

On a class of arithmetical functions

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1. Introduction

Let $\mathcal{H}$ denote the set of all arithmetical functions h such that:

$$(1.1) \quad h: N \times C \rightarrow C$$

$$(1.2) \quad \sum_{x \leq n \leq y} |h(n, z)| \leq R(y-x) + O(y^\alpha (\log x)^{\operatorname{Re} z - q - 2})$$

for $|z| \leq 1$, $0 < \alpha < 1$, where $q \geq 0$ is an arbitrary integer number and $R \geq 1$ is a fixed real number.

$$(1.3) \quad \sum_{n=1}^{\infty} \frac{h(n, z)}{n^s} = g(s, z) \zeta^z(s), \quad \text{for } \operatorname{Re} s > 1, \quad |z| \leq 1$$

(1.4) $g(s, z)$ is a holomorphic function in the region

$$\left\{ \sigma > 1 - \frac{c_1}{\log(|t|+2)}, \quad -\infty < t < +\infty, \quad |z| \leq 1 \right\}; \quad c_1 < \frac{\log 2}{2}$$

$$(1.5) \quad |g(s, z)| \leq K, \quad \text{for } \sigma \geq \sigma_1 > \frac{1}{2}, \quad \sigma_1 < 1, \quad |z| \leq 1.$$

In this paper we shall prove the following theorems:

Theorem 1. Let $h \in \mathcal{H}$, then there exists a sequence of functions $A_j(z)$ defined and continuous in the circle $|z| \leq 1$ and holomorphic in the circle $|z| < 1$ such that for every integer number $q \geq 0$ we have

$$(1.6) \quad \sum_{n \leq x} h(n, z) = \sum_{j=0}^q x A_j(z) (\log x)^{z-j-1} + O(x (\log x)^{\operatorname{Re} z - q - 2})$$

uniformly with respect to $|z| \leq 1$ as $x \rightarrow \infty$, where

$$(1.7) \quad A_j(z) = \frac{B_j(z)}{\Gamma(z-j)}, \quad B_j(z) = \frac{1}{2\pi i} \int_{|s-1|=\delta < 1/2} \frac{H(s, z)}{s(s-1)^j} ds \quad \text{for } j = 0, 1, 2, \dots, q,$$

$$(1.8) \quad H(s, z) = g(s, z) \exp(z \log(s-1) \zeta(s)).$$

Theorem 2. *Let*

$$(i) \quad |h(n, z)| \leq R, \text{ where } R \geq 1 \text{ and } |z| \leq 1,$$

$$(ii) \quad \sum_p \frac{|h(p, z) - z|}{p^\sigma} < \infty, \text{ for } \sigma \geq \sigma_1 > \frac{1}{2}, \quad |z| \leq 1, \quad \sigma_1 < 1,$$

$$(iii) \quad h(k, l, z) = h(k, z) h(l, z) \text{ for all pairs of natural numbers } k, l \text{ such that } (k, l) = 1.$$

If the function h satisfies the condition (i)–(iii) and h is holomorphic with respect to z in the circle $|z| < 1$, then $h \in \mathcal{H}$.

From this theorem we get the following corollary:

Corollary. *Let $h(n, z) = f(n)z^{F(n)}$, where $|z| \leq 1$ and*

$$f: N \rightarrow C, \quad |f(n)| \leq R; \quad R \geq 1$$

$$F: N \rightarrow N \cup \{0\}$$

$f(kl) = f(k)f(l)$, $F(kl) = F(k) + F(l)$ for all pairs of natural numbers k, l such that $(k, l) = 1$ and if $f(p) = F(p) = 1$ for every prime number p , then $h = f(n)z^{F(n)} \in \mathcal{H}$.

We remark that from this Corollary and the Theorem 1 we can get some results which were proved by A. SELBERG in the paper [2] ($h = z^{F(n)}$, $F = \Omega$ or ω) and by H. DELANGE in [1], ($h = f(n)z^{F(n)}$; $f: N \rightarrow \{0, 1\}$).

2. Proof of Theorem 1

In the proof of the Theorem 1 we use the following Lemmas:

Lemma 1. *In the region*

$$\sigma > 1 - \frac{c_1}{\log(|t| + 2)}, \quad |t| \geq 2$$

we have

$$\log \zeta(s) \ll \log \log |t|.$$

Lemma 2. *In the region*

$$\sigma > 1 - \frac{c_1}{\log 4} > \frac{1}{2}, \quad |t| < 2,$$

we have

$$\log(s-1)\zeta(s) = O(1).$$

Applying K. Wiertelak's results ([3]) for the case $K=Q$ and utilizing the standard method for estimating $\log \zeta(s)$ and $\log(s-1)\zeta(s)$ we can get the proofs of Lemma 1 and Lemma 2.

Let $P_z(x) = \sum_{n \leq x} h(n, z)$, then by Perron's classical formula and (3) we get

$$\int_0^x P_z(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^{s+1}}{s(s+1)} g(s, z) \zeta^z(s) ds,$$

where $c > 1$, $x > 0$. Putting $\zeta^z(s)g(s, z) = H(s, z)(s-1)^{-z}$ and

$$\Phi(s, z, x) = \frac{x^{s+1}}{s(s+1)} H(s, z)(s-1)^{-z}$$

we obtain

$$(2.1) \quad \int_0^x P_z(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \Phi(s, z, x) ds.$$

Let $\sigma_1 = 1 - \frac{c_1}{\log 4} = 1 - \eta > \frac{1}{2}$, then $0 < \eta < \frac{1}{2}$ and let $r < \eta$ and $0 < \varepsilon < \arctg \frac{2}{\eta}$. Then by Cauchy's theorem the integral on the right-hand side of (2.1) may be replaced by the integrals $I_1, \dots, I_9$ over the paths $\Gamma_1, \dots, \Gamma_9$ which are defined as follows:

Γ_1 , is the segment $\left\langle c-iT, 1 - \frac{c_1}{\log(|T|+2)} - iT \right\rangle$

Γ_2 , is the curve described by $1 - \frac{c_1}{\log(|t|+2)} + it$ as t increases from $-T$ to -2 .

Γ_3 , is the segment $\langle 1 - \eta - 2i, 1 - \eta - i\eta \operatorname{tg} \varepsilon \rangle$,

Γ_4 , is the segment $\langle 1 - \eta - i\eta \operatorname{tg} \varepsilon, 1 - re^{i\varepsilon} \rangle$

Γ_5 , is the arc of the circle $1 + re^{i\theta}$ described as θ increases from $-\pi + \varepsilon$, to $\pi - \varepsilon$.

Γ_6 , is the segment $\langle 1 - re^{-i\varepsilon}, 1 - \eta + i\eta \operatorname{tg} \varepsilon \rangle$

Γ_7 , is the segment $\langle 1 - \eta + i\eta \operatorname{tg} \varepsilon, 1 - \eta + 2i \rangle$

Γ_8 , is the curve described by $1 - \frac{c_1}{\log(|t|+2)} + it$ as t increases from 2 to T .

Γ_9 , is the segment $\left\langle 1 - \frac{c_1}{\log(T+2)} + iT, c+iT \right\rangle$.

We note that $\Gamma_1, \Gamma_2, \Gamma_8$ and Γ_9 depend only on T , and do not depend on r or ε . For fixed T and r , and for $\varepsilon \rightarrow 0$, Γ_3 and Γ_7 become the segments $\Gamma'_3 = \langle 1 - \eta - 2i, 1 - \eta \rangle$, $\Gamma'_7 = \langle 1 - \eta, 1 - \eta + 2i \rangle$.

For I_4 and I_6 we have

$$\lim_{\varepsilon \rightarrow 0} I_4 = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_4} \Phi(s, z, x) ds = \int_{1-\eta}^{1-r} H(\sigma, z)(1-\sigma)^{-z} (e^{-i\pi})^{-z} \frac{x^{\sigma+1}}{\sigma(\sigma+1)} d\sigma,$$

$$\lim_{\varepsilon \rightarrow 0} I_6 = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_6} \Phi(s, z, x) ds = \int_{1-r}^{1-\eta} H(\sigma, z)(1-\sigma)^{-z} (e^{i\pi})^{-z} \frac{x^{\sigma+1}}{\sigma(\sigma+1)} d\sigma.$$

If $\gamma_r = \{s: |s-1|=r\}$ excluding the point $1-r$, then

$$\lim_{\varepsilon \rightarrow 0} I_4 + I_5 + I_6 = \int_{\gamma_r} \Phi(s, z, x) ds + 2i \sin \pi z \int_r^\eta \frac{H(1-u, z)}{(1-u)(2-u)} u^{-z} x^{2-u} du$$

which does not depend on the choice of r . If $T \rightarrow \infty$ then $I_1 \rightarrow 0$, and $I_9 \rightarrow 0$ so that

$$(2.2) \quad \int_0^x P_z(t) dt = S_z(x) + \omega(x, z)$$

where

$$(2.3) \quad S_z(x) = \frac{\sin \pi z}{\pi} \int_r^\eta \frac{H(1-u, z)}{(1-u)(2-u)} x^{2-u} du + \frac{1}{2\pi i} \int_{\gamma_r} \Phi(s, z, x) ds,$$

$$(2.4) \quad \omega(x, z) = J_2 + J_3 + J_7 + J_8$$

and

$$(2.5) \quad J_2 = \frac{1}{2\pi i} \int_{-\infty}^{-2} \Phi\left(1 - \frac{c_1}{\log(|t|+2)} + it, z, x\right) \left(i + \frac{c_1}{(|t|+2) \log^2(|t|+2)}\right) dt,$$

$$(2.6) \quad J_3 = \frac{1}{2\pi i} \int_{\Gamma_3} \Phi(s, z, x) ds; \quad J_7 = \frac{1}{2\pi i} \int_{\Gamma_7} \Phi(s, z, x) ds,$$

$$(2.7) \quad J_8 = \frac{1}{2\pi i} \int_2^{+\infty} \Phi\left(1 - \frac{c_1}{\log(t+2)} + it, z, x\right) \left(i + \frac{c_1}{(t+2) \log^2(t+2)}\right) dt.$$

Using Lemma 1 we have on Γ_2 and Γ_8

$$\Phi(s, z, x) \ll x^{2-(c_1/\log(|t|+2))} |t|^{-2} \log |t|$$

and we obtain the following estimate:

$$(2.8) \quad J_2 + J_8 \ll x^2 \exp(-2A \sqrt{\log x}) \int_2^{+\infty} t^{-2+\varepsilon} \log t dt$$

where $A = \sqrt{\varepsilon c_3}$, $c_3 > 0$.

Since the integral $\int_2^{+\infty} t^{-2+\varepsilon} \log t dt$ is bounded, from (2.8) we get

$$(2.9) \quad J_2 + J_8 \ll x^2 \exp(-2A \sqrt{\log x}).$$

On Γ_3' and Γ_7' we have

$$\Phi(s, z, x) \ll x^{2-\eta} = x^2 \exp(-\eta \log x)$$

and therefore we obtain

$$(2.10) \quad J_3 + J_7 \ll x^2 \exp(-2A \sqrt{\log x}).$$

By (2.4), (2.9) and (2.10) it follows that

$$(2.11) \quad \omega(x, z) \ll x^2 \exp(-2A \sqrt{\log x}).$$

It can be noted that for $|z| \leq 1$ the function $S_z(x)$ is infinitely differentiable. From (2.3) we get

$$(2.12) \quad S_z''(x) = \frac{\sin \pi z}{\pi} \int_r^\eta H(1-u) u^{-z} x^{-u} du + \frac{1}{2\pi i} \int_{\gamma_r} H(s, z) (s-1)^{-z} x^{s-1} ds.$$

By Lemma 2 and (1.5) it follows that the function $H(s, z) = g(s, z) \exp(z \log(s-1)\zeta(s))$ is bounded in the region $|s-1| \leq \eta$, $|z| \leq 1$. Hence, we have

$$(2.13) \quad \int_r^\eta H(1-u, z) u^{-z} x^{-u} du \ll \int_{r \log x}^{\eta \log x} v^{-1} e^{-v} dv,$$

where

$$u = \frac{v}{\log x}.$$

For $|s-1| = r < \frac{1}{2}$ we obtain

$$(2.14) \quad \int_{\gamma_r} H(s, z) (s-1)^{-z} x^{s-1} ds \ll x^r.$$

Putting $r = \frac{1}{\log x}$ for x sufficiently large from (2.13) and (2.14) we get

$$(2.15) \quad S_z''(x) = O(1).$$

Let $0 < \xi < \frac{x}{2}$, $t \in \langle x, x+\xi \rangle$, then

$$(2.16) \quad \left| \frac{1}{\xi} \int_x^{x+\xi} P_z(t) dt - P_z(x) \right| \leq \frac{1}{\xi} \int_x^{x+\xi} |P_z(t) - P_z(x)| dt.$$

From (2.16), (1.2) we get

$$(2.17) \quad |P_z(t) - P_z(x)| \leq R\xi + O(x^\alpha (\log x)^{\operatorname{Re} z - q - 2})$$

where $0 \leq t - x \leq \xi$ and $0 < \xi < x/2$. From (2.16) and (2.17) we get

$$(2.18) \quad \left| \frac{1}{\xi} \int_x^{x+\xi} P_z(t) dt - P_z(x) \right| \leq R\xi + O(x^\alpha (\log x)^{\operatorname{Re} z - q - 2}).$$

On the other hand we have

$$(2.19) \quad \frac{1}{\xi} \int_x^{x+\xi} P_z(t) dt = S_z'(x) + \xi \int_0^1 (1-u) S_z''(x+u\xi) du + \frac{\omega(x+\xi, z) - \omega(x, z)}{\xi}.$$

By (2.11) and $0 < \xi < \frac{x}{2}$ it follows that

$$(2.20) \quad \omega(x+\xi, z) \ll x^2 \exp(-2A \sqrt{\log x})$$

and similarly from (2.15) we get

$$(2.21) \quad S_z''(x+u\xi) = O(1).$$

From (2.18), (2.20), (2.21) we obtain

$$(2.22) \quad \left| \frac{1}{\xi} \int_x^{x+\xi} P_z(t) dt - S'_z(x) \right| \ll \xi + x(\log x)^{\operatorname{Re} z - q - 2} + \frac{1}{\xi} x^2 \exp(-2A \sqrt{\log x}).$$

Putting $\xi = x \exp(-A \sqrt{\log x})$, from (2.22) we have

$$(2.23) \quad \left| \frac{1}{\xi} \int_x^{x+\xi} P_z(t) dt - S'_z(x) \right| \ll x(\log x)^{\operatorname{Re} z - q - 2}.$$

Since

$$|P_z(x) - S'_z(x)| \leq \left| \frac{1}{\xi} \int_x^{x+\xi} P_z(t) dt - S'_z(x) \right| + \left| P_z(x) - \frac{1}{\xi} \int_x^{x+\xi} P_z(t) dt \right|$$

by (2.23) and (2.18) it follows that

$$(2.24) \quad |P_z(x) - S'_z(x)| \ll x(\log x)^{\operatorname{Re} z - q - 2}.$$

To finish the proof there remains to evaluate $S'_z(x)$. For $|s-1|=r < \frac{1}{2}$, $|z| \leq 1$, we have

$$(2.25) \quad \frac{H(s, z)}{s} = \sum_{j=0}^q B_j(z)(s-1)^j + R_q(s, z)(s-1)^{q+1}$$

for every fixed non-negative integer q , where the $B_j(z)$'s are analytic functions. Cauchy's classical inequality for coefficients of a power series implies that $R_q(s, z) = O(1)$. By (2.25) and (2.3) it follows that

(2.25')

$$S'_z(x) = \sum_{j=0}^q x B_j(z) \left[\frac{\sin \pi(z-j)}{\pi} \int_r^{+\infty} u^{j-z} x^{-u} du + \frac{1}{2\pi i} \int_{\gamma_r} (s-1)^{j-z} x^{s-1} ds \right] + W(x, z)$$

where

$$(2.26) \quad \begin{aligned} W(x, z) = & - \sum_{j=0}^q x B_j(z) \frac{\sin \pi(z-j)}{\pi} \int_{\eta}^{+\infty} u^{j-z} x^{-u} du + \\ & + x \frac{\sin \pi(z-q-1)}{\pi} \int_r^{\eta} R_q(1-u, z) u^{q+1-z} x^{-u} du + \\ & + \frac{x}{2\pi i} \int_{\gamma_r} R_q(s, z) (s-1)^{q+1-z} x^{s-1} ds. \end{aligned}$$

Since, for $0 \leq j \leq q$ we have

$$\frac{\sin \pi(z-j)}{\pi} \int_r^{+\infty} u^{j-z} x^{-u} du + \frac{1}{2\pi i} \int_{\gamma_r} x^{s-1} (s-1)^{j-z} ds = \frac{(\log x)^{z-j-1}}{\Gamma(z-j)},$$

by (2.25') it follows that

$$(2.27) \quad S'_z(x) = \sum_{j=0}^q x B_j(z) \frac{(\log x)^{z-j-1}}{\Gamma(z-j)} + W(x, z).$$

We remark that

$$(2.28) \quad \int_{\eta}^{+\infty} u^{j-z} x^{-u} du \ll (\log x)^{-q-3}, \quad \text{for } 0 \leq j \leq q, \quad |z| \leq 1,$$

$$(2.29) \quad \int_r^{\eta} R_q(1-u, z) u^{q+1-z} x^{-u} du \ll (\log x)^{\operatorname{Re} z - q - 2},$$

$$(2.30) \quad \int_{\gamma_r} R_q(s, z) (s-1)^{q+1-z} x^{s-1} ds \ll (\log x)^{\operatorname{Re} z - q - 2}$$

and therefore by (2.28)–(2.30) and (2.26) it follows that

$$(2.31) \quad W(x, z) \ll x (\log x)^{\operatorname{Re} z - q - 2}.$$

From (2.31), (2.27) and (2.24) we obtain

$$(2.32) \quad P_z(x) = \sum_{j=0}^q x \frac{B_j(z)}{\Gamma(z-j)} (\log x)^{z-j-1} + O(x (\log x)^{\operatorname{Re} z - q - 2})$$

where

$$B_j(z) = \frac{1}{2\pi i} \int_{|s-1|=\delta < 1/2} \frac{H(s, z)}{s(s-1)^{j+1}} ds, \quad \text{for } j = 0, 1, \dots, q$$

and theorem 1 is proved.

3. Proof of Theorem 2

From (i)–(iii) we have

$$\left| \sum_{n=1}^{\infty} \frac{h(n, z)}{n^s} \right| \leq R \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}, \quad \text{for } \sigma > 1 \quad \text{and} \quad |z| \leq 1$$

and

$$(3.0) \quad \sum_{n=1}^{\infty} \frac{h(n, z)}{n^s} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k, z)}{p^{ks}} \right), \quad \operatorname{Re} s = \sigma > 1, \quad |z| \leq 1.$$

We remark that the product

$$(3.1) \quad \prod_p \left(1 - \frac{1}{p^s} \right)^z \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k, z)}{p^{ks}} \right)$$

has the following form

$$(3.2) \quad \prod_p (1 + u_p(s, z)) \exp(-v_p(s, z))$$

where

$$(3.3) \quad u_p(s, z) = \sum_{k=1}^{\infty} \frac{h(p^k, z)}{p^{ks}},$$

$$v_p(s, z) = z \sum_{k=1}^{\infty} \frac{1}{k p^{ks}}.$$

Since $|u_p(s, z)| \leq R \frac{1}{p^\sigma - 1} = U_p$, we get

$$(3.4) \quad \sum_p U_p^2 < \infty \quad \text{for } \sigma \geq \sigma_1 > \frac{1}{2}.$$

In a similar way we obtain

$$(3.5) \quad |u_p(s, z) - v_p(s, z)| \leq \frac{|h(p, z) - z|}{p^\sigma} + \bar{V}_p = V_p,$$

where

$$\bar{V}_p = R_1 \frac{1}{p^{2\sigma}}.$$

From (3.5) and (ii) we get

$$(3.6) \quad \sum_p V_p < \infty, \quad \text{for } \sigma \geq \sigma_1 > \frac{1}{2}.$$

By Delange's Lemma (see [1]) it follows that the product (3.2) is absolutely and uniformly convergent in the region $\sigma \geq \sigma_1 = 1 - \eta > \frac{1}{2}$, $|z| \leq 1$ and is bounded in this region. Let

$$g(s, z) = \prod_p (1 + u_p(s, z)) \exp(-v_p(s, z)),$$

then

$$(3.7) \quad g(s, z) = \prod_p \left(1 - \frac{1}{p^s}\right)^z \left(1 + \sum_{k=1}^{\infty} \frac{h(p^k, z)}{p^{ks}}\right)$$

and by (3.0) and (3.7) it follows that

$$\sum_{n=1}^{\infty} \frac{h(n, z)}{n^s} = g(s, z) \zeta^z(s)$$

so that (1.3) is fulfilled. We have $|g(s, z)| \leq K$ for $\operatorname{Re} s \geq \sigma_1 = 1 - \eta > \frac{1}{2}$, $|z| \leq 1$, thus (1.5) is satisfied. Since $g(s, z)$ is uniformly convergent, $g(s, z)$ is analytic in the region

$$\sigma > 1 - \frac{c_1}{\log(|t| + 2)} > \frac{1}{2}, \quad -\infty < t < +\infty, \quad |z| \leq 1;$$

$c_1 < \frac{\log 2}{2}$ and the theorem is proved.

Corollary 1 is a simple consequence of the Theorem 2.

References

- [1] H. DELANGE, Sur des formules de Atle Selberg, *Acta Arithm.* **19** (1971), 105—146.
- [2] A. SELBERG, Note on a paper by L. G. Sathe. *J. Indian Math. Soc.* **18** (1954), 83—87.
- [3] K. WIETELAK, On the density of some sets of primes, II. *Acta Arithm.* **34** (1978), 197—210.

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