

BV-solutions of a linear functional equation

By CZESŁAW DYJAK (Częstochowa)

Abstract. *BV*-solutions of the linear functional equation $\varphi(x) = g(x)\varphi[f(x)] + h(x)$ are considered and, under some general hypotheses, it is proved that this equation has a unique solution $\varphi \in BV\langle a, b \rangle$ which is given by a series convergent in the sense of the norm in the space $BV\langle a, b \rangle$.

In this paper we consider the solutions of bounded variation (*BV*-solutions) of the linear functional equation

$$(1) \quad \varphi(x) = g(x)\varphi[f(x)] + h(x),$$

where f , g and h are given and φ is an unknown function.

By $BV\langle a, b \rangle$ we denote Banach's functions space $\varphi: \langle a, b \rangle \rightarrow \mathbb{R}$ with the norm defined by the formula

$$(2) \quad \|\varphi\| = |\varphi(a)| + \text{Var}_{\langle a, b \rangle} \varphi,$$

where, as usual,

$$\text{Var}_{\langle a, b \rangle} \varphi := \sup \sum_{i=1}^s |\varphi(x_i) - \varphi(x_{i-1})|,$$

and the supremum is taken over all partitions of the interval $\langle a, b \rangle$.

In the case when $g(x) \equiv \text{const}$, *BV*-solutions of the equation (1) have been considered in the paper [1]. Even in this special case our main result says more. It appears that the series (3) converges in the sense of the *BV*-norm to the solution of the equation (1), whereas in the paper [1] only pointwise convergence is obtained.

We shall prove the following theorem, which generalizes Theorem 1 from the paper [1] above mentioned:

Theorem. *If the following conditions are fulfilled:*

- (i) $f: \langle a, b \rangle \rightarrow \langle a, b \rangle$ is continuous, strictly increasing and $f(a) = a$,
- (ii) $g, h \in BV\langle a, b \rangle$,
- (iii) $\sup_{x \in \langle a, b \rangle} |g(x)| + \text{Var}_{\langle a, b \rangle} g < 1$,

then the equation (1) has in the interval $\langle a, b \rangle$ a unique solution $\varphi \in BV\langle a, b \rangle$. This

solution is given by the series

$$(3) \quad \varphi(x) = \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} g[f^i(x)] h[f^k(x)] + h(x), \quad x \in \langle a, b \rangle$$

which converges in the sense of the BV-norm.

PROOF. In the space $BV\langle a, b \rangle$ we consider the transform

$$(4) \quad T[\varphi](x) = g(x)\varphi[f(x)] + h(x), \quad x \in \langle a, b \rangle.$$

On account of hypotheses (i) and (ii) the transform (4) maps, of course, the space $BV\langle a, b \rangle$ into itself. We shall prove that the transform (4) is a contraction map.

Let $P\langle a, b \rangle$ be the set of all partitions of the form $a = x_0 < x_1 < \dots < x_{s-1} < x_s = b$ of the interval $\langle a, b \rangle$.

Take an arbitrary $\varphi_1, \varphi_2 \in BV\langle a, b \rangle$ and estimate the expression $\|T[\varphi_1] - T[\varphi_2]\|$. Putting $\varphi := \varphi_1 - \varphi_2$ and taking into account (4), (2) and (i), we get

$$\begin{aligned} \|T[\varphi_1] - T[\varphi_2]\| &= \|g\varphi_1[f] + h - g\varphi_2[f] - h\| = \\ &= \|g(\varphi_1[f] - \varphi_2[f])\| = \|g\varphi[f]\| = \\ &= |g(a)\varphi(a)| + \sup_{P\langle a, b \rangle} \sum_{i=1}^s |g(x_i)\varphi[f(x_i)] - g(x_{i-1})\varphi[f(x_{i-1})]| = \\ &= |g(a)| |\varphi(a)| + \sup_{P\langle a, b \rangle} \sum_{i=1}^s |g(x_i)(\varphi[f(x_i)] - \varphi[f(x_{i-1})]) + \\ &\quad + \varphi[f(x_{i-1})](g(x_i) - g(x_{i-1}))| \leq \\ &\leq |g(a)| |\varphi(a)| + \sup_{x \in \langle a, b \rangle} |g(x)| \sup_{P\langle a, b \rangle} \sum_{i=1}^s |\varphi[f(x_i)] - \varphi[f(x_{i-1})]| + \\ &\quad + \sup_{x \in \langle a, b \rangle} |\varphi[f(x)]| \sup_{P\langle a, b \rangle} \sum_{i=1}^s |g(x_i) - g(x_{i-1})| = \\ &= |g(a)| |\varphi(a)| + \sup_{x \in \langle a, b \rangle} |g(x)| \operatorname{Var} \varphi[f] + \sup_{x \in \langle a, b \rangle} |\varphi[f(x)]| \operatorname{Var} g. \end{aligned}$$

Since, of course,

$$\operatorname{Var} \varphi[f] \leq \operatorname{Var} \varphi$$

and

$$\sup_{x \in \langle a, b \rangle} |\varphi[f(x)]| \leq \sup_{x \in \langle a, b \rangle} |\varphi(x)| \leq |\varphi(a)| + \operatorname{Var} \varphi = \|\varphi\|,$$

we have further

$$\begin{aligned} \|T[\varphi_1] - T[\varphi_2]\| &\leq \sup_{x \in \langle a, b \rangle} |g(x)| |\varphi(a)| + \sup_{x \in \langle a, b \rangle} |g(x)| \operatorname{Var}_{\langle a, b \rangle} \varphi + \\ &\quad + \|\varphi\| \operatorname{Var}_{\langle a, b \rangle} g = \sup_{x \in \langle a, b \rangle} |g(x)| \|\varphi\| + \|\varphi\| \operatorname{Var}_{\langle a, b \rangle} g = \\ &= \left(\sup_{x \in \langle a, b \rangle} |g(x)| + \operatorname{Var}_{\langle a, b \rangle} g \right) \|\varphi\| = \left(\sup_{x \in \langle a, b \rangle} |g(x)| + \operatorname{Var}_{\langle a, b \rangle} g \right) \|\varphi_1 - \varphi_2\| \end{aligned}$$

and so, by hypothesis (iii), the transform (4) is a contraction map.

Then, in virtue of Banach's fixpoint theorem, there exists a unique fixpoint of the transform (4) in the space $BV\langle a, b \rangle$ given as the limit of the sequence of successive approximations $\{\varphi_n\}_{n=0,1,\dots}$ convergent in the sense of the norm (2). In other words, there exists a unique function $\varphi \in BV\langle a, b \rangle$, which fulfils equation (1) in the interval $\langle a, b \rangle$. In particular, taking $\varphi_0 \equiv 0$ in $\langle a, b \rangle$ we obtain formula (3).

References

- [1] J. MATKOWSKI, M. C. ZDUN, Solutions of bounded variation of a linear functional equation, *Aequationes Mathematicae*, **10** (1974), 223—235.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
CZĘSTOCHOWA, POLAND

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