i=0}^{n-1} c_i \omega_i^*(x) \right]^2 \leq \sum_{i=0}^{n-1} c_i^2 \sum_{i=0}^{n-1} [\omega_i^*(x)]^2.$$

Thus, and from (2.4), (2.3) we have

$$(2.6) \quad \begin{aligned} \int_{-\infty}^{\infty} [\Psi_n(x)]^2 e^{-x^2} dx &= \sum_{i=0}^{n-1} c_i^2 = \int_{-\infty}^{\infty} \left\{ \sum_{v=1}^n \varepsilon_v l_v(x) \right\}^2 e^{-x^2} dx \leq \\ &\leq \frac{1}{m} \int_{-\infty}^{\infty} \left\{ \sum_{v=1}^v \varepsilon_v l_v(x) \right\}^2 p(x) dx = \frac{1}{m} \int_{-\infty}^{\infty} p(x) dx < \infty. \end{aligned}$$

This value which is also independent of n .

From (2.5) and (2.6) we get

$$(2.7) \quad [\Psi_n(x)]^2 = O(1) \sum_{i=0}^{n-1} [\omega_i^*(x)]^2.$$

Considering [3] (formulas (5.5.1), (5.6.1), (7.6.9) and (7.6.10)) it is easy to verify that

$$(2.8) \quad |\omega_i^*(x)| = O(1)(i+1)^{-1/4}, \quad -A \leq x \leq A$$

where A is an arbitrary real number.

From (2.7) and (2.8) the following inequality

$$(2.9) \quad [\Psi_n(x)]^2 \leq O(1)n^{1/2}, \quad -A \leq x \leq A$$

holds.

Hence, from (2.4) and (2.9) we have

$$|\Psi_n(x_0)| = \lambda_n(x_0) = O(1)n^{1/4}, \quad -A \leq x_0 \leq A.$$

Thus, the theorem is proved, because x_0 is an arbitrary value.

3. We shall in this section consider the Lagrange interpolation polynomial of a continuous function $f(x)$ on the interval $(-\infty, \infty)$.

It is interesting at this point to consider a function

$$(3.1) \quad f(x) = e^{ax^2} \varphi(x) \quad (a > 0, -\infty < x < \infty)$$

where $\varphi(x) \in \text{Lip}_M \gamma$, $\frac{1}{2} < \gamma \leq 1$.

Let us define polynomials $L_n(x, f)$ of degree $n-1$, satisfying the following equalities on the roots of Hermite polynomials

$$(3.2) \quad L_n(x_v; f) = e^{-ax_v^2} f(x_v) = \varphi(x_v) = y_v \\ (v = 1, 2, 3, \dots, n, n = 1, 2, \dots).$$

The polynomials $L_n(x; f)$ have the explicit forms

$$(3.3) \quad L_n(x; f) = \sum_{v=1}^n y_v l_v(x)$$

where

$$(3.4) \quad l_v(x) = \frac{\omega_n^*(x)}{\omega_n^{*'}(x_v)(x-x_v)}.$$

From [3] (formula (5.6.1)) we have, if n is even

$$(3.5) \quad l_v(x) = \frac{\omega_n^*(x)}{\omega_n^{*'}(x_v)(x-x_v)} = \frac{L_{n/2}^{(-1/2)}(x^2)}{2x_v L_{n/2}^{(-1/2)'}(x_v^2)(x-x_v)} \quad (v = 1, 2, \dots, n)$$

and if n is odd

$$(3.6) \quad l_v(x) = \frac{\omega_n^*(x)}{\omega_n^{*'}(x_v)(x-x_v)} = \frac{x L_{(n-1)/2}^{(1/2)}(x^2)}{2x_v^2 L_{(n-1)/2}^{(1/2)'}(x_v^2)(x-x_v)} \quad (v = 1, 2, \dots, n)$$

where $L_k^{(\alpha)}(x)$ is the polynomial Laguerre's.

We shall now prove the following statement.

Theorem 2. *If $f(x)$ satisfies the condition (3.1), then the inequalities*

$$|f(x) - e^{ax^2} L_n(x; f)| \leq O(1) n^{(1/4) - (\gamma/2)}, \quad x \in [-A, A]$$

hold where A is an arbitrary fixed real number.

PROOF. Let x_v denote the roots of polynomials $\omega_n^*(x)$, then the following inequalities are true (SZEGÖ [3])

$$(3.7) \quad c_1 \frac{v}{\sqrt{n}} < x_v < c_2 \frac{v}{\sqrt{n}} \quad \left(0 \leq v \leq \left[\frac{n}{2}\right]\right)$$

and $x_v = -x_{n-v+1}$, if $\left[\frac{n}{2}\right] + 1 \leq v \leq n$, where c_1 and c_2 constants are independent of v and n .

Let $\varphi(x) \in \text{Lip}_M \gamma$, $\frac{1}{2} < \gamma \leq 1$, and $x \in [-x_{[n/2]}, x_{[n/2]}]$, then by using [1] it is easy to verify that there exists a polynomial $Q_n(x; \varphi)$ of degree $n-1$, for which the relations

$$|Q_n(x; \varphi) - \varphi(x)| = O(1) n^{-\gamma/2}$$

hold.

It is well known that

$$(3.9) \quad Q_n(x; \varphi) \equiv \sum_{v=1}^n Q_n(x_v; \varphi) l_v(x)$$

is true.

From (3.9), (3.8) and (3.5) we have, that for $x \in [-A, A]$, n even

$$(3.10) \quad \begin{aligned} |\varphi(x) - L_n(x; \varphi)| &\leq |\varphi(x) - Q_n(x; \varphi)| + \sum_{v=1}^n |Q_n(x_v; \varphi) - \varphi(x_v)| |l_v(x)| \leq \\ &\leq O(1)n^{-\gamma/2} + O(1)n^{-\gamma/2} \sum_{0 \leq x_v \leq 2A} |l_v(x)| + O(1)n^{-\gamma/2} \sum_{x_v > 2A} |L_{n/2}^{(-1/2)}(x^2)| |L_{n/2}^{(-1/2)'}(x_v^2)|^{-1}. \end{aligned}$$

The following inequalities are taken from SZEGŐ [3] for $x \in [-A, A]$

$$(3.11) \quad \sum_{v=1}^n |L_n^{(\alpha)'}(x_v^2)|^{-1} \leq \sqrt{n} \left\{ \sum_{v=1}^n [L_n^{(\alpha)'}(x_v^2)]^{-2} \right\}^{1/2} = O(1)n^{(1/2) - (\alpha/2)}$$

and

$$(3.12) \quad |L_{n/2}^{(-1/2)}(x^2)| = O(1)n^{-1/2}, \quad |xL_{(n-1)/2}^{(1/2)}(x^2)| = O(1).$$

Using (3.1), Theorem 1, (3.10), (3.12) and (3.11) we get for $x \in [-A, A]$, if n is even

$$(3.13) \quad \begin{aligned} |f(x) - e^{ax^2} L_n(x; f)| &= e^{ax^2} |\varphi(x) - L_n(x; \varphi)| = O(1)n^{(1/4) - (\alpha/2)} \\ &(n = 1, 2, 3, \dots). \end{aligned}$$

These relations are also true, if n is odd. The proof is similar to that of (3.13).

Thus Theorem 2 is proved.

References

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