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On the context-freeness of a class of primitive words

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Abstract. Let Q be the set of primitive words over a finite alphabet X having at least two letters. It was conjectured in [2] that intersecting Q with the bounded language $L_n = (ab^*)^n$, we get a context-free language ($a, b \in X, n \in \mathbb{N}$). We proved in [2] that the conjecture is true if n is a product of two prime-powers. Here we generalize this result for the case when n is a product of three prime-powers.

0. Introduction

The properties of primitive words were investigated by several authors. In the papers [1] [2] [3] the still unsolved problem was studied: whether the set Q of all primitive words is non-context-free (we conjecture this). A well-known method to decide on context-freeness is that we investigate not Q itself, but the intersection of Q with a regular language: If Q is context-free, then this intersection must be context-free as well. We considered in [2] the context-freeness of languages $Q_n = Q \cap (ab^*)^n$ and proved that if n is a product of two prime-powers then Q_n is context-free. Our results suggest that Q_n is context-free for an arbitrary positive natural number n, therefore this intersection seems not to be suitable for the proof of the original conjecture on non-context-freeness of Q. However the problem of context-freeness of Q_n may be a touchstone for methods used to prove context-freeness of bounded languages.

1. Preliminaries

Let X be a fixed nonempty alphabet having at least two letters. A *primitive word* (over X) is a nonempty word not of the form w^m for any (nonempty) word w and integer $m \geq 2$. The set of all primitive words over X will be denoted by Q. Let $a, b \in X$, $a \neq b$, $n \in \{1, 2, \dots\}$, and W be an arbitrary subset of the language $(ab^*)^n$. For $w \in W$ let $w = ab^{e_0} \cdots ab^{e_{n-1}}$ and denote the set of all vectors of the form $e(w) = (e_0, \dots, e_{n-1})$ by $E(W)$.

The index set $\underline{n} = \{0, \dots, n-1\}$ will be considered as a "cyclically ordered" set, i.e. the "open intervalls" (i, j) of $\underline{n}$ are defined by $(i, j) = \{k \mid i < k < j\}$ for $i < j$ and by $(i, j) = \{k \mid k < j \text{ or } k > i\}$ for $i > j$. We will use the notations $[i, j), (i, j]$ and $[i, j]$ for the "half closed" and "closed" intervals defined in the usual manner: $[i, j) = \{i\} \cup (i, j)$, $(i, j] = (i, j) \cup \{j\}$ and $[i, j] = \{i\} \cup (i, j) \cup \{j\}$.

We say that the pairs of indices $\{i, j\}$ and $\{k, l\}$ are *crossing* if $k \in (i, j)$ and $l \in (j, i)$ or if $l \in (i, j)$ and $k \in (j, i)$. The subsets R and T of $\underline{n}$ are said to be *non-nested* sets, if there exist two elements i and j of $\underline{n}$ for which $S \subseteq [i, j)$ and $T \subseteq [j, i)$ holds. For the expression "non-nested" we will use the abbreviation n.n.. If there are given more than two subsets of $\underline{n}$, then for the expression *pairwise non-nested* we will use the abbreviation p.n.n.. Addition, summation and multiplication in $\underline{n}$ are meant as (mod n)-operations.

Using minor modifications of known methods in Ginsburg [5], W can be proved context-free by proving that $E(W)$ is a finite union of stratified linear sets. A set $F \subseteq N^s$, where $N = \{0, 1, \dots\}$, $s \geq 1$ is called a *stratified linear* set iff either $F = \emptyset$ or there are $r \geq 1$ and $v_0, \dots, v_r \in N^s$ such that

$$F = \left\{ v_0 + \sum_{i=1}^{r} k_i v_i \mid k_i \geq 0 \right\}$$

and for the vector set $V = \{v_i \mid 1 \leq i \leq r\}$

(1) every $v \in V$ has at most two nonzero components,

(2) if $u = (u_0, \dots, u_{s-1})$ and $w = (w_0, \dots, w_{s-1})$ are two vectors from V and $\{i, j\}$, $\{k, l\}$ are crossing index-pairs then $u_i w_k u_j w_l = 0$.

Sets which are finite unions of stratified linear sets are called *stratified semilinear sets.*

2. Stars, boxes and differences

Let m be a divisor of n and consider the (ordered) subset $S_m = \langle s_0, \dots, s_{m-1} \rangle$ of $\underline{n}$. S_m is an *m-star* if $s_k - s_{k-1} = n/m$ holds for every $1 \leq k \leq m-1$. The index-set $\underline{n}$ may be partitioned into n/m pairwise disjoint m-stars. An m-star will be represented by one of its elements: If $k \in S_m$ then we say that S_m is an $S_m(k)$-star. This notation is ambiguous, e.g. $S_m(k) = S_m(k + l)$ if l is of the form $l = in/m$, $i = 0, \dots, m-1$. If d is a divisor of m and $S_m \cap S_d \neq \emptyset$ then $S_d \subseteq S_m$.

Let $p_1, \ldots, p_\nu$ be pairwise distinct prime divisors of n and let $\xi \in \underline{n}$. We define the *ν-box* $B(\xi; p_1, \ldots, p_\nu)$ as follows:

$$B(\xi; p_1, \ldots, p_\nu) = \Big\{ \xi - \sum_{i=1}^{\nu} \varepsilon_i n/p_i \mid \varepsilon_1 \in \{0,1\}, \ldots, \varepsilon_\nu \in \{0,1\} \Big\}.$$

For $\pi = \{p_1, \ldots, p_\nu\}$ we will use the abbreviation $B(\xi; \pi)$ for $B(\xi; p_1, \ldots, p_\nu)$. If $\pi = \emptyset$ then let $B(\xi; \emptyset) = \{\xi\}$.

To every vector $e = (e_0, \ldots, e_{n-1})$ and ν-box $B = B(\xi; p_1, \ldots, p_\nu)$ there corresponds a *difference* $\Delta(B, e)$ defined by the rule

$$\Delta(B, e) = \sum_{\rho \in B} (-1)^{\sigma(\rho)} e_\rho, \text{ where } \sigma(\rho) = \sum_{i=1}^{\nu} \varepsilon_i, \text{ if } \rho = \xi - \sum_{i=1}^{\nu} \varepsilon_i n/p_i$$

In other words, a difference defined for a vector e and a box B is a signed sum of such components of e the indices of which belong to B, and if the index-pair $\{i, k\}$ is an "edge" of the box B then the corresponding members e_i and e_k of the sum have opposite signs.

Example. Let $n = 105 = 3 \cdot 5 \cdot 7$, and select $p = 3$, $q = 5$ and $r = 7$. Then the set $S_{15} = S_{15}(3) = \langle 3, 10, 17, 24, 31, 38, 45, 52, 59, 66, 73, 80, 87, 94, 101\rangle$ is a 15-star and the 5-star $S_5 = S_5(3) = \langle 3, 24, 45, 66, 87\rangle$ is a substar of S_{15}.

Let $\xi = 77$, $\nu = 3$, then the 3-box $B(77; 3, 5, 7)$ is the folowing set:

ε_1 ε_2 ε_3	$\rho = \xi \ - (\varepsilon_1 \cdot n/3 + \varepsilon_2 \cdot n/5 + \varepsilon_3 \cdot n/7)$	$\sigma(\rho)$
0 0 0	$\rho = 77 - (0 \cdot 35 + 0 \cdot 21 + 0 \cdot 15) = 77$	0
0 0 1	$\rho = 77 - (0 \cdot 35 + 0 \cdot 21 + 1 \cdot 15) = 62$	1
0 1 0	$\rho = 77 - (0 \cdot 35 + 1 \cdot 21 + 0 \cdot 15) = 56$	1
0 1 1	$\rho = 77 - (0 \cdot 35 + 1 \cdot 21 + 1 \cdot 15) = 41$	2
1 0 0	$\rho = 77 - (1 \cdot 35 + 0 \cdot 21 + 0 \cdot 15) = 42$	1
1 0 1	$\rho = 77 - (1 \cdot 35 + 0 \cdot 21 + 1 \cdot 15) = 27$	2
1 1 0	$\rho = 77 - (1 \cdot 35 + 1 \cdot 21 + 0 \cdot 15) = 21$	2
1 1 1	$\rho = 77 - (1 \cdot 35 + 1 \cdot 21 + 1 \cdot 15) = 6$	3

B(77;3,5,7)={6,21,27,41,42,56,62,77}

The difference $\Delta(B, e)$ corresponding to the box B is the following:

$$\Delta(B, e) = e_{77} - e_{62} - e_{56} + e_{41} - e_{42} + e_{27} + e_{21} - e_6$$

In order to prove that a given subset of N^n is stratified linear the following result is useful:

Lemma 1. *For $i = 0, \dots, n-1$ let the δ_i be arbitrarily prescribed "signs", i.e. let $\delta_i \in \{0, 1, -1\}$ and consider the set $E = \{e = (e_0, \dots, e_{n-1}) \mid \delta_0 e_0 + \dots + \delta_{n-1} e_{n-1} \neq 0\}$. Then E is a stratified semilinear set.*

For the proof of Lemma 1 see [4].

Corollary 2. *Let B be an arbitrary box.*
If $E(B) = \{e = (e_0, \dots, e_{n-1}) \mid \Delta(B, e) \neq 0\}$, then $E(B)$ is a stratified semilinear set.

Corollary 3. *Let $B_1, \dots, B_z$ be a collection of pairwise non-nested boxes. Then the set*

$$E(B_1, \dots, B_z) = \{e \mid \Delta(B_1, e) \neq 0, \dots, \Delta(B_z, e) \neq 0\}$$

is a stratified semilinear set.

Let $n = {p_1}^{r_1} \dots {p_s}^{r_s}$ and $\Pi = \{\pi_1, \dots, \pi_\gamma\}$ be a partition of the set $\{p_1, \dots, p_s\}$, and let ξ_i be an arbitrary element of $\underline{n}$. To every pair $(\xi_i; \pi_i)$ there corresponds a box $B(\xi_i; \pi_i)$. For every partition Π we consider such collections of $B(\xi_i, \pi_i)$-s which are pairwise non-nested sets. The union – over the pairs (ξ_i, π_i) for fixed $\{\pi_1, \dots, \pi_\gamma\}$ – of the corresponding $E(B(\xi_1, \pi_1), \dots, B(\xi_\gamma, \pi_\gamma))$-s is denoted by $E(\Pi)$:

$$E(\Pi) = \bigcup \{E(B(\xi_1, \pi_1), \dots, B(\xi_\gamma, \pi_\gamma)) \mid B(\xi_1, \pi_1), \dots, B(\xi_\gamma, \pi_\gamma) \text{ are p.n.n. sets}\}.$$

By Corollary 3. the vector set $E(\Pi)$ is stratified semilinear for every partition Π.

Example. Let $n = 30$, i.e. $p = 2$, $q = 3$ and $r = 5$. The partitions of the set $\{2, 3, 5\}$ are as follows: $\Pi_1 = \{\{2\}, \{3\}, \{5\}\}$, $\Pi_2 = \{\{2\}, \{3, 5\}\}$, $\Pi_3 = \{\{3\}, \{2, 5\}\}$, $\Pi_4 = \{\{2, 3\}, \{5\}\}$ and $\Pi_5 = \{\{2, 3, 5\}\}$.

$$E(\Pi_1) = \bigcup \{\{e = (e_0, \dots, e_{n-1}) \mid e_{\xi_1} - e_{\xi_1 - 15} \neq 0,\ e_{\xi_2} - e_{\xi_2 - 10} \neq 0,\ e_{\xi_3} - e_{\xi_3 - 6} \neq 0\} \mid \{\xi_1, \xi_1 - 15\}, \{\xi_2, \xi_2 - 10\} \text{ and } \{\xi_3, \xi_3 - 6\} \text{ are p.n.n. sets}\}.$$

$E(\Pi_2) = \emptyset$ since the boxes $B(\xi_1; 2) = \{\xi_1, \xi_1 - 15\}$ and $B(\xi_2; 3, 5) = \{\xi_2, \xi_2 - 6, \xi_2 - 10, \xi_2 - 16\}$ are for every choice of ξ_1 and ξ_2 nested sets. Similarly, $E(\Pi_3) = \emptyset$.

$E(\Pi_4) = \bigcup \{\{e = (e_0, \dots, e_{n-1}) \mid e_{\xi_1} - e_{\xi_1 - 10} - e_{\xi_1 - 15} + e_{\xi_1 - 25} \neq 0$, $e_{\xi_2} - e_{\xi_2 - 6} \neq 0 \mid \{\xi_1, \xi_1 - 10, \xi_1 - 15, \xi_1 - 25\}$ and $\{\xi_2, \xi_2 - 6\}$ are n.n. sets$\}$.

$E(\Pi_5) = \bigcup\{\{e = (e_0, \dots, e_{n-1}) \mid e_\xi - e_{\xi-6} - e_{\xi-10} - e_{\xi-15} + e_{\xi-16} + e_{\xi-21} + e_{\xi-25} - e_{\xi-1} \neq 0\} \mid \xi \in \underline{n}\}$.

Chains of boxes. Let $B_1 = B(\xi; p_1, \dots, p_s)$ and q a fixed element of the set $\pi = \{p_1, \dots, p_s\}$. Consider the sequence $\Lambda = \Lambda(B_1, q) = (B_1, \dots, B_\tau)$ where $B_i = B(\xi + (i-1)n/q; \pi)$ if $i = 1, \dots, \tau$. We will refer to Λ as a *chain of boxes.* If $\tau = q$ then we will say that Λ is a *full chain.*

In our proofs we will frequently use the following

Lemma 4. *Let $\Lambda = \Lambda(B_1, q) = (B_1, \dots, B_q)$ be a full chain of boxes. For $i = 1, \dots, q$ we consider the differences $\Delta(B_i, e)$ corresponding to B_i. Then*

$$\sum_{i=1}^{q} \Delta(B_i, e) = 0 \quad \textit{holds for every} \mid e \in N^n.$$

Proof. Let $B_i = B(\xi_i; \pi)$ and $q \in \pi$. Then

$$\Delta(B_i, e) = \Delta(B(\xi_i; \pi\backslash\{q\}), e) - \Delta(B(\xi_i - n/q; \pi\backslash\{q\}), e)$$

holds by the definition of $\Delta(B_i, e)$. This means, that in the sum

$$\sum_{i=1}^{q} \Delta(B_i, e) = \sum_{i=1}^{q} (\Delta(B(\xi_i; \pi\backslash\{q\}), e) - \Delta(B(\xi_{i-1}; \pi\backslash\{q\}), e))$$

every term $\Delta(B(\xi_i; \pi\backslash\{q\}), e)$ appears twice but with opposite signs.

Example. Let $n = 105 = 3 \cdot 5 \cdot 7$, and consider the 2-box $B(58; 3, 5) = \{58, 37, 23, 2\}$. The chain $\Lambda(B(58; 3, 5), 3)$ is the following:
$\Lambda(B(58; 3, 5), 3) = \{\{58, 37, 23, 2\}, \{93, 72, 58, 37\}, \{23, 2, 93, 72\}\}$.
The sum of the corresponding differences is

$$(e_{58} - e_{37} - e_{23} + e_2) + (e_{93} - e_{72} - e_{58} + e_{37}) + (e_{23} - e_2 - e_{93} + e_{72}) = 0.$$

We need to define some special subsets of a given chain Λ of boxes, consisting of such members of Λ which are non-nested relative to a given pair $\{i, j\}$ of elements in $\underline{n}$: $\Lambda\,]i, j[\, = \{B \mid B \in \Lambda,\ B$ and $\{i, j\}$ are non-nested sets$\}$.

3. The main theorem

This section is devoted to the proof of the following:

Theorem 1. *Let $a, b \in X$ $a \neq b$ and $n = p^{f_1} q^{f_2} r^{f_3}$, where p, q and r are pairwise different prime numbers, $f_1, f_2, f_3 \geq 1$. Let further $L = (ab^*)^n$. Then $Q \cap L$ is a context-free language.*

Proof. Without loss of generality we may assume that $p < q < r$. As we have seen in the special case $p = 2$, $q = 3$ and $r = 5$, the set $\{p, q, r\}$ has five different partitions: $\Pi_1 = \{\{p\}, \{q\}, \{r\}\}$, $\Pi_2 = \{\{p\}, \{q, r\}\}$, $\Pi_3 = \{\{q\}, \{p, r\}\}$, $\Pi_4 = \{\{r\}, \{p, q\}\}$ and $\Pi_5 = \{\{p, q, r\}\}$. Let

$$E(n) = \bigcup \{E(\Pi_i) \mid i = 1, \ldots, 5\}. \tag{2.1}$$

We will prove that

$$E(Q \cap L) = E(n) \text{ if } pq \neq 6 \text{ and} \tag{2.2}$$

$$E(Q \cap L) = E(n) \cup C \text{ if } pq = 6, \tag{2.3}$$

where $C = \bigcup \{\{e = (e_0, \ldots, e_{n-1}) \mid e_{j_2} - e_{i_1} \neq 0, e_{j_1} - e_{i_2} \neq 0, e_{j_3} - e_{i_3} \neq 0\} \mid i_2 - i_1 = i_1 - j_2 = j_2 - j_1 = n/6, j_3 - i_3 = n/r$, the sets $\{i_1, i_2, j_1, j_2\}$ and $\{j_3, i_3\}$ are n.n. sets$\}$.

If $e \in N^n \backslash E(Q \cap L)$ then the function φ defined on $\underline{n}$ by the rule $\varphi(i) = e_i$ is an n/p, n/q, or n/r-periodic function of i. Using this fact it is easy to show that $e \notin E(n)$ and – in case of $pq = 6$ – that $e \notin (E(n) \cup C)$. Therefore $E(n) \subseteq E(Q \cap L)$ if $pq \neq 6$ and $E(n) \cup C \subseteq E(Q \cap L)$ if $pq = 6$.

In contradiction to (2.2) and (2.3) let us now assume that

$$e^* = ({e_0}^*, \ldots, e^*_{n-1}) \in E(Q \cap L) \backslash E(n)$$

holds if $pq \neq 6$, or

$$e^* = (e^*_0, \ldots, e^*_{n-1}) \in E(Q \cap L) \backslash (E(n) \cup C) \quad \text{holds if } pq = 6.$$

Step 1. Since $e^* \in E(Q \cap L)$, there exists an index-pair $\{i, j\}$ such that $j - i = n/r$ and $e^*_j - e^*_i \neq 0$ holds. Let $S_r(i) = \langle s_0, \ldots, s_{r-1} \rangle$. From the definition of $S_r(i)$ it follows that $j \in S_r(i)$. We will show that there exists another index-pair $\{k, l\}$ with the same properties, i.e. such that $l - k = n/r$ and $e^*_l - e^*_k \neq 0$ holds. Let us consider the equality $(e^*_{s_1} - e^*_{s_0}) + \ldots + (e^*_{s_0} - e^*_{s_{r-1}}) = 0$. If on the left side of the equality one term differs from zero, then another such term must exist as well.

Step 2. We say that the pq-star $S_{pq} = \langle s_0, \ldots, s_{pq-1} \rangle$ is a *rigid star* relative to the vector $e = (e_0, \ldots, e_{n-1})$ if for the elements $s_\alpha, s_{\alpha+q}, s_\beta$, and $s_{\beta+q}$ of S_{pq}

$$e_{s_{\alpha+q}} - e_{s_\alpha} = e_{s_{\beta+q}} - e_{s_\beta} \quad \text{holds whenever} \quad \alpha \equiv \beta \pmod{p}. \tag{2.4}$$

In Steps 1–7 we will show that every S_{pq}-star of $\underline{n}$ is a rigid star relative to the vector e^*.

Case 1. In the following Steps 3–5 let $p = 2$.

Step 3. Let $\{i, j\}$ and $\{k, l\}$ as in Step 1, and consider the $2q$-star $S_{2q} = \langle s_0, \dots, s_{2q-1}\rangle$. Denote the set of all one-boxes of the form $B(\xi, 2) = \{\xi,\ \xi - n/2\}$ contained in S_{2q} by Φ and consider the subsets of Φ consisting of such boxes $B = \{s_\alpha, s_{\alpha+q}\}$, for which B and $\{i, j\}$ are non-nested sets by $\Phi(i, j)$ (in case of $\{k, l\}$ and $\{s_\alpha, s_{\alpha+q}\}$ by $\Phi(k, l)$ respectively). We will say that the star S_{2q} is *well-positioned* relative to the intervals $[i, j]$ and $[k, l]$ if

$$\Phi = \Phi(i, j) \cup \Phi(k, l) \tag{2.5}$$

$$\Phi(i, j) \cap \Phi(k, l) \neq \emptyset. \tag{2.6}$$

We will show that

(2.7) *If the 2q-star S_{2q} is well-positioned relative to $[i, j]$ and $[k, l]$ then S_{2q} is a rigid star relative to the vector e^*.*

Let us consider the chain $\Lambda = \Lambda(B_1, q) = (B_1, \dots, B_\tau)$. Here B_1 and B_τ satisfy the following conditions:

(2.8) $B_1 = B(\sigma + n/2 + n/q; 2, q) = \{\sigma + n/2 + n/q, \sigma + n/2, \sigma + n/q, \sigma\}$ and σ is the element of the set $S_{2q}\backslash[i, j]$ which lies – according to its cyclic order – nearest to j.

(2.9) $B_\tau = B(\xi; 2, q)$, where ξ is that element of the set $S_{2q}\backslash[i, j]$ which lies – according to its cyclical order – nearest to i.

Let us consider the vector set

$$E(\Pi_4) = \{\{e = (e_0, \dots, e_{n-1}) \mid \Delta(B(\xi_1; 2, q), e) \neq 0, \Delta(B(\xi_2; r), e) \neq 0\} \mid B(\xi_1; 2, q) \text{ and } B(\xi_2; r) \text{ are n.n. sets}\}.$$

Here $B(\xi_1; 2, q) = \{\xi_1,\ \xi_1 - n/q, \xi_1 - n/2, \xi_1 - n/q - n/2\}$, $B(\xi_2; r) = \{\xi_2,\ \xi_2 - n/r\}$ holds by the definition of boxes, while $\Delta(B(\xi_1; 2, q), e) = e_{\xi_1} - e_{\xi_1 - n/q} - e_{\xi_1 - n/2} + e_{\xi_1 - n/q - n/2}$ and $\Delta(B(\xi_2; r), e) = e_{\xi_2} - e_{\xi_2 - n/r}$ holds by the definition of differences.

It is easy to show that – for every $m \in \{1, \dots, \tau\}$ – the sets B_m and $B(j; r) = \{i, j\}$ are non-nested sets, therefore

$$\{e = (e_0, \dots, e_{n-1}) \mid \Delta(B_m, e) \neq 0, \Delta(B(j; r), e) \neq 0\} \subset E(\Pi_4).$$

The vector e^* is choosen such that $e^* \notin E(n)$, therefore $e^* \notin \{e = (e_0, \dots, e_{n-1}) \mid \Delta(B_m, e) \neq 0, \Delta(B(j; r), e) \neq 0\}$ holds as well. But i and j are such that $\Delta(B(j; r), e^*) = e^*_j - e^*_i \neq 0$, hence $\Delta(B_m, e^*) = 0$ for every $m \in \{1, \dots \tau\}$. Using this fact it is easy to show that (2.4) holds for the elements of $\Phi(i, j)$. By similar arguments as in the case of $\Phi(i, j)$, (2.4) can be proved for the elements of $\Phi(k, l)$ as well. Finally using (2.5) and (2.6) we can check the validity of (2.4) for the elements of Φ.

Step 4. Let S_{2qr} be an arbitrary $2qr$-star of $\underline{n}$ and let us represent S_{2qr} by its greatest element $(n-s)$: $S_{2qr} = S_{2qr}(n-s)$. Without loss of generality we may assume that (in Step 1) i, j, k and l are chosen such that $i = 0$, and $k - j < i - l$ holds. We will show that if $q < z < r$, then the $2q$-star $S_{2q}(-s + z(n/(2qr))$ is well-positioned relative to $[i, j]$ and $[k, l]$ (See for the definition Step 3). Let $\{\phi_1, \phi_2\} \in \Phi$. We prove that if $\phi_1 \in [i, j]$ then $\phi_2 \notin [k, l]$.

Assume indirectly that $\phi_1 \in [i, j]$ and $\phi_2 \in [k, l]$. Using (2.10) it is easy to see that $n/2 \leq \phi_2 < n/2 + n/2r$. But then $0 \leq \phi_1 < n/2r$ holds for $\phi_1 = \phi_2 - n/2$, contradicting the fact that $S_{2q} \cap [0, n/2r] = \emptyset$ by the choice of z. We conclude that if $\{\phi_1, \phi_2\} \notin \Phi(i, j)$ then $\{\phi_1, \phi_2\} \in \Phi(k, l)$ and therefore (2.5) is valid. It is easy to prove that $|S_{2q} \cap [i, j]| \leq 1$ and $|S_{2q} \cap [k, l]| \leq 1$ therefore $|\Phi(i, j) \backslash \Phi(k, l)| + |\Phi(k, l) \backslash \Phi(i, j)| \leq 2$. According to (2.5) $\Phi = (\Phi(i, j) \backslash \Phi(k, l)) \cup (\Phi(i, j) \backslash \Phi(k, l)) \cup (\Phi(i, j) \cap \Phi(k, l))$ holds and therefore $|\Phi(i, j) \cap \Phi(k, l)| \geq |\Phi| - 2 = q - 2 > 0$. Thus (2.6) is valid.

Case 2. In Steps 5-6 let $p > 2$.

Step 5. Without loss of generality we may assume that (in Step 1) the indices i, j, k and l are chosen such that $l = n - 1$ and $k - j \leq i - j$ hold. Let S_{pqr} be an arbitrary pqr-star and $S_{pq} = S_{pq}(\theta)$ be a pq-substar of S_{pqr} such that the element θ satisfies the inequalities $k - n/pqr \leq \theta < k$. Consider the full chain $\Lambda_\rho = \Lambda(B(\xi; p, q), \rho) = \{B_1, \ldots, B_\rho\}$ where $\xi \in S_{pq}$ and $\rho \in \{p, q\}$. The subsets $\Lambda_\rho(i, j)$ and $\Lambda_\rho(k, l)$ of boxes in $\Lambda(B_1, \rho)$ are defined by $\Lambda_\rho(i, j) = \Lambda_\rho\,]i, j[$ and $\Lambda_\rho(k, l) = \Lambda_\rho\,]k, l\,[$ respectively.

Let $\xi \in S_{pq}$ and $B = B(\xi + n/p; p)$ be an arbitrary one-box in S_{pq}. We say that B is *q-reducible* if there exists a one-box $B(\eta + n/p; p)$ such that $\xi \equiv \eta \pmod{n/q}$, $0 \leq \eta < n/q$ and $\Delta(B, e^*) = \Delta(B(\eta + n/p; p), e^*)$.

It is easy to see that for $\rho \in \{p, q\}$ and $(\mu, \nu) \in \{(i, j), (k, l)\}$ $\Lambda_\rho(\mu, \nu)$ is a chain of boxes. We show that for every $B \in \Lambda_\rho(\mu, \nu)$, $\Delta(B, e^*) = 0$ holds. Note that $e^* \notin \{\{(e_0, \ldots, e_{n-1}) \mid \Delta(B, \underline{e}) \neq 0, e_\mu - e_\nu \neq 0\} \mid B$ and $\{\mu, \nu\}$ are n.n. sets$\}$ by the definition of e^*. But B and $\{\mu, \nu\}$ are n.n. sets and $e_\mu * - e_\nu * \neq 0$ therefore $\Delta(B, e^*) = 0$. Let $\Lambda_q(\mu, \nu) = \Lambda(B(\sigma + n/p + n/q; p, q), q) = \{C_1, \ldots, C_\tau\}$, then for $1 \leq \vartheta \leq \tau$ $\Delta(C_\vartheta, e^*) = 0$ i.e. $e^*_{\sigma + (\vartheta - 1)n/q + n/p} - e^*_{\sigma + (\vartheta - 1)n/q} = e^*_{\sigma + \vartheta n/q + n/p} - e^*_{\sigma + \vartheta n/q}$ holds. We conclude that if for suitable ϑ and one-box $B = B(\xi_1; p)$ $B \subset C_\vartheta$ holds, then B is q-reducible.

Step 6. Let $\Omega = \Lambda(B(\xi, p), p) = \{B_1, \ldots, B_p\}$ be a full chain of one-boxes in S_{pq}. According to the result of Step 5 and using the fact that $n/p > n/r$ we can state that all but possibly one element of Ω are q-reducible. Without loss of generality we may assume that $B_1, \ldots$, and B_{p-1} are q-reducible. We will show that B_p is q-reducible as well. Let us consider the function ψ which is defined on the set of all one-boxes of the form $B(\xi + n/p; p)$ in S_{pq} as follows:

$$\psi(B(\xi + n/p; p)) = B(\eta + n/p; p) \text{ where } \eta \equiv \xi \pmod{n/q}$$
$$\text{and } 0 \leq \eta < n/q.$$

The q-reducibility of $B_1, \ldots$ and B_{p-1} means that $\Delta(B_m, e^*) = \Delta(\psi(B_m), e^*)$ holds if $m = 1, \ldots, p-1$. By Proposition 6

$$\Delta(B_p, e^*) = -\sum_{m=1}^{p-1} \Delta(B_m, e^*). \tag{2.14}$$

To prove that $\Delta(B_p, e^*) = \Delta(\psi(B_p), e^*)$ it is enough to show that

$$\sum_{m=1}^{p} \Delta(\psi(B_m), e^*) = \sum_{m=0}^{p-1} \Delta(B(\xi_0 + mn/pq + n/p, p), e^*) = 0, \tag{2.15}$$

where ξ_0 is the smallest element of S_{pq}.

Let us consider the full chain $\Omega' = \Lambda(B(\theta + n/q; p), p) = \{B'_1, \ldots, B'_p\}$, where $k - n/pqr \leq \theta < k$ holds (see the definition of θ in Step 5). Here the one-boxes $B'_2, \ldots, B'_p$ are q-reducible by the result of Step 5. Box $B(\theta + n/q; p, q)$ and set $\{k, l\}$ are n.n. sets, therefore $\Delta(B(\theta + n/q; p, q), e^*) = 0$, hence B'_1 is q-reducible as well. It follows by Proposition 6 that

$$\sum_{m=1}^{p} \Delta(\psi(B'_m), e^*) = \sum_{m=0}^{p-1} \Delta(B(\xi_0 + mn/pq + n/p, p), e^*) = 0 \tag{2.16}$$

and therefore (2.15) is valid.

Step 7. In Steps 1–6 we proved that every pqr-star contains a rigid pq-star as a substar. Let S_{pqr} be an arbitrary pqr-star of $\underline{n}$, and $S_{pq}(s)$ a rigid substar of S_{pqr}. We prove that all pq-substars of S_{pqr} are rigid stars. For $m = 0, \ldots, r-1$ let us consider the pq-stars $S_{pq}(s + mn/r)$. Assume that there exists an m_0 for which $S_{pq}(s + (m_0 - 1)n/r)$ is rigid, but $S_{pq}(s + m_0 n/r)$ is not, i.e.: there exists a j_0 such that $j_0 \in S_{pq}(s + m_0 n/r)$ and $e^*_{j_0} - e^*_{j_0 - n/p} \neq e^*_{j_0 - n/q} - e^*_{j_0 - n/q - n/p}$ holds. It is easy to see that $j_0 - n/r \in$

$S_{pq}(s+(m_0-1)n/r)$ and therefore $e^*_{j_0-n/r} - e^*_{j_0-n/r-n/p} = e^*_{j_0-n/r-n/q} - e^*_{j_0-n/r-n/q-n/p}$ holds by the rigidity of $S_{pq}(s+(m_0-1)n/r)$. But then $\Delta(B(j_0;p,q,r),e^*) \neq 0$, therefore $e^* \in \Pi_5$, which is a contradiction.

Step 8. Using the fact that $e^* \in Q$ it is easy to prove that there exist boxes $B_r = B(\xi_r;r), B_q = B(\xi_q;q)$ and $B_p = B(\xi_p;p)$, such that $\Delta(B_r,e^*) \neq 0$, $\Delta(B_q,e^*) \neq 0$ and $\Delta(B_p,e^*) \neq 0$ hold. Let us fix the box B_r and consider for $\mu = 1,\ldots,p$ the boxes $B_q(\mu) = B(\xi_q+(\mu-1)n/p;q)$ and for $\nu = 1,\ldots,q$ the boxes $B_p(\nu) = B(\xi_p+(\nu-1)n/q;p)$. Using the fact that every pq-star is a rigid star it is easy to prove that for every $\mu \in \{1,\ldots,p\}$, $\Delta(B_q(\mu),e^*) = \Delta(B_q(1),e^*) \neq 0$ and for every $\nu \in \{1,\ldots,q\}$ $\Delta(B_p(\nu),e^*) = \Delta(B_p(1),e^*) \neq 0$. An elementary computation shows that if $pq \neq 6$ then there exist indices μ_0 and ν_0 such that the boxes $B_q(\mu_0)$, $B_p(\nu_0)$ and B_r are p.n.n. sets. But then $e^* \in E(\Pi_1)$, again a contradiction. Similarly, the case $pq = 6$ leads to the contradiction that $e^* \in C$. $\square$

4. Conclusions

The proof of Theorem 1 has some ad hoc elements. To get a development in the general case the systematic investigation of properties of boxes and differences seems to be necessary.

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