

On P -Sasakian manifolds which admit certain tensor fields

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0. Introduction. G. P. POKHARIYAL and R. S. MISHRA ([5]) have introduced new tensor fields named as W_2 — and E -tensor fields in a Riemannian manifold and studied their properties. Next, G. P. POKHARIYAL has studied some properties of these tensor fields in a Sasakian manifold ([6]).

On the other hand, I. SATO introduced the notion of an almost paracontact, a P -Sasakian and an SP -Sasakian manifold and he had a lot of very interesting results about such manifolds ([2], [3]).

In this paper, we shall study P -Sasakian manifolds which admit the W_2 — or the E -tensor field that satisfy certain conditions. In § 1, we recall the notion of a P -Sasakian and an SP -Sasakian manifold and the essential properties of these manifolds. In § 2, we shall get some formulas of the W_2 - and the E -tensor fields in a P -Sasakian manifold. In § 3, we shall show that a W_2 -symmetric P -Sasakian manifold is of constant curvature and an E -symmetric P -Sasakian manifold is an SP -Sasakian manifold of constant curvature or an η -Einstein P -Sasakian manifold with the harmonic structure vector field. In the last § 4, we shall prove that an E -recurrent P -Sasakian manifold is necessarily of constant curvature.

In this paper, we always assume that manifolds and tensor fields are differentiable.

1. Preliminaries. An n -dimensional P -Sasakian manifold M^n is a Riemannian manifold with positive definite Riemannian metric g which admits a unit 1-form η satisfying

$$(1.1) \quad \begin{aligned} \text{a) } & \nabla_\mu \eta_\lambda - \nabla_\lambda \eta_\mu = 0, \\ \text{b) } & \nabla_\nu \nabla_\mu \eta_\lambda = -g_{\nu\mu} \eta_\lambda - g_{\nu\lambda} \eta_\mu + 2\eta_\nu \eta_\mu \eta_\lambda, \end{aligned}$$

where ∇_λ denotes the covariant differentiation with respect to g and the indices $\nu, \mu, \dots, \lambda$ run over the range $1, 2, \dots, n$. If we put

$$(1.2) \quad \xi^\lambda = g^{\lambda\epsilon} \eta_\epsilon, \quad \varphi_\mu^\lambda = \nabla_\mu \xi^\lambda,$$

then we have

$$(1.3) \quad \begin{aligned} \eta_\epsilon \xi^\epsilon &= 1, \quad \varphi_\epsilon^\lambda \xi^\epsilon = 0, \quad \eta_\epsilon \varphi_\mu^\epsilon = 0, \\ \varphi_{\mu\lambda} &= \varphi_{\lambda\mu} (\varphi_{\mu\lambda} = \varphi_\mu^\epsilon g_{\epsilon\lambda}), \quad \text{rank}(\varphi_\mu^\lambda) = n-1, \\ \varphi_\mu^\epsilon \varphi_\epsilon^\lambda &= \delta_\mu^\lambda - \eta_\mu \xi^\lambda, \quad g_{\gamma\epsilon} \varphi_\mu^\gamma \varphi_\lambda^\epsilon = g_{\mu\lambda} - \eta_\mu \eta_\lambda. \end{aligned}$$

In an n -dimensional P -Sasakian manifold M^n , the following identities are established ([4]);

$$(1.4) \quad R_{\omega\nu\mu}{}^\varepsilon \eta_\varepsilon = g_{\omega\mu} \eta_\nu - g_{\nu\mu} \eta_\omega, \quad R_{\mu\varepsilon} \zeta^\varepsilon = -(n-1)\eta_\mu,$$

$$(1.5) \quad R_\mu{}^\varepsilon \varphi_{\varepsilon\lambda} - R_{\mu\varepsilon\gamma\lambda} \varphi^{\varepsilon\gamma} = (n-2) \varphi_{\mu\lambda} - \varphi g_{\mu\lambda} + 2\varphi \eta_\mu \eta_\lambda, \\ R_\mu{}^\varepsilon \varphi_{\varepsilon\lambda} = R_\lambda{}^\varepsilon \varphi_{\varepsilon\mu},$$

$$(1.6) \quad R_{\omega\nu\varepsilon\gamma} \varphi_\mu{}^\varepsilon \varphi_\lambda{}^\gamma = R_{\omega\nu\mu\lambda} + g_{\nu\lambda} g_{\omega\mu} - g_{\omega\lambda} g_{\nu\mu} + \varphi_{\omega\lambda} \varphi_{\nu\mu} \\ - \varphi_{\nu\lambda} \varphi_{\omega\mu} + 2(g_{\nu\mu} \eta_\omega \eta_\lambda - g_{\omega\mu} \eta_\nu \eta_\lambda + g_{\omega\lambda} \eta_\nu \eta_\mu - g_{\nu\lambda} \eta_\omega \eta_\mu),$$

where $R_{\omega\nu\mu\lambda}$ and $R_{\mu\lambda}$ denote the Riemannian curvature tensor and the Ricci tensor with respect to g , respectively, and $\varphi = \text{trace}(\varphi_\mu{}^\lambda)$.

In a P -Sasakian manifold M^n , if the unit 1-form η satisfies the condition

$$(1.7) \quad \nabla_\mu \eta_\lambda = \varepsilon(-g_{\mu\lambda} + \eta_\mu \eta_\lambda), \quad (\varepsilon = \pm 1),$$

then the manifold M^n is called an SP -Sasakian manifold ([1]). From (1.7), we can easily see that an SP -Sasakian manifold is characterized by $\varphi^2 = (n-1)^2$ ([1]).

Remark. At first, I. SATO ([2]) defined an SP -Sasakian manifold as a P -Sasakian manifold satisfying $\nabla_\mu \eta_\lambda = -g_{\mu\lambda} + \eta_\mu \eta_\lambda$.

A P -Sasakian manifold is called an η -Einstein manifold if the Ricci tensor $R_{\mu\lambda}$ is written by the form

$$(1.8) \quad R_{\mu\lambda} = a g_{\mu\lambda} + b \eta_\mu \eta_\lambda$$

for certain scalar fields a and b . The scalar fields a and b are related by $a+b = -(n-1)$ and $na+b=R$, where R denotes the scalar curvature respect to g . The following proposition was proved in [4];

Proposition 1.1. *An n -dimensional ($n \neq 3$) non-Einstein η -Einstein P -Sasakian manifold has a constant scalar curvature R if and only if the vector field ξ is harmonic.*

2. W_2 - and E -tensor fields in a P -Sasakian manifold. Let M^n be an n -dimensional P -Sasakian manifold. We define tensor fields W_2 and E in M^n as follows;

$$(2.1) \quad W_{2\omega\nu\mu\lambda} = R_{\omega\nu\mu\lambda} + \frac{1}{n-1} (g_{\omega\mu} R_{\nu\lambda} - g_{\nu\mu} R_{\omega\lambda}),$$

and

$$(2.2) \quad E_{\omega\nu\mu\lambda} = \frac{1}{2} (W_{2\omega\nu\mu\lambda} - W_{2\omega\nu\lambda\mu}) = R_{\omega\nu\mu\lambda} + \frac{1}{2(n-1)} (g_{\omega\mu} R_{\nu\lambda} - \\ - g_{\omega\lambda} R_{\nu\mu} + g_{\nu\lambda} R_{\omega\mu} - g_{\nu\mu} R_{\omega\lambda}).$$

In this section, we state the properties of W_2 - and E -tensor fields in M^n . The proof of the following two propositions are trivial. So, we omit their proof.

Proposition 2.1. *In an n -dimensional P -Sasakian manifold, we have*

$$(2.3) \quad \left\{ \begin{array}{l} \text{a) } W_{2\omega\nu\mu\epsilon} \xi^\epsilon = 0, \quad \text{b) } W_{2\epsilon\nu\mu\lambda} \xi^\epsilon = \left(g_{\nu\lambda} + \frac{1}{n-1} R_{\nu\lambda} \right) \eta_\mu \\ \text{c) } W_{2\tau\kappa\gamma\epsilon} \varphi_\omega^\tau \varphi_\nu^\kappa \varphi_\mu^\gamma \varphi_\lambda^\epsilon = W_{2\omega\nu\mu\lambda} + \left\{ \left(g_{\omega\lambda} + \frac{1}{n-1} R_{\omega\lambda} \right) \eta_\nu - \right. \\ \left. - \left(g_{\nu\lambda} + \frac{1}{n-1} R_{\nu\lambda} \right) \eta_\omega \right\} \eta_\mu = W_{2\omega\nu\mu\lambda} + (W_{2\epsilon\omega\nu\lambda} - W_{2\epsilon\nu\omega\lambda}) \xi^\epsilon \eta_\mu. \end{array} \right.$$

Proposition 2.2. *In an n -dimensional P -Sasakian manifold M^n , the E -tensor field satisfies*

$$(2.4) \quad \left\{ \begin{array}{l} \text{a) } E_{\omega\nu\mu\epsilon} \xi^\epsilon = \frac{1}{2} (g_{\omega\mu} \eta_\nu - g_{\nu\mu} \eta_\omega) + \frac{1}{2(n-1)} (R_{\omega\mu} \eta_\nu - R_{\nu\mu} \eta_\omega), \\ \text{b) } E_{\epsilon\nu\mu\lambda} \xi^\epsilon = \frac{1}{2} (g_{\nu\lambda} \eta_\mu - g_{\nu\mu} \eta_\lambda) + \frac{1}{2(n-1)} (R_{\nu\lambda} \eta_\mu - R_{\nu\mu} \eta_\lambda), \\ \text{c) } E_{\epsilon\nu\mu\gamma} \xi^\epsilon \xi^\gamma = -\frac{1}{2} \left(g_{\nu\mu} + \frac{1}{n-1} R_{\nu\mu} \right). \end{array} \right.$$

3. Symmetric P -Sasakian manifolds.

Definition 3.1. An n -dimensional P -Sasakian manifold M^n is called W_2 -symmetric if it satisfies

$$(3.1) \quad \nabla_\alpha W_{2\omega\nu\mu\lambda} = 0,$$

and an n -dimensional P -Sasakian manifold is called E -symmetric if it satisfies

$$(3.2) \quad \nabla_\alpha E_{\omega\nu\mu\lambda} = 0.$$

Now, let M^n be an n -dimensional W_2 -symmetric P -Sasakian manifold. Then we have from (3.1) and the second Bianchi identity ([7])

$$(3.3) \quad R_{\tau\kappa\omega}{}^\epsilon W_{2\epsilon\nu\mu\lambda} + R_{\tau\kappa\nu}{}^\epsilon W_{2\omega\epsilon\mu\lambda} + R_{\tau\kappa\mu}{}^\epsilon W_{2\omega\nu\epsilon\lambda} + R_{\tau\kappa\lambda}{}^\epsilon W_{2\omega\nu\mu\epsilon} = 0.$$

Transvecting (3.3) with ξ^λ and taking account of (1.4) and (2.3), we have

$$(3.4) \quad \eta_\alpha W_{2\omega\nu\mu\tau} - \eta_\tau W_{2\omega\nu\mu\alpha} = 0,$$

from this, we obtain

$$(3.5) \quad W_{2\omega\nu\mu\lambda} = 0,$$

that is, the manifold M^n is W_2 -flat. (3.5) implies

$$(3.6) \quad R_{\omega\nu\mu\lambda} = -\frac{1}{n-1} (g_{\omega\mu} R_{\nu\lambda} - g_{\nu\mu} R_{\omega\lambda}).$$

Transvecting (3.6) with $g^{\omega\lambda}$, we get

$$(3.7) \quad R_{\nu\mu} = \frac{R}{n} g_{\nu\mu}.$$

By virtue of (1.4) and (3.7), we obtain $R = -n(n-1)$. Thus (3.6) and (3.7) imply

$$(3.8) \quad R_{\omega\nu\mu\lambda} = -(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda}).$$

Thus we have

Theorem 3.1. *An n -dimensional W_2 -symmetric P -Sasakian manifold is of constant curvature -1 .*

Next, let M^n be an n -dimensional E -symmetric P -Sasakian manifold. Then we have from (3.2) and the second Bianchi identity

$$(3.9) \quad R_{\tau\kappa\omega}{}^\varepsilon E_{\varepsilon\nu\mu\lambda} + R_{\tau\kappa\nu}{}^\varepsilon E_{\omega\varepsilon\mu\lambda} + R_{\tau\kappa\mu}{}^\varepsilon E_{\omega\nu\varepsilon\lambda} + R_{\tau\kappa\lambda}{}^\varepsilon E_{\omega\nu\mu\varepsilon} = 0.$$

Transvecting (3.9) with $\xi^\kappa \xi^\lambda$ and taking account of (1.2), (1.3), (1.4) and (2.4), we get

$$(3.10) \quad R_{\omega\nu\mu\lambda} = -\frac{1}{2}(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda} + g_{\omega\lambda}\eta_\nu\eta_\mu - g_{\nu\lambda}\eta_\omega\eta_\mu) + \\ + \frac{1}{2(n-1)}(g_{\nu\mu}R_{\omega\lambda} - g_{\omega\mu}R_{\nu\lambda} - \eta_\nu\eta_\mu R_{\omega\lambda} + \eta_\omega\eta_\mu R_{\nu\lambda}).$$

The equation (3.10) implies

$$(3.11) \quad R_{\mu\lambda} = \frac{R-(n-1)^2}{2n-1}g_{\mu\lambda} - \frac{R+n(n-1)}{2n-1}\eta_\mu\eta_\lambda$$

that is, the manifold M^n is of η -Einstein. Substituting (3.11) into (3.10), the curvature tensor $R_{\omega\nu\mu\lambda}$ is written by

$$(3.12) \quad R_{\omega\nu\mu\lambda} = \frac{R-(n-1)(3n-2)}{2(n-1)(2n-1)}(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda}) - \\ - \frac{R+n(n-1)}{2(n-1)(2n-1)}(g_{\omega\lambda}\eta_\nu\eta_\mu - g_{\nu\lambda}\eta_\omega\eta_\mu + g_{\nu\mu}\eta_\omega\eta_\lambda - g_{\omega\mu}\eta_\nu\eta_\lambda).$$

Substituting (3.11) and (3.12) into (1.5) and by the straightforward calculation, we have

$$(3.13) \quad (R-7n^2+11n-4)\varphi(\eta_\omega\eta_\lambda - g_{\omega\lambda}) + (2n-1)\{R-(n-1)(3n-4)\}\varphi_{\omega\lambda} = 0.$$

Thus it is sufficient to consider the following two cases;

(i) $\varphi \neq 0$. Then, transvecting (3.13) with $g^{\omega\lambda}$, we have

$$(3.14) \quad -(n-1)(R-7n^2+11n-4) + (2n-1)\{R-(n-1)(3n-4)\} = 0$$

By virtue of (3.13) and (3.14), we get

$$(n-1)\varphi_{\omega\lambda} + \varphi(\eta_\omega\eta_\lambda - g_{\omega\lambda}) = 0,$$

that is,

$$(3.15) \quad \varphi_{\omega\lambda} = \varepsilon(-g_{\omega\lambda} + \eta_\omega\eta_\lambda).$$

Moreover, the equation (3.14) gives us

$$(3.16) \quad R = -n(n-1).$$

Thus we have from (3.12) and (3.16)

$$R_{\omega\nu\mu\lambda} = -(g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda}),$$

that is, the manifold M^n is of constant curvature -1 .

(ii) $\varphi=0$. Then we have from (3.13)

$$(3.17) \quad R = (n-1)(3n-4).$$

Substituting (3.17) into (3.12), the curvature tensor $R_{\omega\nu\mu\lambda}$ is written by

$$(3.18) \quad R_{\omega\nu\mu\lambda} = \frac{-1}{2n-1} (g_{\omega\lambda}g_{\nu\mu} - g_{\omega\mu}g_{\nu\lambda}) - \frac{2(n-1)}{2n-1} (g_{\omega\lambda}\eta_\nu\eta_\mu - g_{\nu\lambda}\eta_\omega\eta_\mu + g_{\nu\mu}\eta_\omega\eta_\lambda - g_{\omega\mu}\eta_\nu\eta_\lambda).$$

Thus we have

Theorem 3.2. For an n -dimensional E -symmetric P -Sasakian manifold M^n , we have the following two cases;

(i) If $\varphi \neq 0$, then the manifold M^n is SP -Sasakian with constant curvature -1 , and

(ii) If $\varphi=0$, then the vector field ξ^λ is harmonic, the manifold M^n is η -Einstein and the curvature tensor $R_{\omega\nu\mu\lambda}$ is given by (3.18).

4. Recurrent P -Sasakian manifolds.

Definition 4.1. An n -dimensional P -Sasakian manifold M^n is called W_2 -recurrent if it satisfies

$$(4.1) \quad \nabla_\kappa W_{2\omega\nu\mu\lambda} = \theta_\kappa W_{2\omega\nu\mu\lambda}$$

for certain non-zero vector field θ_κ , and an n -dimensional P -Sasakian manifold M^n is called E -recurrent if it satisfies

$$(4.2) \quad \nabla_\kappa E_{\omega\nu\mu\lambda} = \theta_\kappa E_{\omega\nu\mu\lambda}$$

for a certain non-zero vector field θ_κ .

It is trivial that a W_2 -recurrent P -Sasakian manifold is an E -recurrent one.

Let M^n be an n -dimensional E -recurrent P -Sasakian manifold. Then we have (4.2) and this equation means

$$(4.3) \quad \begin{aligned} \nabla_\kappa R_{\omega\nu\mu\lambda} + \frac{1}{2(n-1)} (g_{\omega\mu}\nabla_\kappa R_{\nu\lambda} - g_{\omega\lambda}\nabla_\kappa R_{\nu\mu} + g_{\nu\lambda}\nabla_\kappa R_{\omega\mu} - g_{\nu\mu}\nabla_\kappa R_{\omega\lambda}) = \\ = \theta_\kappa \{R_{\omega\nu\mu\lambda} + \frac{1}{2(n-1)} (g_{\omega\mu}R_{\nu\lambda} - g_{\omega\lambda}R_{\nu\mu} + g_{\nu\lambda}R_{\omega\mu} - g_{\nu\mu}R_{\omega\lambda})\}. \end{aligned}$$

Transvecting (4.3) with $g^{\omega\lambda}$, we get

$$(4.4) \quad n\nabla_{\times} R_{\nu\mu} = n\theta_{\times} R_{\nu\mu} - R\theta_{\times} g_{\nu\mu} + g_{\nu\mu} \nabla_{\times} R.$$

Moreover, transvecting (4.4) with $g^{\nu\mu}$, we obtain

$$(4.5) \quad \nabla_{\times} R = R\theta_{\times} + n(n-1)\theta_{\times}.$$

By virtue of (4.3), (4.4) and (4.5), we have

$$(4.6) \quad \nabla_{\times} R_{\omega\nu\mu\lambda} = \theta_{\times} (R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}).$$

Covariant differentiation of (4.6) gives

$$(4.7) \quad \begin{aligned} \nabla_{\tau} \nabla_{\times} R_{\omega\nu\mu\lambda} &= \nabla_{\tau} \theta_{\times} (R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}) + \\ &+ \theta_{\tau} \theta_{\times} (R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}). \end{aligned}$$

From this, we obtain

$$\begin{aligned} R_{\tau\kappa\omega}{}^{\varepsilon} R_{\varepsilon\nu\mu\lambda} + R_{\tau\kappa\nu}{}^{\varepsilon} R_{\omega\varepsilon\mu\lambda} + R_{\tau\kappa\mu}{}^{\varepsilon} R_{\omega\nu\varepsilon\lambda} + R_{\tau\kappa\lambda}{}^{\varepsilon} R_{\omega\nu\mu\varepsilon} = \\ = (\nabla_{\tau} \theta_{\times} - \nabla_{\times} \theta_{\tau}) (R_{\omega\nu\mu\lambda} + g_{\omega\lambda} g_{\nu\mu} - g_{\omega\mu} g_{\nu\lambda}). \end{aligned}$$

Transvection of the above equation with $\xi^{\times} \xi^{\lambda}$ gives that the manifold M^n is of constant curvature -1 . Thus we have

Theorem 4.1. *An E -recurrent P -Sasakian manifold is of constant curvature -1 .*

Corollary 4.2. *A W_2 -recurrent P -Sasakian manifold is of constant curvature -1 .*

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