

On functional equations and measures of information I

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§ 1. Introduction

In analysing the additivity property of Shannon's entropy one comes across the functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m f(q_j).$$

Recently Z. DARÓCZY and A. JÁRAI [1] solved this equation with $n=m=2$ for (Lebesgue) measurable functions f .

The purpose of the present paper is to obtain the measurable solutions $f_i:]0, 1[\rightarrow \mathbb{R}$ of the functional equation

(1.1)

$$f_1(pq) + f_2(p(1-q)) + f_3((1-p)q) + f_4((1-p)(1-q)) = f_5(p) + f_6(q) \quad (p, q \in]0, 1[)$$

which is a generalization of the equation considered in [1].

§ 2. Some preliminary results

Let X be a set and $(Y, \mathcal{B}, m)$ be a finite measure space. Let $\mathcal{D}$ be the family of all subsets $D \subseteq X \times Y$ satisfying the property

$$\inf_{x \in X} m(D_x) > 0, \quad \text{where } D_x := \{y \in Y \mid (x, y) \in D\} \in \mathcal{B}.$$

Let $\mathcal{F}(D) = \{g: D \rightarrow Y\}$ be the family of all functions g such that for all $\varepsilon > 0$, there exists $\delta > 0$ so that for each $E \in \mathcal{B}$ with $m(E) < \delta$ and each $x \in X$, the set $g_x^{-1}(E) := \{y \in Y \mid g(x, y) \in E\}$ is contained in some set B (depending upon x and $E \in \mathcal{B}$ with $m(B) < \varepsilon$). We present the following Lemma without a proof.

Lemma 1. *Let $D \in \mathcal{D}$ and $g_i \in \mathcal{F}(D)$ for $i=1, 2, \dots, n$. Then there exists $\delta > 0$ so that for each $E \in \mathcal{B}$ with $m(E) < \delta$,*

$$\bigcup_{i=1}^n (g_i)_x^{-1}(E) \subsetneq D_x \quad \text{for every } x \in X,$$

that is, for each $x \in X$ there exists $y_x \in D_x$ such that

$$g_i(x, y_x) \notin E \quad \text{for all } i = 1, 2, \dots, n.$$

Using Lemma 1 we can prove the following Theorem.

Theorem 2. Let $D \in \mathcal{D}$, $g_i \in \mathcal{F}(D)$, $h_i: Y \rightarrow R$ and $h: X \rightarrow R$ be functions satisfying the functional equation

$$h(x) = \sum_{i=1}^n h_i(g_i(x, y)) \quad (x, y) \in D.$$

If the functions h_i are measurable, then h is bounded on X .

The proofs of the above results are analogous to those used in [1].

We now proceed to show that if f_i 's are (Lebesgue) measurable functions satisfying the functional equation (1.1), then they are locally bounded. We shall make use of the following lemma ([2], Lemma 3, p. 210) in Natanson.

Lemma 3. Let $h: [\alpha, \beta] \rightarrow R$ be strictly monotone and suppose $E \subset [\alpha, \beta]$ is such that $|h'(x)| \geq b$ holds on E for some $b \geq 0$, then $m^*h(E) \geq bm^*E$, where m^* is the (Lebesgue) outer measure.

We can rewrite (1.1) as

$$(2.1) \quad \begin{aligned} f_1(x) = & -f_2(g_1(x, y)) - f_3(g_2(x, y)) - f_4(g_3(x, y)) + \\ & + f_5(g_4(x, y)) + f_6(g_5(x, y)), \quad (x, y) \in T \end{aligned}$$

where $g_1(x, y) = \frac{x}{y} - x$, $g_2(x, y) = y - x$, $g_3(x, y) = 1 + x - y - \frac{x}{y}$, $g_4(x, y) = \frac{x}{y}$, $g_5(x, y) = y$ and $T = \{(x, y) | 0 < x < y < 1\}$.

Let $X = [\alpha, \beta] \subset]0, 1[$ ($\alpha < \beta$) be an arbitrary closed interval. We shall show that f_1 is bounded on X .

Let $Y =]0, 1[$ and $\mathcal{B}$ be the class of all Lebesgue measurable subsets of Y with Lebesgue measure m . We can choose a sufficiently small $\eta > 0$ such that the set

$$D = \{(x, y) \in T | x \in X, (y + \eta)^2 \leq x\}$$

is in $\mathcal{D}$.

Since $\frac{\partial}{\partial y} g_i(x, y)$ ($i = 1, 2, \dots, 5$) are continuous and non-vanishing on the closure $\bar{D}$ of D in R^2 , and since $\bar{D}$ is compact, there exists $b > 0$ such that

$$\left| \frac{\partial}{\partial y} g_i(x, y) \right| \geq b \quad \text{for all } (x, y) \in \bar{D}.$$

Thus for each $x \in X$,

$$\left| \frac{\partial}{\partial y} g_i(x, y) \right| \geq b \quad \text{for all } y \in \overline{D_x},$$

and therefore the functions $g_i(x, \cdot)$ are strictly monotone on $\overline{D_x}$. So, by Lemma 3,

$$m^*(g_i(x, E)) \geq bm^*E \quad \text{for all } E \subset D_x.$$

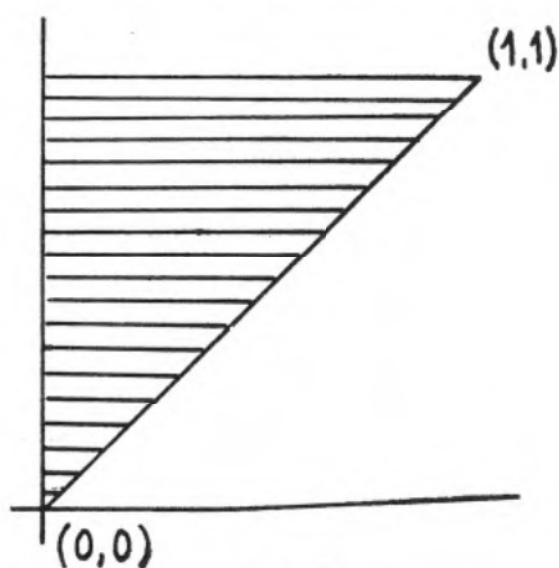


Figure-T

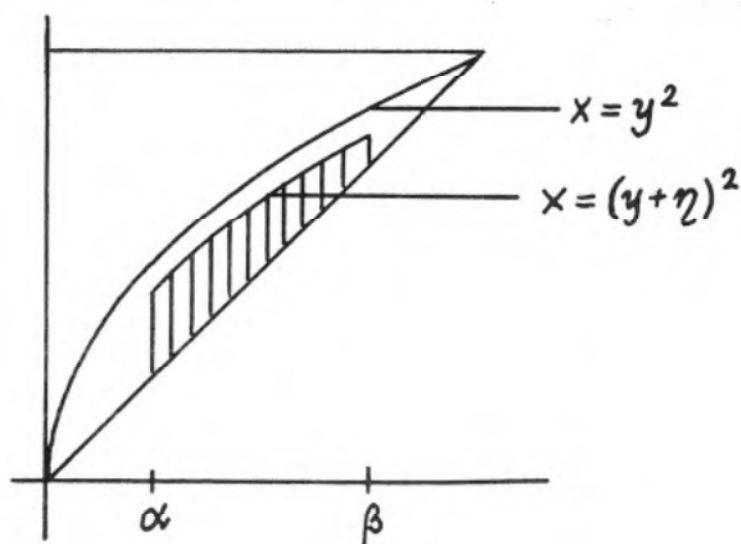


Figure-D

Now we claim that $g_i \in \mathcal{F}(D)$ and thereby the boundedness of f_1 on X follows from Theorem 2. In fact, let $\varepsilon > 0$ be given. Choose $\delta = b\varepsilon > 0$. Then for any $F \in \mathcal{B}$ with $m(F) < \delta$ and for each $x \in X$, we have

$$m^*((g_i)_x^{-1}(F)) \leq \frac{1}{b} m^*(g_i(x, (g_i)_x^{-1}(F))) \leq \frac{1}{b} m^*(F) < \varepsilon.$$

Hence $(g_i)_x^{-1}(F)$ is contained in some measurable set in Y of measure less than ε .

As the closed interval $[\alpha, \beta]$ is arbitrary, this proves the local boundedness of f_1 on $]0, 1[$. Since the f_i 's ($i=1, 2, 3, 4$) play symmetric roles in (1.1) we have indeed established the following theorem.

Theorem 4. *If $f_i:]0, 1[\rightarrow R$ are measurable functions satisfying (1.1), then they are all locally bounded.*

From Theorem 4, it follows that the measurable solutions f_i 's of (1.1) are locally integrable and then by adapting similar techniques employed in ([1], Theorem 3) (which is now considered as standard in the theory of functional equations), it can be shown that the derivatives of all orders of the f_i 's exist on $]0, 1[$.

§ 3. Main results

We provide the measurable solutions of (1.1) through a sequence of auxiliary results. The first two propositions follow from ([1], Lemma 7, Theorem 4 and Lemma 5).

Proposition 5. *A measurable function $F:]0, 1[\rightarrow R$ satisfies the equation*

$$F(pq) + F(p(1-q)) + F((1-p)q) + F((1-p)(1-q)) = 0$$

for all $p, q \in]0, 1[$ if, and only if,

$$F(p) = 4ap - a \quad \text{for all } p \in]0, 1[,$$

where a is an arbitrary constant.

Proposition 6. *The measurable solutions $F:]0, 1[\rightarrow R$ of*

$$F(pq) - F(p(1-q)) - F((1-p)q) + F((1-p)(1-q)) = 0$$

for all $p, q \in]0, 1[$ are given by

$$F(p) = ap^2 - ap + b \log p + c, \quad p \in]0, 1[$$

where a, b, c are arbitrary constants.

Remark. From now on all functions occurring in this section are from $]0, 1[$ into R and all equations displayed are supposed to hold for all variables in $]0, 1[$ (unless otherwise specified).

Proposition 7. *The measurable solutions of*

$$(3.1) \quad F(pq) + G(p(1-q)) + G((1-p)q) + F((1-p)(1-q)) = 0$$

are given by

$$(3.2) \quad \begin{cases} F(p) = bp^2 + (2a-b)p - c \log p - d - a, \\ G(p) = -bp^2 + (2a+b)p + c \log p + d, \end{cases}$$

where a, b, c, d are arbitrary constants.

PROOF. Replacing p by $1-p$ in (3.1) and adding the resultant to (3.1), we obtain

$$(F+G)(pq) + (F+G)(p(1-q)) + (F+G)((1-p)q) + (F+G)((1-p)(1-q)) = 0,$$

from which it follows by Proposition 5 that

$$(3.3) \quad F(p) + G(p) = 4ap - a,$$

where a is a constant. With the use of (3.3) and (3.1) we get

$$H(pq) - H(p(1-q)) - H((1-p)q) + H((1-p)(1-q)) = 0$$

where $H(p) := G(p) - 2ap$. By Proposition 6, we obtain

$$H(p) = -bp^2 + bp + c \log p + d$$

which in turn yields the asserted form of G in (3.2). This, together with (3.3) yields the sought for form of F in (3.2).

Proposition 8. *The solutions of*

$$F(pq) + G(p(1-q)) - G((1-p)q) - F((1-p)(1-q)) = 0$$

are given by

$$F = a \quad \text{and} \quad G = b$$

where a, b are arbitrary constants.

PROOF. From

$$F(pq) - F((1-p)(1-q)) = G((1-p)q) - G(p(1-q))$$

and the symmetry of the left side and the antisymmetry of the right side in p and q we get

$$F(pq) - F((1-p)(1-q)) = 0 = G((1-p)q) - G(p(1-q)).$$

Putting $u = pq$ in the equation of F we can rewrite it as

$$F(u) = F\left(1 - p - \frac{u}{p} + u\right) \quad \text{for all } p \in]u, 1[$$

for each $u \in]0, 1[$. This is equivalent to

$$F(u) = F(t) \quad \text{for all } t \in]0, (1 - \sqrt{u})^2[$$

for each $u \in]0, 1[$. This constancy of F on the intervals $]0, (1 - \sqrt{u})^2[$ for each $u \in]0, 1[$ implies that F is constant on $]0, 1[$. Similarly G is also constant.

Proposition 9. *The measurable solutions of*

$$(3.4) \quad f_1(pq) + f_2(p(1-q)) + f_3((1-p)q) + f_4((1-p)(1-q)) = 0$$

are given by

$$(3.5) \quad \begin{cases} f_i(p) = bp^2 + (a-b)p - c \log p + d_i, & i = 1, 4, \\ f_i(p) = -bp^2 + (a+b)p + c \log p + d_i, & i = 2, 3, \end{cases}$$

where a, b, c, d_i 's are constants with $a + d_1 + d_2 + d_3 + d_4 = 0$.

PROOF. Replace p by $1-p$ and q by $1-q$ in (3.4), then add and subtract the resultant to (3.4) respectively to get

$$(3.6) \quad (f_1+f_4)(pq) + (f_2+f_3)(p(1-q)) + (f_2+f_3)((1-p)q) + (f_1+f_4)((1-p)(1-q)) = 0,$$

$$(3.7) \quad (f_1-f_4)(pq) + (f_2-f_3)(p(1-q)) - (f_2-f_3)((1-p)q) - (f_1-f_4)((1-p)(1-q)) = 0.$$

By applying Propositions 7 and 8 to (3.6) and (3.7) respectively we get

$$(f_1+f_4)(p) = 2bp^2 + (2a-2b)p - 2c \log p + e_1,$$

$$(f_2+f_3)(p) = -2bp^2 + (2a+2b)p + 2c \log p + e_2$$

with $a+e_1+e_2=0$, and

$$(f_1-f_4)(p) = e_3,$$

$$(f_2-f_3)(p) = e_4.$$

This proves (3.5) with $d_1=(e_1+e_3)/2$, $d_4=(e_1-e_3)/2$, $d_2=(e_2+e_4)/2$ and $d_3=(e_2-e_4)/2$.

Theorem 10. *The measurable solutions of the functional equation (1.1) are given by*

$$f_i(p) = 4ap^3 + (b-9a)p^2 + c_i p + cp \log p + e_i \log p + d_i, \quad i = 1, 4$$

$$f_i(p) = 4ap^3 - (b+9a)p^2 + c_i p + cp \log p + e_i \log p + d_i, \quad i = 2, 3$$

$$f_5(p) = -6ap^2 + (6a-2b+c_2-c_4)p + c[p \log p + (1-p) \log (1-p)] + (e_1+e_2) \log p + (e_3+e_4) \log (1-p) + d_5,$$

$$f_6(p) = -6ap^2 + (6a-2b+c_3-c_4)p + c[p \log p + (1-p) \log (1-p)] + (e_1+e_3) \log p + (e_2+e_4) \log (1-p) + d_6$$

where a, b, c, c_i, d_i, e_i , are constants with

$$4b + c_1 - c_2 - c_3 + c_4 = 0 \quad \text{and} \quad b - 5a + c_4 + d_1 + d_2 + d_3 + d_4 = d_5 + d_6.$$

PROOF. From section 2 we may assume that the f_i 's have derivatives of all orders. Differentiate (1.1) twice, first with respect to p and then with respect to q , to obtain

$$F_1(pq) - F_2(p(1-q)) - F_3((1-p)q) + F_4((1-p)(1-q)) = 0,$$

where

$$(3.8) \quad F_i(p) := pf_i''(p) + f_i'(p), \quad (i = 1, 2, 3, 4).$$

By Proposition 9 the F_i 's are given by

$$(3.9) \quad \begin{aligned} F_i(p) &= Bp^2 + (A-B)p - C \log p + D_i, \quad i = 1, 4, \\ F_i(p) &= Bp^2 - (A+B)p - C \log p - D_i, \quad i = 2, 3, \end{aligned}$$

where A, B, C, D_i are constants with $A + D_1 + D_2 + D_3 + D_4 = 0$. Using (3.8) and (3.9) we get

$$\begin{aligned} f_i(p) &= \frac{B}{9} p^3 + \frac{A-B}{4} p^2 - C(p \log p - 2p) + \\ &\quad + D_i p + e_i \log p + d_i, \quad i = 1, 4, \\ f_i(p) &= \frac{B}{9} p^3 - \frac{A+B}{4} p^2 - C(p \log p - 2p) - \\ &\quad - D_i p + e_i \log p + d_i, \quad i = 2, 3, \end{aligned}$$

where e_i, d_i are constants. This establishes the asserted form of f_i 's for $i = 1, 2, 3, 4$, which in turn, from (1.1), yields the form of f_5 and f_6 . This proves the theorem.

Corollary 11. *The measurable solutions of the functional equation (1.1) with $f_1 = f_2 = f_3 = f_4 = f$ are given by*

$$\begin{aligned} f(p) &= 4ap^3 - 9ap^2 + c_0 p + cp \log p + e_0 \log p + d, \\ f_i(p) &= -6ap^2 + 6ap + c[p \log p + (1-p) \log (1-p)] + \\ &\quad + 2e_0 \log p(1-p) + d_i, \quad i = 5, 6, \end{aligned}$$

where a, c_0, c, e_0, d, d_i are constants with $-5a + c_0 + 4d_0 = d_5 + d_6$.

Corollary 12. *The measurable solutions of the functional equation*

$$\begin{aligned} f(pq) + f(p(1-q)) + f((1-p)q) + f((1-p)(1-q)) = \\ = f(p) + f(1-p) + f(q) + f(1-q) \end{aligned}$$

for all $p, q \in]0, 1[$ are given by

$$f(p) = 4ap^3 - 9ap^2 + 5ap + cp \log p + d$$

where a, c, d are constants. (This is Theorem 5 in [1]).

References

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