

Convolution of temperate distributions

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Let $\|\cdot\|_0$ be the norm in the space $L^2(R^n)$ of square integrable functions. We put $L_q = \{\varphi: R^n \rightarrow C; \|\varphi\|_q^2 = \sum_{|\alpha+\beta| \leq q} \|x^\alpha D^\beta \varphi\|_0^2 < +\infty\}$, $q \in N$. Here $D^\beta \varphi$ is the generalized derivative defined by Sobolev in [1]. Each space L_q , and its dual L_{-q} , is Hilbert. The $\text{proj} \lim_{q \rightarrow \infty} L_q = \mathcal{S}$ is the space of rapidly decreasing functions, and the $\text{ind} \lim_{q \rightarrow \infty} L_{-q} = \mathcal{S}'$ is the space of temperate distributions. The Fourier transformation $\mathcal{F}: L_k \rightarrow L_k$ is a topological isomorphism for any integer k .

For $\varphi \in \mathcal{S}$, $f \in \mathcal{S}'$, we use the dilation $d_\lambda \varphi(x) = \varphi(\lambda x)$, $x \in R^n$, $\lambda \in R$, $\lambda \neq 0$, $(d_\lambda f)\varphi = |\lambda|^{-n} f(d_{\lambda^{-1}} \varphi)$ and translation $(\tau_h \varphi)(x) = \varphi(x-h)$, $x, h \in R^n$, $(\tau_h f)\varphi = f(\tau_{-h} \varphi)$. It is convenient to introduce the weight-function $W(x) = (1+|x|^2)^{1/2}$, $x \in R^n$.

Definition. Let $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. The C^∞ -function $x \mapsto f(\tau_x d_{-1} \varphi)$ is called convolution of f and φ and is denoted by $f * \varphi$. For each $p, q \in N$, we define

$$0_{p,q}^* = \{f \in \mathcal{S}'; \|f\|_{p,q}^* = \sup\{\|f * \varphi\|_q; \varphi \in \mathcal{S}, \|\varphi\|_p \leq 1\} < +\infty\}.$$

The space $\mathcal{S}$ is dense in L_p . Hence for each $f \in 0_{p,q}^*$ the continuous map $\varphi \mapsto f * \varphi: (\mathcal{S}, \|\cdot\|_p) \rightarrow L_q$ can be continuously extended to L_p . Therefore $0_{p,q}^*$ is a subspace of $\mathcal{L}(L_p, L_q)$. We denote the norm of $\mathcal{L}(L_p, L_q)$ restricted to $0_{p,q}^*$ by $\|\cdot\|_{p,q}^*$.

The identity maps $\text{id}: 0_{p,q+1}^* \rightarrow 0_{p,q}^* \rightarrow 0_{p+1,q}^*$, $p, q \in N$ are continuous. Hence the limits $0_q^* = \text{ind} \lim_{p \rightarrow \infty} 0_{p,q}^*$ and $0^* = \text{proj} \lim_{q \rightarrow \infty} 0_q^*$ make sense.

Definition. Let $f \in 0_{p,q}^*$ and $g \in L_{-q}$. The functional $\varphi \mapsto g(d_{-1} f * \varphi): L_p \rightarrow C$ is called the convolution of f and g . We denote it by $f * g$.

For $f \in 0_{p,q}^*$ and $g \in \mathcal{S}$ we have two definitions of $f * g$. They coincide. Also, if f and g are both functions, our convolution $f * g$ is the same as the convolution defined by an integral.

Theorem 1. A linear map $L: L_{-q} \rightarrow L_{-p}$, resp. $L: L_{-q} \rightarrow \mathcal{S}'$, $p, q \in N$, is translation invariant and continuous iff there exists a unique $f \in 0_{p,q}^*$, resp. $f \in 0_q^*$, such that $Lg = f * g$, $g \in L_{-q}$.

PROOF. The only if part is evident. Let $L: L_{-q} \rightarrow L_{-p}$ be translation invariant and continuous. Its conjugate $L^*: L_p \rightarrow L_q$ is then also translation invariant and continuous. Hence $L^* \varphi \in C^\infty(R^n)$ for any $\varphi \in \mathcal{S}$. The continuous functional $\varphi \mapsto (L^* d_{-1} \varphi)(0): \mathcal{S} \rightarrow C$ defines a distribution $f \in \mathcal{S}'$.

For any $\varphi \in \mathcal{S}$, $x \in \mathbb{R}^n$, we have $(f * \varphi)(x) = f(\tau_x d_{-1} \varphi) = (L^* d_{-1} \tau_x d_{-1} \varphi)(0) = (L^* \tau_{-x} \varphi)(0) = \tau_{-x}(L^* \varphi)(0) = (L^* \varphi)(x)$. Further,

$$\|f\|_{p,q}^* = \sup \{\|L^* \varphi\|_q, \varphi \in \mathcal{S}, \|\varphi\|_p \leq 1\} = \|L^*\|, \text{ i.e., } f \in 0_{p,q}^*.$$

The uniqueness of f is evident and the respective part of the theorem immediately follows.

Definition. Let $\mathcal{F}$ be the Fourier transformation with the kernel $\exp(-2\pi i x, y)$ and Δ the Laplace operator. Each map $f \mapsto (1 - \Delta)^{-k} f = \mathcal{F}^{-1}(d_{2\pi} W^{-2k} \cdot \mathcal{F} f): \mathcal{S}' \rightarrow \mathcal{S}'$, $k \in \mathbb{N}$, is injective. We define spaces $(1 - \Delta)^{-k} L_{-q} = \{(1 - \Delta)^{-k} f; f \in L_{-q}\}$ and $(1 - \Delta)^k L_q = \{f \in \mathcal{S}'; (1 - \Delta)^{-k} f \in L_q\}$, $k, q \in \mathbb{N}$, and provide them with the topologies which make the operator $(1 - \Delta)^{-k}$ a topological isomorphism.

For each $k, q \in \mathbb{N}$, the spaces $(1 - \Delta)^{-k} L_{-q}$ and $(1 - \Delta)^k L_q$ are Hilbert and mutually dual with the duality form:

$$\langle u, v \rangle \mapsto ((1 - \Delta)^k u), ((1 - \Delta)^{-k} v): (1 - \Delta)^{-k} L_{-q} \times (1 - \Delta)^k L_q \rightarrow \mathbb{C}.$$

Lemma. $\mathcal{S}$ is dense in each $(1 - \Delta)^{-k} L_{-q}$.

PROOF. The operator $(1 - \Delta)^{-k}: \mathcal{S} \rightarrow \mathcal{S}$ is bijective. Since $\mathcal{S}$ is dense in L_{-q} , $\mathcal{S} = (1 - \Delta)^{-k} \mathcal{S}$ is dense in $(1 - \Delta)^{-k} L_{-q}$.

Proposition. For any $k, q \in \mathbb{N}$, we have:

- (1) $L_q \subset (1 - \Delta) L_q \subset (1 - \Delta)^2 L_q \subset \dots$ with all maps $\text{id}: (1 - \Delta)^k L_q \rightarrow (1 - \Delta)^{k+1} L_q$ continuous.
- (2) $L_{-q} \supset (1 - \Delta)^{-1} L_{-q} \supset (1 - \Delta)^{-2} L_{-q} \supset \dots$ with all maps $\text{id}: (1 - \Delta)^{-k-1} L_{-q} \rightarrow (1 - \Delta)^{-k} L_{-q}$ continuous.
- (3) $\text{ind} \lim_{k \rightarrow \infty} (1 - \Delta)^k L_q$ is the strong dual of the Fréchet space $\text{proj} \lim_{k \rightarrow \infty} (1 - \Delta)^{-k} L_{-q}$.

PROOF. (1) & (2) are evident.

(3) follows from [3, CH IV, Th. 4.4] since, by lemma,

$$\text{proj} \lim_{k \rightarrow \infty} (1 - \Delta)^{-k} L_{-q} \text{ is dense in each } (1 - \Delta)^{-k} L_{-q}.$$

Theorem 2. $\text{ind} \lim_{k \rightarrow \infty} (1 - \Delta)^k L_q = 0_q^*$, $q \in \mathbb{N}$.

PROOF. Take $f \in 0_{p,q}^*$ and put $r = 1 + \left\lfloor \frac{1}{2} n \right\rfloor$, $k = 1 + \left\lfloor \frac{1}{2} (p + r) \right\rfloor$. Then

$W^{-2k} \in L_p$, $\varphi = \mathcal{F} d_{2\pi} W^{-2k} \in L_p$, and $g = f * \varphi \in L_q$. Since $\mathcal{F} g = \mathcal{F} f \cdot d_{2\pi} W^{-2k}$, we have $g = (1 - \Delta)^{-k} f$ and $f \in (1 - \Delta)^k L_q$. The maps

$$f \mapsto f * \varphi \mapsto (1 - \Delta)^k (f * \varphi) = f: 0_{p,q}^* \rightarrow L_q \rightarrow (1 - \Delta)^k L_q$$

are both continuous, hence $\text{id}: 0_{p,q}^* \rightarrow (1 - \Delta)^k L_q$ and $\text{id}: 0_q^* \rightarrow \text{ind} \lim_{k \rightarrow \infty} (1 - \Delta)^k L_q$ are continuous too.

In the sequel we use the inequality $|x| \leq W(x-y) \cdot W(y)$, $x, y \in \mathbb{R}^n$. Take $k, q \in \mathbb{N}$, $g \in L_q$, $\varphi \in \mathcal{S}$, and put $\psi = (1-\Delta)^k \varphi \in \mathcal{S}$. Then there are constants A, B, C such that

$$\begin{aligned} \|(1-\Delta)^k g * \varphi\|_q^2 &= \|g * \psi\|_q^2 = \sum_{|\alpha+\beta| \leq q} \int_{\mathbb{R}^n} x^{2\alpha} |g * D^\beta \psi|^2 dx = \\ &= \sum_{|\alpha+\beta| \leq q} \int_{\mathbb{R}^{2n}} x^{2\alpha} g(x-y) D^\beta \psi(y) \overline{g(x-z) D^\beta \psi(z)} dx dy dz \leq \\ &\leq \sum_{|\alpha+\beta| \leq q} \int_{\mathbb{R}^{2n}} W^{|\alpha|}(y) D^\beta \psi(y) \overline{W^{|\alpha|}(z) D^\beta \psi(z)} \int_{\mathbb{R}^n} W^{|\alpha|}(x-y) g(x-y) \\ &\quad W^{|\alpha|}(x-z) \overline{g(x-z)} dx dy dz \leq A \|g\|_q^2 \sum_{|\alpha+\beta| \leq q} \|W^{|\alpha|} D^\beta \psi\|_0^2 \leq B \|\psi\|_q^2 \leq C \|\varphi\|_{q+2k}^2. \end{aligned}$$

Hence $(1-\Delta)^k g \in 0_{q+2k, q}^*$ and

$$\text{id}: (1-\Delta)^k L_q \rightarrow 0_{q+2k, q}^* \quad \text{as well as} \quad \text{id}: \text{ind} \lim_{k \rightarrow \infty} (1-\Delta)^k L_q \rightarrow 0_q^*$$

are continuous.

Consequences:

- (1) $\text{id}: 0_q^* \rightarrow \mathcal{S}'$ is continuous,
- (2) 0_q^* is strong dual of the Fréchet space $\text{proj} \lim_{k \rightarrow \infty} (1-\Delta)^{-k} L_{-q}$.

Theorem 3. Each 0_q^* is a complete, reflexive, and bornological space.

PROOF. The space $\text{proj} \lim_{k \rightarrow \infty} (1-\Delta)^{-k} L_{-q}$ is reflexive, see [3, CH IV, §§ 5.5, 5.6, 5.8]. The strong dual 0_q^* is reflexive and complete. 0_q^* is bornological as an inductive limit of Hilbert spaces $(1-\Delta)^k L_q$.

Theorem 4. Let $f \in \mathcal{S}'$. Then

- (1) $f \in 0_q^*$ iff $f = \sum_{\alpha \in A} D^\alpha f_\alpha$, where $f_\alpha \in L_q$, $\alpha \in A$, and $A \subset \mathbb{N}^n$ is finite.
- (2) $f \in 0_q^*$ iff the map $\varphi \mapsto W^q(f * \varphi): \mathcal{S} \rightarrow L^2(\mathbb{R}^n)$ is continuous.

PROOF. (1) Take $\alpha \in \mathbb{N}^n$, integer $k \geq \frac{1}{2}|\alpha|$, and $f \in L_q$. We have $\mathcal{F}f \in L_q$ and $g = \mathcal{F}^{-1}((2\pi i x)^\alpha W^{-2k}(2\pi x) \mathcal{F}f) \in L_q$. Then $(1-\Delta)^{-k} D^\alpha f = \mathcal{F}^{-1}(d_{2\pi} W^{-2k} \mathcal{F}(D^\alpha f)) = g \in L_q$ and $D^\alpha f \in (1-\Delta)^k L_q \subset 0_q^*$. The inverse implication follows from Theorem 2.

(2) Let $f \in 0_q^*$. Then the maps $\varphi \mapsto f * \varphi \mapsto W^q(f * \varphi): \mathcal{S} \rightarrow L_q \rightarrow L^2(\mathbb{R}^n)$ are continuous.

Assume $\varphi \mapsto W^q(f * \varphi): \mathcal{S} \rightarrow L^2(\mathbb{R}^n)$ to be continuous. For each $\alpha, \beta \in \mathbb{N}^n$, $|\alpha+\beta| \leq q$, the maps

$$\varphi \mapsto D^\beta \varphi \mapsto x^\alpha (f * D^\beta \varphi) = x^\alpha D^\beta (f * \varphi): \mathcal{S} \rightarrow \mathcal{S} \rightarrow L^2(\mathbb{R}^n)$$

are continuous. Hence $\varphi \mapsto f * \varphi: \mathcal{S} \rightarrow L_q$ is continuous too and $f \in 0_q^*$.

Theorem 5. *The inductive topology of 0_q^* is generated by the family of seminorms $f \mapsto \|f * \varphi\|_q$, $\varphi \in \mathcal{S}$.*

PROOF. Denote by T the topology generated by these seminorms and by T_i the inductive topology of 0_q^* . Since each seminorm $f \mapsto \|f * \varphi\|_q: (0_q^*, T_i) \rightarrow \mathbb{R}$ is continuous, T is weaker than T_i .

Take $U \in T_i$, $0 \in U$. By Theorem 2 there is a bounded set $B \subset \text{proj} \lim_{k \rightarrow \infty} (1 - \Delta)^{-k} L_{-q}$ such that the polar $B^0 \subset U$. By [6, Lemma 4] there exists $\varphi \in \mathcal{S}$ such that $B \subset \varphi * D$, where D is the unit ball in L_{-q} .

For $v \in (1 - \Delta)^k L_q$ and $\psi \in \mathcal{S}$, we have $\langle \varphi * \psi, v \rangle = (1 - \Delta)^k \varphi * \psi((1 - \Delta)^{-k} v) = \psi((1 - \Delta)^k d_{-1} \varphi * (1 - \Delta)^{-k} v) = \psi(d_{-1} \varphi * v)$. Since $\mathcal{S}$ is dense in L_{-q} we can, for any $f \in 0_q^*$, write

$$\begin{aligned} & \sup \{ |\langle g, f \rangle|; g \in B \} \equiv \\ & \equiv \sup \{ |\langle \varphi * h, f \rangle|; h \in D \} = \sup \{ |\langle \varphi * \psi, f \rangle|; \psi \in \mathcal{S}, \|\psi\|_{-q} \leq 1 \} = \\ & = \sup \{ |\psi(d_{-1} \varphi * f)|; \psi \in \mathcal{S}, \|\psi\|_{-q} \leq 1 \} = \|d_{-1} \varphi * f\|_q. \end{aligned}$$

Hence U contains $\{f \in 0_q^*; \|d_{-1} \varphi * f\|_q \leq 1\} \in T$.

For $p, q \in N$, we put $0_{p,q} = \{f: \mathbb{R}^n \rightarrow \mathbb{C}; \varphi \mapsto f\varphi: L_q \rightarrow L_p \text{ continuous}\}$. Each $0_{p,q}$ with the topology of $\mathcal{L}(L_p, L_q)$ is a Banach space. We denote its norm by $\|\cdot\|_{p,q}$.

Theorem 6. *The Fourier transformation $\mathcal{F}: 0_{p,q}^* \rightarrow 0_{p,q}$, $p, q \in N$, is a topological isomorphism.*

PROOF. For convenience, put $A_k = \sup \{\|\mathcal{F}f\|_k; f \in L_k, \|f\|_k \leq 1\}$, $k \in N$. Take $f \in 0_{p,q}^*$, $\lambda > 0$, and choose $\psi \in \mathcal{S}$ such that $\mathcal{F}\psi(x) = 1$ for $|x| < \lambda$. Then $f * \psi \in L_q$ and $\mathcal{F}(f * \psi) = \mathcal{F}f \cdot \mathcal{F}\psi \in L_q$ are both functions. Thus $\mathcal{F}f$, restricted to $\{x \in \mathbb{R}^n; |x| < \lambda\}$, is a function too. Since λ was arbitrary, $\mathcal{F}f$ is a function.

For any $\varphi \in \mathcal{S}$, $\|\mathcal{F}f \cdot \mathcal{F}\varphi\|_q \leq A_q \|f * \varphi\|_q \leq A_q \|f\|_{p,q}^* \|\varphi\|_p \leq A_p A_q \|f\|_{p,q}^* \|\mathcal{F}\varphi\|_p$. Hence $\mathcal{F}f \in 0_{p,q}$ and $\mathcal{F}: 0_{p,q}^* \rightarrow 0_{p,q}$ is continuous.

Let $g \in 0_{p,q}$. By [5, Lemma 3], $0_{p,q} \subset L_{q-p-r}$, $r = 1 + \left\lfloor \frac{1}{2}n \right\rfloor$. This implies $0_{p,q} \subset \mathcal{S}'$ and $g \in \mathcal{S}'$. For any $\varphi \in \mathcal{S}$, we have $\varphi W^n \in L_p$. Then $g\varphi W^n \in L_q$ and the integral in the next formula is absolutely convergent.

$$\begin{aligned} \mathcal{F}^{-1}(g\varphi)(x) &= \int_{\mathbb{R}^n} g(t) \varphi(t) W^n(t) W^{-n}(t) \exp(2\pi i x, t) dt = \\ &= g_t(\varphi(t) \exp(2\pi i x, t)) = \mathcal{F}^{-1} g(\tau_x \mathcal{F}\varphi) = \mathcal{F}^{-1} g(\tau_x d_{-1} \mathcal{F}^{-1} \varphi) = \\ &= (\mathcal{F}^{-1} g * \mathcal{F}^{-1} \varphi)(x). \end{aligned}$$

Further

$$\begin{aligned} \|\mathcal{F}^{-1} g * \mathcal{F}^{-1} \varphi\|_q &= \|\mathcal{F}^{-1}(g\varphi)\|_q \equiv \\ &\equiv A_q \|g\varphi\|_q \leq A_q \|g\|_{p,q} \|\varphi\|_p \leq A_p A_q \|g\|_{p,q} \|\mathcal{F}^{-1} \varphi\|_p. \end{aligned}$$

Hence $\mathcal{F}^{-1} g \in 0_{p,q}^*$ and $\mathcal{F}^{-1}: 0_{p,q} \rightarrow 0_{p,q}^*$ is continuous.

Theorem 7. The convolution is continuous on $0_q^* \times L_{-q}$ for any $q \in N$, and not continuous on $0^* \times \mathcal{S}'$.

PROOF. Let U be an absolutely convex 0-neighborhood in $\mathcal{S}'$ and B the unit ball in L_{-q} . Then $U \cap L_{-p}$ is a 0-neighborhood in L_{-q} for any $p \in N$. Since the convolution is continuous on $0_{p,q}^* \times L_{-q}$, there exists a ball B_p in $0_{p,q}^*$ such that $B_p * B \subset U \cap L_{-p}$. The absolutely convex hull A of $\bigcup \{B_p; p \in N\}$ is a 0-neighborhood in 0_q^* and $A * B \subset U$.

Denote by 0 the $\text{proj} \lim_{q \rightarrow \infty} \text{ind} \lim_{p \rightarrow \infty} 0_{p,q}$. It is proved in [7] that the multiplication

$(f, g) \mapsto fg: 0 \times \mathcal{S}' \rightarrow \mathcal{S}'$ is not continuous. Hence the convolution $(f, g) \mapsto f * g: 0^* \times \mathcal{S}' \rightarrow \mathcal{S}'$ is not continuous, either.

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