

On additive arithmetical functions with values in topological groups I.

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1. Throughout this paper we shall use the following standard notations: $\mathbf{N}$ =natural numbers; $\mathbf{Z}$ =rational integers; $\mathbf{Q}_x$ =multiplicative group of positive rationals; $\mathbf{R}_x$ =multiplicative group of positive reals; $\mathbf{Q}$ =additive group of rationals; $\mathbf{R}$ =additive group of reals; T =one-dimensional circle group (torus); each of them in the usual topology.

Let G be an Abelian group. We shall say that a mapping $\varphi: \mathbf{N} \rightarrow G$ is a completely additive function, if

$$(1.1) \quad \varphi(mn) = \varphi(m) + \varphi(n) \quad \forall m, n \in \mathbf{N}$$

holds.

If we consider G as a multiplicative group, then the mapping $V: \mathbf{N} \rightarrow G$ satisfying the relation

$$(1.2) \quad V(mn) = V(m)V(n) \quad \forall m, n \in \mathbf{N}$$

is called a completely multiplicative function.

We can extend the domain of φ and V to $\mathbf{Q}_x$ by the relations

$$(1.3) \quad \varphi\left(\frac{m}{n}\right) := \varphi(m) - \varphi(n), \quad V\left(\frac{m}{n}\right) = V(m)V^{-1}(n),$$

uniquely. Furthermore the relations

$$(1.4) \quad \varphi(rs) = \varphi(r) + \varphi(s), \quad V(rs) = V(r)V(s)$$

hold. So the extensions of the functions φ, V define a $\mathbf{Q}_x \rightarrow G$ homomorphism

Let now G be an Abelian topological group, $\varphi: \mathbf{Q}_x \rightarrow G$ be a homomorphism. We shall say that φ is continuous at the point 1, if $r_v \in \mathbf{Q}_x, r_v \rightarrow 1$ implies that

$$(1.5) \quad \varphi(r_v) \rightarrow 0.$$

Lemma 1. *Let G be an additively written closed Abelian topological group, $\varphi: \mathbf{Q}_x \rightarrow G$ be a homomorphism that is continuous at the point 1. Then its domain can be extended by the relation*

$$(1.6) \quad \varphi(\alpha) := \lim_{\substack{r_v \rightarrow \alpha \\ r_v \in \mathbf{Q}_x}} \varphi(r_v) \quad (\alpha \in \mathbf{R}_x)$$

uniquely. The so obtained $\varphi: \mathbf{R}_x \rightarrow G$ is a continuous homomorphism, consequently

$$(1.7) \quad \varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta) \quad (\forall \alpha, \beta \in \mathbf{R}_x)$$

holds.

PROOF. Let $\alpha \in \mathbf{R}_x$, $r_v \rightarrow \alpha$ be an arbitrary sequence of rationals. Since $r_v/r_\mu \rightarrow 1$ as $v, \mu \rightarrow \infty$, therefore

$$\varphi\left(\frac{r_v}{r_\mu}\right) = \varphi(r_v) - \varphi(r_\mu) \rightarrow 0 \quad \text{as } v, \mu \rightarrow \infty,$$

consequently $\varphi(r_v)$ is a Cauchy-sequence, and so it is convergent. Hence it follows immediately that the limit is well defined. The further assertions in the lemma are obvious consequences of this. ■

We are interested in such completely additive functions for which

$$(1.8) \quad \Delta\varphi(n) := \varphi(n+1) - \varphi(n) \rightarrow 0 \quad (n \rightarrow \infty)$$

holds.

An old theorem of P. ERDŐS asserts that for $G = \mathbf{R}$ (1.8) implies that φ is a constant multiple of $\log$, in other words that φ is a continuous homomorphism $\mathbf{Q}_x \rightarrow \mathbf{R}$ [1].

A very recent, until now unpublished result due to E. WIRSING [2] asserts that: if $G = T$ written additively, then (1.8) implies that φ is a constant multiple of $\log \pmod{2\pi}$, i.e. that φ is a continuous homomorphism $\mathbf{Q}_x \rightarrow T$.

Now we state this theorem in multiplicative form as

Lemma 2. Let $T = \{z \in \mathbf{C} \mid |z| = 1\}$ be the unit circle, and $V: \mathbf{N} \rightarrow T$ be a completely multiplicative function, such that

$$(1.9) \quad \delta V(n) := V(n+1)V^{-1}(n) \rightarrow 1 \quad (\in T) \quad (n \rightarrow \infty).$$

Then $V(n) = n^{i\tau}$, τ is a real number.

This is the crucial point of the proof of our

Theorem 1. Let G be an additively written, metrically compact Abelian topological group. Let $\varphi: \mathbf{N} \rightarrow G$ be a completely additive function satisfying the condition (1.8). Then its extension $\varphi: \mathbf{Q}_x \rightarrow G$ defined by (1.3) is continuous at 1, consequently its extension $\varphi: \mathbf{R}_x \rightarrow G$ defined by (1.6) is a continuous homomorphism.

2. PROOF OF THEOREM 1. Let us assume that (1.1), (1.8) hold. Let $\chi: G \rightarrow T$ be any continuous character,

$$V(n) := \chi(\varphi(n)).$$

Then

$$\delta V(n) = V(n+1)V(n)^{-1} = \chi(\Delta\varphi(n)) \rightarrow \chi(0) = 1 \quad (n \rightarrow \infty).$$

Furthermore V is completely multiplicative, and so by Lemma 2, $V(n) = e^{i\tau \log n}$ ($\tau \in \mathbf{R}$).

Now we prove that $\varphi: \mathcal{Q}_x \rightarrow G$ is continuous in 1. Let $N_j/M_j \rightarrow 1$, $N_j, M_j \in \mathbf{N}$ ($j \rightarrow \infty$). We consider $A_j = \varphi(N_j) - \varphi(M_j)$. Since G is metrically therefore it is sequentially compact. Then there exists a convergent subsequence, $A_{j_i} \rightarrow B (\in G)$. Then $\chi(A_{j_i}) \rightarrow \chi(B)$. By Lemma 2 we get

$$\chi(A_{j_i}) = \exp \left(i\tau \log \frac{N_{j_i}}{M_{j_i}} \right) \rightarrow 1.$$

So $\chi(B) = 1$ for each continuous character χ , consequently $B = O(\in G)$, and so $\varphi: \mathcal{Q}_x \rightarrow G$ is continuous in 1. Lemma 1 completes the proof of the theorem. ■

3. Some remarks

1. It is known that every locally compact, compactly generated Abelian group G is topologically isomorphic with $\mathbf{R}^a \times \mathbf{Z}^b \times F$ for some nonnegative integers a and b and some compact Abelian group F . For the proof see [4] Theorem 9.8 p. 90. Theorem 1 and theorem of ERDŐS imply immediately

Theorem 2. *Let G be an additively written, metrizable, locally compact, compactly generated Abelian topological group. Let us assume that the conditions (1.1), (1.8) for $\varphi: \mathbf{N} \rightarrow G$ hold. Then the assertions stated in Theorem 1 remain true.*

PROOF. Since $G = \mathbf{R}^a \times \mathbf{Z}^b \times F$, it is enough to prove the assertion for $G = \mathbf{R}, \mathbf{Z}, F$. This was proved for $G = \mathbf{R}, F$ earlier. Furthermore for $G = \mathbf{Z}$, $\Delta\varphi \rightarrow 0$ implies that $\varphi(n) \equiv 0$. ■

This completes the proof.

2. Let G be an Abelian topological group. G is said to be solenoidal, if there exists a continuous homomorphism of $\mathbf{R}$ into G , such that the image $\tau(\mathbf{R})$ is dense in G . The mapping $v \rightarrow e^v$ is a topological isomorphism of $\mathbf{R}$ onto $\mathbf{R}_x$, so we may change $\mathbf{R}$ by $\mathbf{R}_x$ in the definition of solenoidal groups. It is known that a compact Abelian group is solenoidal if and only if it is connected and its cardinality is not greater than the continuum. For the proof see [4] Theorem 25.18.

Let us assume that the conditions of Theorem 1 or 2 are satisfied. Let $H := \varphi(\mathbf{R}_x)$ be the image of the continuous homomorphism. Then the mapping $\varphi: \mathbf{R}_x \rightarrow H$ is a topological isomorphism, or H^- (H^- denotes the closure of H) is a compact subgroup in G . (See [4], Theorem 9.1.)

These theorems allow us to give a quite complete characterization of the sets that can occur as images for a suitable continuous homomorphism $\varphi: \mathbf{R}_x \rightarrow G$.

For a mapping $\varphi: \mathbf{N} \rightarrow G$ under the conditions (1.1), (1.8) let $K_\varphi = \text{closure of } \varphi(\mathbf{R}_x)$, where the extension of the domain of φ to $\mathbf{R}_x$ is defined by (1.6).

Theorem 3. 1. *Let G be a metrically compact Abelian group, S be a subgroup in G . Then there exists a completely additive function $\varphi: \mathbf{N} \rightarrow G$ satisfying (1.8) and $K_\varphi = S$ if and only if S is compact, connected and the cardinality of it is not greater than the continuum.*

2. Let G be a metrizable, locally compact, compactly generated Abelian group, S be a subgroup in G . Then there exists a completely additive function $\varphi: \mathbb{N} \rightarrow G$ satisfying (1.8) and $K_\varphi = S$ if and only if

- a) S is topologically isomorphic to $\mathbb{R}$, or
- b) S is compact, connected and the cardinality of it is not greater than the continuum.

3. Let G be an arbitrary Abelian topological group, $\varphi, \psi: \mathbb{N} \rightarrow G$ be completely additive functions, such that

$$(3.1) \quad \psi(n+1) - \varphi(n) \rightarrow 0 \quad (n \rightarrow \infty).$$

Then $\psi(n) = \varphi(n) \quad \forall n \in \mathbb{N}$.

This assertion is almost obvious. Let $H(n) := \psi(n) - \varphi(n)$.

Starting from the relation

$$\begin{aligned} \psi(n+1) - \varphi(n) &= -H(2) + \psi(2n+2) - \varphi(2n) = \\ &= -H(2) + \psi(2n+2) - \varphi(2n+1) - H(2n+1) + \psi(2n+1) - \varphi(2n), \end{aligned}$$

from (3.1) we deduce that

$$(3.2) \quad H(2n+1) + H(2) \rightarrow 0 \quad (n \rightarrow \infty).$$

Let m be an arbitrary odd number. From (3.2) we get that $H((2n+1)m) \rightarrow -H(2)$ ($n \rightarrow \infty$), $H(2n+1) \rightarrow -H(2)$ ($n \rightarrow \infty$), and so that $H(m) = 0$. So $H(2n+1) = 0$ ($\forall n$), and so $H(2) = 0$.

4. In a recent joint paper [5] we proved the following assertion.

Let G be a metrically compact Abelian group. Let us assume that the completely additive function $\varphi: \mathbb{N} \rightarrow G$ satisfies the following requirement: if $n_1 < n_2 < \dots$ is such an infinite subsequence in $\mathbb{N}$ for which $\lim_k \varphi(n_k)$ exists, then there exists $\lim_k \varphi(n_k + 1)$ as well. Then (1.8) holds.

4. Additive functions defined over the Gaussian integers

Let $\mathbb{F}$ be the nonzero Gaussian integers, $\mathbb{K}$ be the multiplicative group of nonzero Gaussian rationals, $\mathbb{C}_x$ be the multiplicative group of nonzero complex numbers, $\mathbb{C}$ be the additive group of complex numbers.

Let G be an Abelian group for the addition. A mapping $\varphi: \mathbb{F} \rightarrow G$ is said to be an additive function, if

$$(4.1) \quad \varphi(ab) = \varphi(a) + \varphi(b) \quad (\forall a, b \in \mathbb{F})$$

holds.

Let $\beta \in \mathbb{F}$ be fixed,

$$(4.2) \quad \Delta_\beta \varphi(\alpha) := \varphi(\alpha + \beta) - \varphi(\alpha) \quad (\alpha, \alpha + \beta \in \mathbb{F}).$$

We should like to determine those φ for which

$$(4.3) \quad \Delta_\beta \varphi(\alpha) \rightarrow 0 \quad (|\alpha| \rightarrow \infty)$$

holds.

Since

$$\Delta_\beta \varphi(\alpha\beta) = \Delta_1 \varphi(\alpha),$$

therefore we may restrict ourselves to the case $\beta=1$,

$$(4.4) \quad \Delta\varphi(\alpha) := \varphi(\alpha+1) - \varphi(\alpha) \rightarrow 0 \quad (|\alpha| \rightarrow \infty).$$

It is easy to prove that for $G=\mathbf{R}$ (4.4) implies that

$$\varphi(\alpha) = c \log |\alpha|.$$

Let us consider now the case $G=T$, written multiplicatively. E. WIRSING proved [3] the next theorem, which we quote now as

Lemma 4. *Let $V: \mathbf{F} \rightarrow T$ be a completely multiplicative function, such that*

$$(4.5) \quad V(\alpha+1)V^{-1}(\alpha) \rightarrow 1 \quad \text{as } |\alpha| \rightarrow \infty, \quad \alpha \in \mathbf{F}.$$

Then $V(\alpha) := e^{i\tau \log |\alpha|} \cdot e^{ik \arg \alpha}$, where τ a real constant, k is a rational integer.

By repeating the argument used in Section 1 and 2 from Lemma 4 we can deduce easily

Theorem 4. *Let G be a metrically compact Abelian group, $\varphi: \mathbf{F} \rightarrow G$ be a completely additive function, satisfying (4.4). Then the domain of φ can be extended on $\mathbf{K}$ by*

$$\varphi\left(\frac{\alpha}{\beta}\right) := \varphi(\alpha) - \varphi(\beta) \quad (\alpha, \beta \in \mathbf{F}),$$

and on $\mathbf{C}_x$ by

$$(4.6) \quad \varphi(\gamma) = \lim_{\substack{P_n \rightarrow \gamma \\ P_n \in \mathbf{K}}} \varphi(P_n), \quad (\gamma \in \mathbf{C}_x)$$

uniquely. The mapping $\varphi: \mathbf{C}_x \rightarrow G$ is a continuous homomorphism.

Inversely, let $\varphi: \mathbf{C}_x \rightarrow G$ be a continuous homomorphism. Then the restriction of φ on $\mathbf{F}$ defines a completely additive function, for which (4.4) holds.

References

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