

On one-sided invertibility of linear coercive Fourier series operators

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Abstract

Let C_π^∞ be the linear space of all smooth periodic functions. The paper considers the linear operators L from C_π^∞ into itself which possess the formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$, that is, there exists a linear operator $L': C_\pi^\infty \rightarrow C_\pi^\infty$ such that

$$\int_W (L\varphi)(x) \psi(x) dx = \int_W \varphi(x) (L'\psi)(x) dx$$

for all $\varphi, \psi \in C_\pi^\infty$, where W is the (2π) -cube in $\mathbb{R}^n$. One shows that these operators can always be expressed in the form

$$(1) \quad (L\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L(x, l) \varphi_l e^{i(l, x)} \quad \text{for } \varphi \in C_\pi^\infty,$$

where the mapping $x \rightarrow L(x, l)$ lies in C_π^∞ for each $l \in \mathbb{Z}^n$ and the derivative $D_x^\alpha L(x, l)$ is tempered with a polynomial. On the other hand supposing that for a given function $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ the mapping $x \rightarrow L(x, l)$ lies in C_π^∞ and that there exist constants $\mu \in \mathbb{R}$ and $0 \leq \delta < 1$ such that

$$(2) \quad |D_x^\alpha L(x, l)| \leq (1 + |l|^2)^{(\mu + \delta|\alpha|)/2} \quad \text{for all } l \in \mathbb{Z}^n, x \in \mathbb{R}^n,$$

one verifies that the operator L defined by (1) maps C_π^∞ into itself and that the formal transpose L' of L exists.

For the operators $L: C_\pi^\infty \rightarrow C_\pi^\infty$ which own the formal transpose L' , the existence of the one-sided inverse K of the maximal extension $L'_k^\# : H_k^\pi \rightarrow H_k^\pi$ is proved under the following coercivity condition (posed on L')

$$(3) \quad \|L'\varphi\|_{1/k^\nu} \geq C_1 \|\varphi\|_{k^\nu/k^\nu} - C_2 \|\varphi\|_{1/k^\nu} \quad \text{for all } \varphi \in C_\pi^\infty,$$

where $C_1 > 0$ and $C_2 \geq 0$. The spaces H_k^π are certain weighted subspaces of the space D'_π of all periodic distributions and the norm in the space H_k^π is denoted by $\|\cdot\|_k$. The weight function k^ν is assumed to obey the condition that $k^\nu(l) \rightarrow \infty$ with $|l| \rightarrow \infty$, and k^ν is defined by $k^\nu(l) = k(-l)$. Furthermore, sufficient criteria under which the one-sided inverse K can be expressed as the extension of a Fourier series operator are revealed.

1. Introduction

Let $L(x, D)$ be a linear partial differential operator with smooth periodic coefficients. Then $L(x, D)\varphi$ can be expressed in the form

$$(1.1) \quad (L(x, D)\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L(x, l) \varphi_l e^{i(l, x)}$$

for all φ lying in the space C_π^∞ of all smooth periodic functions. With the help of the symbol $L(x, l)$ one is able to show existence, uniqueness and regularity results for the distributional equation $L(x, D)u = f$, where u and f lie in the space D'_π of all periodic distributions. The exposition is usually done in the frame of the Hilbert spaces H_π^s of generalized trigonometric polynomials. For the elliptic case we refer to [2], pp. 131—299 and [4], pp. 95—124. In the contribution [6] one has considered (global) hypoelliptic operators and in [7] one has exposed the existence and uniqueness theory for t -coercive operators. For the generalizations we refer also to [3], where the operators of the following form

$$(1.2) \quad (A(D)\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} A(l) \varphi_l e^{i(l, x)}$$

have been studied, where A is a mapping $\mathbb{Z}^n \rightarrow \mathbb{C}$.

This paper generalizes the notion of a partial differential operator by examining the operators defined by

$$(1.3) \quad (L(x, D)\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L(x, l) \varphi_l e^{i(l, x)}$$

for φ lying in C_π^∞ . Here $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ is a mapping satisfying some tempering criteria (cf. Theorem 2.3). We call these operators Fourier series operators.

For the operators (1.3) which own the formal transpose $L'(x, D): C_\pi^\infty \rightarrow C_\pi^\infty$, we will construct a one-sided continuous inverse K of the maximal operator L'_k under the assumption that a certain a priori estimate holds for $L'(x, D)$ (cf. Theorem 3.5 and Corollary 3.6). In addition we prove that the operator K can be expressed as the extension of a Fourier series operator in the case when " k is large enough" (cf. Corollary 4.3) and in the case when $L(x, D)$ is hypoelliptic (cf. Theorem 5.1).

2. Spaces H_k^π and Fourier series operators

2.1. Let W be a cube of $\mathbb{R}^n$ such that

$$(2.1) \quad W = \{x \in \mathbb{R}^n \mid -\pi < x_j < \pi \text{ for } j \in \{1, \dots, n\}\}.$$

Furthermore, denote by C_π^∞ the linear subspace of all those $C^\infty(\mathbb{R}^n)$ -functions which are periodic with respect to W . In the space C_π^∞ we set a locally convex topology defined by the semi-norms $q_\sigma: C_\pi^\infty \rightarrow \mathbb{R}$ such that

$$(2.2) \quad q_\sigma(\psi) = \sup_{x \in W} |(D^\sigma \psi)(x)|, \quad \sigma \in \mathbb{N}_0^n.$$

It is well-known that C_π^∞ is a Frechet space and that ψ belongs to C_π^∞ if and only if it owns the form

$$(2.3) \quad \psi(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} \psi_l e^{i(l, x)},$$

where the scalars $\psi_l \in \mathbb{C}$ satisfy the condition

$$(2.4) \quad \sup_{l \in \mathbb{Z}^n} |\psi_l| (1 + |l|^2)^{s/2} \leq C_{s, \psi}$$

for every $s \in \mathbb{R}$ (cf. [1], p. 131).

The periodic distribution is a continuous linear form $C_\pi^\infty \rightarrow \mathbb{C}$. Let D'_π denote the linear space of all periodic distributions. Then T lies in D'_π if and only if there exist $t \in \mathbb{N}_0$ and $C > 0$ such that

$$(2.5) \quad |T\psi| \leq C \sum_{|\sigma| \leq t} q_\sigma(\psi) \quad \text{for all } \psi \in C_\pi^\infty.$$

We use weak dual topology in D'_π .

Let $k: \mathbb{Z}^n \rightarrow \mathbb{R}$ be a positive function such that there exist constants $C > 0$ and $N > C$ with which

$$(2.6) \quad k(l+z) \leq C(1+|l|^2)^{N/2} k(z) \quad \text{for all } l, z \in \mathbb{Z}^n.$$

Denote by K_π the family of these mappings. Clearly one has for all $l \in \mathbb{Z}^n$

$$(2.7) \quad k(0)C^{-1}(1+|l|^2)^{-N/2} \leq k(l) \leq k(0)C(1+|l|^2)^{N/2}.$$

Furthermore one sees that the functions k_1+k_2 , k_1k_2 and k^s are lying in K_π for all k_1 and $k_2 \in K_\pi$ and $s \in \mathbb{R}$. The basic example about the elements of K_π is the function $k_s: \mathbb{Z}^n \rightarrow \mathbb{R}$ defined by $k_s(l) = (1+|l|^2)^{s/2}$, $s \in \mathbb{R}$.

Definition 2.1. A distribution $T \in D'_\pi$ belongs to the space H_k^π if and only if

$$(2.8) \quad \left(\lambda_n \sum_{l \in \mathbb{Z}^n} |T_l k(l)|^2 \right)^{1/2} < \infty,$$

where $\lambda_n := (2\pi)^{-n}$ and

$$(2.9) \quad T_l := T(e^{-i(l,x)}).$$

The mapping $T \rightarrow \|T\|_k := \left(\lambda_n \sum_{l \in \mathbb{Z}^n} |T_l k(l)|^2 \right)^{1/2}$ is clearly a norm in H_k^π . By using the properties of l_2 -spaces one sees that the linear space H_k^π is a separable and reflexive Banach space. Furthermore it can be equipped with a scalar product defined by

$$(2.10) \quad (u, v)_k = \lambda_n \sum_{l \in \mathbb{Z}^n} \bar{u}_l v_l k(l)^2.$$

The space C_π^∞ is a dense subspace of H_k^π for each $k \in K_\pi$ and then the space H_k^π can be interpreted as a completion of C_π^∞ with respect to the scalar product $(\cdot, \cdot)_k: C_\pi^\infty \times C_\pi^\infty \rightarrow \mathbb{C}$ such that

$$(\varphi, \psi) = \lambda_n \sum_{l \in \mathbb{Z}^n} \bar{\varphi}_l \psi_l k(l)^2,$$

where φ_l is the Fourier coefficient of φ defined by

$$\varphi_l := \varphi(e^{-i(l,x)}) := \int_W \varphi(x) e^{-i(l,x)} dx.$$

Let $k^v \in K_\pi$ be defined through $k^v(l) = k(-l)$. One has for all φ and $\psi \in C_\pi^\infty$

$$(2.11) \quad \varphi(\psi) := \int_W \varphi(x) \psi(x) dx = \lambda_n \sum_{l \in \mathbb{Z}^n} \varphi_l \psi_{-l}$$

and then by Hölder's inequality we obtain

$$(2.12) \quad |\varphi(\psi)| \leq \|\varphi\|_k \|\psi\|_{1/k'},$$

where $\|\varphi\|_k$ is the norm induced by the scalar product (2.10). In view of the inequality (2.12) one sees that

$$(2.13) \quad |T(\varphi)| \leq \|T\|_k \|\varphi\|_{1/k'} \quad \text{for all } T \in H_k^\pi \text{ and } \varphi \in C_\pi^\infty.$$

This means that the topology of H_k^π is stronger than the topology induced by D'_π . Thus we have established that

$$(2.14) \quad C_\pi^\infty \subset H_k^\pi \subset D'_\pi \quad \text{for each } k \in K_\pi,$$

where both inclusions are topological.

Let $H_k^{\pi*}$ be the dual space of H_k^π . Then due to the Frechet—Riesz Theorem one has

Theorem 2.2. *Assume that k lies in K_π . Then for every $L \in H_k^{\pi*}$ there exists $V \in H_{1/k'}^\pi$ such that*

$$(2.15) \quad L\varphi = V(\varphi) \quad \text{for all } \varphi \in C_\pi^\infty$$

and $\|L\| = \|V\|_{1/k'}$. Conversely the linear form $L: C_\pi^\infty \rightarrow C_\pi^\infty$ defined by $L\varphi = V(\varphi)$ with $V \in H_{1/k'}^\pi$ can be (uniquely) extended to a continuous linear form on H_k^π .

We finally formulate an algebraic criterion for the compactness of the imbedding $\lambda: H_{k^*}^\pi \rightarrow H_k^\pi$, where k and k^* lie in K_π (cf. [4], pp. 111—112).

Theorem 2.3. *The imbedding λ is compact if and only if the weight functions k and k^* obey*

$$(2.16) \quad k(l)/k^*(l) \rightarrow 0 \quad \text{for } |l| \rightarrow \infty.$$

PROOF. At first we suppose that (2.16) holds. Let $\{T_n\} \subset H_{k^*}^\pi$ be a sequence such that

$$(2.17) \quad \|T_n\|_{k^*} \leq M \quad \text{for all } n \in \mathbb{N}.$$

Then one has for all $l \in \mathbb{Z}^n$ and $n \in \mathbb{N}$

$$(2.18) \quad (\lambda_n)^{1/2} |(T_n)_l k^*(l)| \leq \|T_n\|_{k^*} \leq M.$$

The inequality (2.18) implies that one is able to find a subsequence $\{T_{n_j}\}$ such that

$$(2.19) \quad \sum_{|l| \geq \varrho} |(T_{n_j} - T_{n_k})_l k(l)|^2 \rightarrow 0 \quad \text{with } j, k \rightarrow \infty$$

for each $\varrho \geq 0$ (cf. [4], pp. 111—112).

Let ε be an arbitrary positive number and let $\varrho \in \mathbb{R}$ such that $k(l)/k^*(l) \leq \varepsilon$ for $|l| \geq \varrho$. Then we obtain

$$(2.20) \quad \|T_{n_j} - T_{n_k}\|_k \leq 2^{1/2} (2M\varepsilon + \sum_{|l| \geq \varrho} |T_{n_j} - T_{n_k})_l k(l)|^2)^{1/2}.$$

Hence due to (2.19) one sees that $\{T_{n_j}\}$ is a Cauchy sequence in H_k^π .

The converse is easy to see by applying the assumption to the sequence $\{u_j\} \subset C_\pi^\infty$ defined through

$$(2.21) \quad u_j(x) = \lambda_n e^{i(l_j, x)} / k^\sim(l_j),$$

where $\{l_j\} \subset \mathbf{Z}^n$ is an (arbitrary) sequence such that $|l_j| \rightarrow \infty$ with $j \rightarrow \infty$. ■

2.2. Suppose that L is a linear operator $C_\pi^\infty \rightarrow C_\pi^\infty$ such that its formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$ exists, in other words, one can find a linear operator $L': C_\pi^\infty \rightarrow C_\pi^\infty$ satisfying the relation

$$(2.22) \quad (L\varphi)(\psi) := \int_W (L\varphi)(x) \psi(x) dx = (\varphi)(L'\psi).$$

In the sequel we show that this kind of operator L can always be expressed in the form

$$(2.23) \quad (L(x, D)\varphi)(x) = \lambda_n \sum_{l \in \mathbf{Z}^n} L(x, l) \varphi_l e^{i(l, x)} \quad \text{for } x \in \mathbf{R}^n,$$

where $L(\cdot, \cdot)$ is a mapping $\mathbf{R}^n \times \mathbf{Z}^n \rightarrow \mathbf{C}$ such that the function $x \rightarrow L(x, l)$ belongs to C_π^∞ for each $l \in \mathbf{Z}^n$ and that the derivative $D_x^\alpha L(x, l)$ is tempered with a polynomial in l .

Lemma 2.4. Suppose that L is a continuous linear operator $C_\pi^\infty \rightarrow C_\pi^\infty$. Then L can be expressed as a Fourier series operator defined by

$$(2.24) \quad (L\varphi)(x) = \lambda_n \sum_{l \in \mathbf{Z}^n} L(x, l) \varphi_l e^{i(l, x)} =: (L(x, D)\varphi)(x),$$

where

1° the mapping $x \rightarrow L(x, l)$ lies in C_π^∞ for each $l \in \mathbf{Z}^n$

and

2° for each $\alpha \in \mathbf{N}_0^n$ one can find constants $C_\alpha > 0$ and $\mu_\alpha \in \mathbf{R}$ such that

$$(2.25) \quad |D_x^\alpha L(x, l)| \leq C_\alpha (1 + |l|^2)^{\mu_\alpha/2} \quad \text{for all } x \in W \text{ and } l \in \mathbf{Z}^n.$$

Conversely, suppose that $L(\cdot, \cdot): \mathbf{R}^n \times \mathbf{Z}^n \rightarrow \mathbf{C}$ is a mapping satisfying 1° and 2°. Then the relation (2.24) introduces a linear continuous operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$.

PROOF. A. Let L be a continuous linear mapping $C_\pi^\infty \rightarrow C_\pi^\infty$. Then for each $\alpha \in \mathbf{N}_0^n$ there exists $N_\alpha \in \mathbf{N}$ such that for all $\varphi \in C_\pi^\infty$

$$(2.26) \quad \sup_{x \in W} |D_x^\alpha (L\varphi)(x)| \leq C_\alpha \sum_{|\beta| \leq N_\alpha} \sup_{x \in W} |D_x^\beta (\varphi)(x)|$$

(cf. [8], p. 42). Applying this inequality with $\varphi = e^{i(l, x)}$ we obtain

$$(2.27) \quad \begin{aligned} & \sup_{x \in W} |D_x^\alpha (L(e^{i(l, x)}))(x) e^{-i(l, x)}| \leq \\ & \leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \sup_{x \in W} |D_x^\gamma (L(e^{i(l, x)}))(x)| |l|^{|\alpha - \gamma|} \leq \\ & \leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_\gamma \sum_{|\beta| \leq N_\gamma} |l|^{|\beta|} |l|^{|\alpha - \gamma|} \leq C_\alpha k_{\mu_\alpha}(l), \end{aligned}$$

where we denoted $(1 + |l|^2)^{\mu_\alpha/2} = k_{\mu_\alpha}(l)$.

Define now a relation through

$$(2.28) \quad (L\varphi)(x) = g(x)(\varphi), \quad \text{for } \varphi \in C_\pi^\infty \quad \text{and} \quad x \in \mathbb{R}^n.$$

Then $g(x)$ lies in D'_π (for a fixed $x \in \mathbb{R}^n$): Clearly $g(x)$ is a well-defined linear form $C_\pi^\infty \rightarrow \mathbb{C}$. Furthermore, the convergence

$$\varphi_n \rightarrow \varphi \quad \text{in } C_\pi^\infty$$

implies (in view of (2.26)) that

$$(2.29) \quad \sup_{x \in W} |(L\varphi_n)(x) - (L\varphi)(x)| \rightarrow 0$$

and then

$$g(x)(\varphi_n) = (L\varphi_n)(x) \rightarrow (L\varphi)(x) = g(x)(\varphi).$$

This says that $g(x)$ belongs to D'_π .

The expression (2.28) yields us

$$(2.30) \quad \begin{aligned} (L\varphi)(x) &= \sum_{l \in \mathbb{Z}^n} (g(x))_{-l} \varphi_l = \lambda_n \sum_{l \in \mathbb{Z}^n} (\lambda_n^{-1} (g(x))_{-l} e^{-i(l, x)}) \varphi_l e^{i(l, x)} =: \\ &=: \lambda_n \sum_{l \in \mathbb{Z}^n} L(x, l) \varphi_l e^{i(l, x)}, \end{aligned}$$

where we used the fact that

$$(2.31) \quad T\varphi = \sum_{l \in \mathbb{Z}^n} T_{-l} \varphi_l \quad \text{for } T \in D'_\pi \quad \text{and} \quad \varphi \in C_\pi^\infty.$$

The mapping $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ such that

$$(2.32) \quad \begin{aligned} L(x, l) &= \lambda_n^{-1} (g(x))_{-l} e^{-i(l, x)} = \\ &= \lambda_n^{-1} g(x)(e^{i(l, \cdot)}) e^{-i(l, x)} = \lambda_n^{-1} L(e^{i(l, \cdot)})(x) e^{-i(l, x)} \end{aligned}$$

is well-defined and the mapping $x \mapsto L(x, l)$ lies in C_π^∞ . By taking into account relations (2.27) and (2.32) we obtain the validity of (2.25). Hence the first part of the proof is ready.

B. We now assume that the mapping $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ satisfies 1^0 and 2^0 . The distribution $\varphi \in D'_\pi$ lies in C_π^∞ if and only if for each $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$(2.33) \quad |\varphi_l| \leq C_N (1 + |l|^2)^{-N/2} \quad \text{for all } l \in \mathbb{Z}^n$$

(cf. the subsection 2.1). Hence one sees in view of (2.25) that the sum (with $\varphi \in C_\pi^\infty$)

$$\lambda_n \sum_{l \in \mathbb{Z}^n} D_x^\alpha (L(x, l) \varphi_l e^{i(l, x)})$$

is (absolutely) uniformly convergent for each $\alpha \in \mathbb{N}_0^n$. This yields that $D_x^\alpha (L\varphi)(x)$

exists and that

$$\begin{aligned}
 (2.34) \quad |D_x^\alpha(L\varphi)(x)| &= \lambda_n \left| \sum_{l \in \mathbb{Z}^n} D_x^\alpha(L(x, l) e^{i(l, x)}) \varphi_l \right| \leq \\
 &\leq \lambda_n \sum_{l \in \mathbb{Z}^n} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D_x^\beta L(x, l) |l|^{\alpha-\beta} |\varphi_l| \leq \\
 &\leq \lambda_n \sum_{l \in \mathbb{Z}^n} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C_\beta k_{\mu_\beta}(l) k_{|\alpha-\beta|}(l) |\varphi_l| \leq \\
 &\leq C'_\alpha \sum_{l \in \mathbb{Z}^n} (1 + |l|^2)^{N'_\alpha} |\varphi_l| / k_{n+1}(l) \leq \\
 &\leq C''_\alpha \sum_{l \in \mathbb{Z}^n} \sum_{|\gamma| \leq N'_\alpha} l^{2\gamma} |\varphi_l| / k_{n+1}(l) \leq \\
 &\leq C''_\alpha \sum_{l \in \mathbb{Z}^n} \sum_{|\gamma| \leq N'_\alpha} |(D_x^{2\gamma} \varphi)_l| / k_{n+1}(l) = \\
 &\leq C''_\alpha \lambda_n \sum_{|\gamma| \leq N'_\alpha} \sup_{x \in W} |(D_x^{2\gamma} \varphi)(x)| \sum_{l \in \mathbb{Z}^n} 1/k_{n+1}(l),
 \end{aligned}$$

where N'_α is a sufficiently large natural number. This completes the proof. ■

We are now ready to verify

Theorem 2.5. Suppose that the linear operator L defined on C_π^∞ obeys

- (i) L maps C_π into itself,
- (ii) there exists the formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$ of L .

Then $L: C_\pi^\infty \rightarrow C_\pi^\infty$ is continuous so that it can be expressed as a Fourier series operator (2.24), where the mapping $L(\cdot, \cdot)$ satisfies 1° and 2° .

PROOF. We show that $L: C_\pi^\infty \rightarrow C_\pi^\infty$ is a closed operator. Let $\{\varphi_n\} \subset C_\pi^\infty$ be a sequence such that

$$(2.35) \quad \varphi_n \rightarrow \varphi \quad \text{in } C_\pi^\infty$$

and

$$(2.36) \quad L\varphi_n \rightarrow \psi \quad \text{in } C_\pi^\infty.$$

In virtue of the assumption (ii) for each $\Phi \in C_\pi^\infty$

$$(2.37) \quad (L\varphi_n)(\Phi) = \varphi_n(L'\Phi) \rightarrow \varphi(L'\Phi) = L\varphi(\Phi)$$

and then by (2.36) $L\varphi = \psi$. The closedness of L proves (by the Closed Graph Theorem) that L is continuous. The last assertions follow from Lemma 2.4 and then the proof is complete. ■

A sufficient criterion under which the assumptions (i) and (ii) of Theorem 2.5 hold is given by the following

Theorem 2.6. Assume that the mapping $x \rightarrow L(x, l)$ lies in C_π^∞ for each $l \in \mathbb{Z}^n$ and that one can find numbers $\mu \in \mathbb{R}$ and $0 \leq \delta < 1$ such that for each $\alpha \in \mathbb{N}_0^n$ there exists a constant $C_\alpha > 0$ with which the inequality

$$(2.38) \quad |D_x^\alpha L(x, l)| \leq C_\alpha (1 + |l|^2)^{(\mu + \delta|\alpha|)/2}$$

for all $l \in \mathbb{Z}^n$, $x \in W$ holds. Then we have

- 3^o the linear operator given through (2.23) maps C_π^∞ into itself,
 4^o the formal transpose $L'(x, D): C_\pi^\infty \rightarrow C_\pi^\infty$ of $L(x, D)$ exists and

$$(2.39) \quad (L'(x, D)\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} \left(\int_W L(y, -l) \varphi(y) e^{-i(l, y)} dy \right) e^{i(l, x)}$$

for every $\varphi \in C_\pi^\infty$.

PROOF. In virtue of (2.38) the sum $\sum_{l \in \mathbb{Z}^n} L(y, l) \varphi_l e^{i(l, y)}$ is uniformly convergent. Hence we get via a direct computation

$$(2.40) \quad \begin{aligned} (L(x, D)\psi)(\varphi) &= \int_W (L(y, D)\psi)(y) \varphi(y) dy = \\ &= \int_W \left(\lambda_n \sum_{l \in \mathbb{Z}^n} L(y, l) \psi_l e^{i(l, y)} \right) \varphi(y) dy = \\ &= \lambda_n \sum_{l \in \mathbb{Z}^n} \psi_l \left(\int_W L(y, l) \varphi(y) e^{i(y, l)} dy \right) = \\ &= \lambda_n \sum_{l \in \mathbb{Z}^n} \psi_l (L'(x, D)\varphi)_{-l} = \psi(L'(x, D)\varphi). \end{aligned}$$

We now check that $L'(x, D)\varphi$ defined through (2.39) lies in C_π^∞ . Employing (2.33) our only task is to show that for every $\tau \in \mathbb{N}_0^n$

$$|l^\tau (L'(x, D)\varphi)_l| \leq C_{\tau, \varphi} \quad \text{for every } l \in \mathbb{Z}^n.$$

The condition (2.38) yields

$$(2.41) \quad \begin{aligned} |l^\alpha (L'(x, D)\varphi)_l| &= \left| \int_W L(y, l) \varphi(y) D_y^\alpha (e^{i(l, y)}) dy \right| \leq \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int_W |D_y^\gamma L(y, l) (D_y^{\alpha-\gamma} \varphi)(y)| dy \leq \\ &\leq \sum_{\gamma \leq \alpha} \sup_{y \in W} |D_y^\gamma L(y, l)| \|D_y^{\alpha-\gamma} \varphi\|_{L^1(W)} \leq \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_\gamma (1 + |l|^2)^{(\mu + \delta|\gamma|)/2} \|D_y^{\alpha-\gamma} \varphi\|_{L^1(W)} \leq \\ &\leq C_{\alpha, \varphi} (1 + |l|^2)^{(\mu + \delta|\alpha|)/2}. \end{aligned}$$

The inequality (2.41) tells us that for every $m \in \mathbb{N}$ one is able to find a constant $C_{m, \varphi} > 0$ such that

$$(2.42) \quad |l^\tau (L'(x, D)\varphi)_l| \leq C_{m, \varphi} (1 + |l|^2)^{(\mu + \delta|\tau| + (\delta-1)m)/2}.$$

Since $\delta < 1$ our assertion follows from (2.42) by choosing m large enough. ■

2.3. Define a linear operator $L_k: H_k^\pi \rightarrow H_k^\pi$ via the requirement (here L is further linear operator $C_\pi^\infty \rightarrow C_\pi^\infty$ satisfying the condition (ii) of Theorem 2.5)

$$(2.43) \quad \begin{cases} D(L_k) = C_\pi^\infty, \\ L_k \varphi = L\varphi \quad (= L(x, D)\varphi) \quad \text{for } \varphi \in D(L_k). \end{cases}$$

Then L_k is densely defined and because of the existence of the formal transpose L' , one sees that L_k is closable in H_k^π . This is based on the fact that the topology of H_k^π is stronger than the topology of D'_π . Let $L_k^\sim$ be the smallest closed extension of L_k (cf. [8], pp. 77—79).

Furthermore, let $L_k'^\#$ be a linear operator $H_k^\pi \rightarrow H_k^\pi$ defined by

$$(2.44) \quad \begin{cases} D(L_k'^\#) = \{u \in H_k^\pi \mid \text{there exists an element } f \in H_k^\pi \text{ such that} \\ u(L_k'\varphi) = f(\varphi) \text{ for all } \varphi \in C_\pi^\infty\}, \\ L_k'^\# u = f. \end{cases}$$

The operator $L_k'^\#$ is closed and $L_k^\sim \subset L_k'^\#$ (that is, $L_k'^\#$ is an extension of $L_k^\sim$).

3. The construction of a one-sided inverse of $L_k'^\#$

3.1. We at first establish some semi-Fredholm properties of the minimal operator $L_k^\sim$. Afterwards we give the existence results for the solutions of the maximal equation $L_k'^\# u = f$ (by employing the duality between H_k^π and $H_{1/k}^{\pi^*}$). Let $L_k^*: H_k^{\pi^*} \rightarrow H_k^{\pi^*}$ be the dual operator of L_k . The kernel (the range) is denoted by $N(L_k^\sim)$ (and $R(L_k^\sim)$, resp.). We show

Theorem 3.1. *Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ obeys condition (ii) of Theorem 2.5 and that there exist constants $C_1 > 0$ and $C_2 \geq 0$ such that*

$$(3.1) \quad \|L\varphi\|_k \cong C_1 \|\varphi\|_{kk^\sim} - C_2 \|\varphi\|_k \quad \text{for all } \varphi \in C_\pi^\infty,$$

where $k^\sim \in K_\pi$ is chosen so that

$$(3.2) \quad k^\sim(l) \rightarrow \infty, \quad \text{with } |l| \rightarrow \infty.$$

Then the operator $L_k^\sim$ is a semi-Fredholm operator with

$$(3.3) \quad \dim N(L_k^\sim) < \infty.$$

PROOF. In virtue of (3.1) for all $u \in D(L_k^\sim)$

$$(3.4) \quad C_1 \|u\|_{kk^\sim} \leq \|L_k^\sim u\|_k + C_2 \|u\|_k.$$

Hence every bounded sequence of $N(L_k^\sim)$ possesses a convergent subsequence (since the imbedding $\lambda: H_{kk^\sim}^\pi \rightarrow H_k^\pi$ is compact). This shows the validity of (3.3).

Furthermore, by taking a sequence $\{L_k^\sim u_n\}$ in H_k^π , where $\{u_n\}$ is bounded in $D(L_k^\sim)$, one sees due to (3.4) that $\{u_n\}$ has a convergent subsequence $\{u_{n_j}\}$. Since $\|L_k^\sim u_{n_j} - f\|_k \rightarrow 0$ with some $f \in H_k^\pi$ with $j \rightarrow \infty$, the limit u of $\{u_{n_j}\}$ lies in $D(L_k^\sim)$ and $L_k^\sim u = f$. Hence $L_k^\sim(B)$ is closed when B is closed and bounded in $D(L_k^\sim)$, which implies that $R(L_k^\sim)$ is closed (cf. [5], pp. 99—100). This completes the proof. ■

Corollary 3.2. *Let L be such as in Theorem 3.1. Then the relations*

$$(3.5) \quad R(L_k^\sim) = N(L_k^*)^\perp := \{f \in H_k^\pi \mid Tf = 0 \text{ for all } T \in N(L_k^*)\}$$

and

$$(3.6) \quad R(L_k^*) = {}^\perp N(L_k^\sim) := \{T \in H_k^{\pi*} \mid Tu = 0 \text{ for all } u \in N(L_k^\sim)\}$$

hold.

This is a standard consequence of the properties of semi-Fredholm operators by taking into account that $L_k^{\sim*} = L_k^*$.

3.2. Suppose that the inclusion

$$(3.7) \quad D(L_{1/(kk^\sim)}^{\sim\prime}) \subset H_{1/k}^\pi$$

holds, where $k^\sim \in K_\pi$ satisfies

$$(3.8) \quad k^\sim(l) \geq 1 \quad \text{for all } l \in \mathbb{Z}^n.$$

Then $L_{1/(kk^\sim)}^{\sim\prime}$ is a closed operator $H_{1/k}^\pi \rightarrow H_{1/(kk^\sim)}^\pi$ so that the dual operator $L_{1/(kk^\sim)}^{\sim\prime*}$ is a closed operator $H_{1/(kk^\sim)}^{\pi*} \rightarrow H_{1/k}^{\pi*}$. Let J_k be the isometrical isomorphism $H_k^\pi \rightarrow H_{1/k}^{\pi*}$ for $k \in K_\pi$ (established in Theorem 2.2). In the sequel we exhibit the connection between the operators $L_k^{\sim\prime\#}$ and $L_{1/(kk^\sim)}^{\sim\prime*}$.

Theorem 3.3. *Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ obeys the condition (ii) of Theorem 2.5 and that the inclusion (3.7) holds with $k^\sim \geq 1$. Then the relation*

$$(3.9) \quad L_k^{\sim\prime\#}|_{H_{kk^\sim}^\pi} = J_k^{-1} \circ (L_{1/(kk^\sim)}^{\sim\prime*}) \circ J_{kk^\sim}$$

is valid.

PROOF. *A.* At first we show that $L_k^{\sim\prime\#}|_{H_{kk^\sim}^\pi} \subset J_k^{-1} \circ (L_{1/(kk^\sim)}^{\sim\prime*}) \circ J_{kk^\sim}$. Suppose that u lies in $H_{kk^\sim}^\pi \cap D(L_k^{\sim\prime\#})$ and that $L_k^{\sim\prime\#}u = f$. Let $F = J_k f$ (and $U = J_{kk^\sim}u$) belong to $H_{1/k}^{\pi*}$ (and to $H_{1/(kk^\sim)}^{\pi*}$, resp.) such that

$$(3.10) \quad F\varphi = f(\varphi) \quad \text{for all } \varphi \in C_\pi^\infty$$

and

$$(3.11) \quad U\varphi = u(\varphi) \quad \text{for all } \varphi \in C_\pi^\infty.$$

The relation $L_k^{\sim\prime\#}u = f$ implies that

$$(3.12) \quad F\varphi = f(\varphi) = u(L_k^{\sim\prime}\varphi) = U(L_{1/(kk^\sim)}^{\sim\prime}\varphi)$$

for all $\varphi \in C_\pi^\infty = D(L_{1/(kk^\sim)}^{\sim\prime})$. Hence U lies in $D(L_{1/(kk^\sim)}^{\sim\prime*})$ and

$$(3.13) \quad L_{1/(kk^\sim)}^{\sim\prime*}(J_{kk^\sim}u) = L_{1/(kk^\sim)}^{\sim\prime*}U = F = J_k f,$$

which proves the first part of the assertion.

B. On the other hand we assume that u lies in $D(J_k^{-1} \circ (L'_{1/(kk\sim)})^* \circ J_{kk\sim})$. Then u lies in $H_{kk\sim}^\pi$ and for all $\varphi \in C_\pi^\infty$

$$(3.14) \quad \begin{aligned} u(L'_k \varphi) &= (J_{kk\sim} u)(L'_k \varphi) = (J_{kk\sim} u)(L'_{1/(kk\sim)} \varphi) = \\ &= (L'_{1/(kk\sim)} (J_{kk\sim} u))(\varphi) = J_k^{-1} (L'_{1/(kk\sim)} (J_{kk\sim} u))(\varphi), \end{aligned}$$

since $J_{kk\sim} u$ belongs to $D(L'_{1/(kk\sim)})$. Hence the proof is complete. ■

Corollary 3.4. *Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ obeys condition (ii) of Theorem 2.5. Then the range $R(L'_k^\#)$ is closed in H_k^π if and only if the range $R(L'_{1/k^\vee})$ is closed in $H_{1/k^\vee}^\pi$.*

PROOF. The assertion follows immediately from Theorem 3.3 with $k^\sim \equiv 1$ and from the fact that $R(L'_{1/k^\vee})$ is closed if and only if $R(L'_{1/k^\vee})$ is closed. ■

3.3. In the sequel we construct a one-sided continuous inverse for $L'_k^\#$ on $R(L'_k^\#)$. More precisely, we show that (under certain conditions) there exists a continuous operator $K: R(L'_k^\#) \rightarrow H_k^\pi$ satisfying

$$(3.15) \quad L'_k^\# (Kf) = f \quad \text{for all } f \in R(L'_k^\#).$$

Theorem 3.5. *Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ obeys the condition (ii) of Theorem 2.5 and that there exist constants $C_1 > 0$ and $C_2 \geq 0$ such that*

$$(3.16) \quad \|L' \varphi\|_{1/k^\vee} \leq C_1 \|\varphi\|_{k^\sim/k^\vee} + C_2 \|\varphi\|_{1/k^\vee}$$

for all $\varphi \in C_\pi^\infty$, where $k^\sim$ satisfies (3.2). Then there exists a continuous linear operator $K: R(L'_k^\#) \rightarrow H_k^\pi$ with the property

$$(3.17) \quad L'_k^\# (Kf) = f \quad \text{for all } f \in R(L'_k^\#).$$

PROOF. In virtue of Theorem 3.1, the assumption (3.16) implies that $R(L'_{1/k^\vee})$ is closed. Furthermore, one sees by Theorem 3.3 that

$$(3.18) \quad L'_k^\# = J_k^{-1} \circ (L'_{1/k^\vee})^* \circ J_k.$$

As a Hilbert space the space H_k^π can be expressed as the orthogonal sum

$$(3.19) \quad H_k^\pi = N(L'_k^\#) \oplus N,$$

where N is closed in H_k^π . Define now a linear operator $\mathcal{L}: H_k^\pi \rightarrow H_k^\pi$ as the restriction $L'_k^\#|_N$. Then the kernel $N(\mathcal{L})$ is $\{0\}$.

Since the range $R(L'_{1/k^\vee})$ is closed it follows that the range $R(L'_{1/k^\vee})$ is closed in $H_{1/k^\vee}^\pi$. Hence due to (3.18) the range $R(\mathcal{L}) = R(L'_k^\#)$ is closed in H_k^π . Furthermore the closedness of N implies that the operator $\mathcal{L}: H_k^\pi \rightarrow R(L'_k^\#)$ is closed. In view of the Closed Graph Theorem the operator $K := \mathcal{L}^{-1}: R(L'_k^\#) \rightarrow H_k^\pi$ is continuous. Thus the proof is complete. ■

Corollary 3.6. Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ obeys the condition (ii) of Theorem 2.5 and that there exists $\varkappa > 0$ such that

$$(3.20) \quad \|L'\varphi\|_{1/(kk^\sim)^\vee} \cong \varkappa \|\varphi\|_{1/k^\vee} \quad \text{for all } \varphi \in C_\pi^\infty,$$

where $k^\sim \cong 1$. Then there exists a continuous linear operator $K: H_k^\pi \rightarrow H_{kk^\sim}^\pi$ such that

$$(3.21) \quad L_k'^\# (Kf) = f \quad \text{for all } f \in H_k^\pi.$$

PROOF. The inequality (3.20) yields that $R(L_{1/(kk^\sim)^\vee}^{\sim})$ is closed in $H_{1/(kk^\sim)^\vee}^\pi$ and that $N(L_{1/(kk^\sim)^\vee}^{\sim}) = \{0\}$. The space $H_{kk^\sim}^\pi$ can be expressed as the sum

$$H_{kk^\sim}^\pi = N(L_{kk^\sim}^{\sim}) \oplus N$$

where N is closed in $H_{kk^\sim}^\pi$. One sees (as in the proof of Theorem 3.5) that the inverse of $L_k'^\#|_N: N \rightarrow H_k^\pi$ satisfies the required conditions. ■

Corollary 3.7. Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ obeys the condition (ii) of Theorem 2.5 and that there exists a constant $\gamma > 0$ such that for all $\varphi \in C_\pi^\infty$

$$(3.22) \quad \|L\varphi\|_k \cong \gamma \|\varphi\|_{kk^\sim}$$

and

$$(3.23) \quad \|L'\varphi\|_{1/(kk^\sim)^\vee} \cong \gamma \|\varphi\|_{1/k^\vee},$$

where $k^\sim \cong 1$. Then there exists a continuous linear operator $E: H_k^\pi \rightarrow H_{kk^\sim}^\pi$ with the properties

$$(3.24) \quad L_k'^\# (Ef) = f \quad \text{for all } f \in H_k^\pi$$

and

$$(3.25) \quad E(L_k^\sim u) = u \quad \text{for all } u \in D(L_k^\sim).$$

PROOF. In virtue of the inequalities (3.22) and (3.23) the ranges $R(L_k^\sim)$ and $R(L_{1/(kk^\sim)^\vee}^{\sim})$ are closed and

$$(3.26) \quad N(L_k^\sim) = N(L_{1/(kk^\sim)^\vee}^{\sim}) = \{0\}.$$

As a Hilbert space the space $H_{kk^\sim}^\pi$ can be expressed as the orthogonal sum

$$(3.27) \quad H_{kk^\sim}^\pi \in R(L_k^\sim) \oplus R,$$

and the projection $p: H_k^\pi \rightarrow R(L_k^\sim)$ is continuous.

Define a linear operator $E: H_k^\pi \rightarrow H_{kk^\sim}^\pi$ via the formula

$$(3.28) \quad Ef = L_k^{\sim -1}(pf) + K((I-p)f),$$

where K is the operator $H_k^\pi \rightarrow H_{kk^\sim}^\pi$ constructed in Corollary 3.6. Then E is continuous and since

$$L_k'^\# (L_k^{\sim -1}(pf)) = pf$$

and

$$(I-p)f = 0 \quad \text{for all } f \in R(L_k^\sim),$$

a direct computation shows the relations (3.24)–(3.25). This finishes the proof. ■

4. On properties of the one-sided inverse when k is "large"

4.1. Let L be (as in the previous chapter) a linear operator $C_\pi^\infty \rightarrow C_\pi^\infty$ possessing the formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$. As we have shown in Theorem 3.5, the inequality

$$(4.1) \quad \|L'\varphi\|_{1/k^\vee} \cong C_1 \|\varphi\|_{k^\sim/k^\vee} - C_2 \|\varphi\|_{1/k^\vee} \quad \text{for all } \varphi \in C_\pi^\infty,$$

where $k^\sim \in K_\pi$ satisfies (3.2), is sufficient to guarantee the existence of a linear continuous operator $K: R(L_k'^\#) \rightarrow H_k^\pi$ such that

$$L_k'^\#(Kf) = f \quad \text{for all } f \in R(L_k'^\#).$$

Furthermore, in Corollary 3.6 we showed that the inequality

$$(4.2) \quad \|L'\varphi\|_{1/(kk^\sim)^\vee} \cong \varkappa \|\varphi\|_{1/k^\vee} \quad \text{for all } \varphi \in C_\pi^\infty,$$

where $k^\sim \cong 1$, implies the existence of a linear continuous operator $K: H_k^\pi \rightarrow H_{kk^\sim}^\pi$ such that

$$L_k'^\#(Kf) = f \quad \text{for all } f \in H_k^\pi.$$

In this chapter we turn to the problem of seeking sufficient conditions under which the operator K can be expressed as a Fourier series operator

$$(4.3) \quad (K\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} K(x, l) \varphi_l e^{i(l, x)} \quad \text{for } \varphi \in C_\pi^\infty \cap R(L_k'^\#),$$

where the mapping $K(\cdot, \cdot)$ obeys certain regularity properties. The essential tool will be the boundedness of K on $R(L_k'^\#)$.

Let μ_n be a positive number such that

$$\mu_n := \inf \left\{ \lambda > 0 \mid \sum_{l \in \mathbb{Z}^n} 1/(1 + |l|^2)^\lambda < \infty \right\}.$$

The linear space of all m -times continuously differentiable periodic functions $\mathbb{R}^n \rightarrow \mathbb{C}$ will be denoted by C_π^m . First we show

Lemma 4.1. *Suppose that S is a continuous linear operator $H_k^\pi \rightarrow H_k^\pi$, where $k \in K_\pi$ satisfies the inequality*

$$(4.4) \quad k_{\gamma+m}(l) := (1 + |l|^2)^{(\gamma+m)/2} \leq Ck(l) \quad \text{for all } l \in \mathbb{Z}^n$$

with constants $C > 0$, $\gamma > \mu_n$ and $m \in \mathbb{N}_0$. Then there exists a mapping $g: W \rightarrow H_k^{\pi*}$ such that

- 1° the function $x \rightarrow g(x)(v)$ lies in C_π^m for each $v \in H_k^\pi$,
- 2° for all $v \in H_k^\pi$ and $\varphi \in C_\pi^\infty$ one has

$$(4.5) \quad (Sv)(\varphi) = \int_W g(x)(v) \varphi(x) dx,$$

3° for all $x, y \in W$ one has

$$(4.6) \quad \|g(x) - g(y)\| \leq M \|S\| \left(\lambda_n \sum_{l \in \mathbb{Z}^n} |e^{i(l, x)} - e^{i(l, y)}|^2 / k_{(\gamma+m)}^2(l) \right)^{1/2}$$

4° the mapping $g: W \rightarrow H_k^{\pi*}$ is continuous and bounded on W .

PROOF. A. For every $\varphi \in C_\pi^\infty$ we obtain by (4.4)

$$\begin{aligned}
 (4.7) \quad |(D_x^\alpha \varphi)(x)| &= \lambda_n \left| \sum_{l \in \mathbb{Z}^n} l^\alpha \varphi_l e^{i(l, x)} \right| \leq \\
 &\leq \lambda_n \sum_{l \in \mathbb{Z}^n} |\varphi_l| k_{\gamma+|\alpha|}(l) (|l|^{|\alpha|}/k_{\gamma+|\alpha|}(l)) \leq \\
 &\leq (\lambda_n \sum_{l \in \mathbb{Z}^n} (|l|^{|\alpha|}/k_{\gamma+|\alpha|}(l))^2)^{1/2} \|\varphi\|_{k_{\gamma+|\alpha|}} \leq C' \|\varphi\|_k
 \end{aligned}$$

for each $|\alpha| \leq m$, where we applied Hölder's inequality. Hence for every $u \in H_k^\pi$ there exists a function f_u in C_π^m such that

$$(4.8) \quad u(\varphi) = \int_W f_u(x) \varphi(x) dx \quad \text{for all } \varphi \in C_\pi^\infty$$

(since C_π^m is complete equipped with the usual norm topology).

B. Define now a relation g through

$$(4.9) \quad g(x)(v) := f_{Sv}(x) \quad \text{for } x \in W.$$

Then for all $x \in W$ and $v \in H_k^\pi$ one has by (4.7) (with $\alpha=0$)

$$(4.10) \quad |g(x)(v)| = |f_{Sv}(x)| \leq C' \|Sv\|_k \leq C' \|S\| \|v\|_k.$$

In virtue of (4.7) the function $g(x): v \rightarrow g(x)(v)$ is well-defined for each $x \in W$ and by (4.10) $g(x)$ lies in $H_k^{\pi*}$. Similarly it is clear (because of (4.9)) that the functions $x \rightarrow g(x)(v)$ (for a fixed $v \in H_k^\pi$) and $x \rightarrow g(x)$ are well-defined. Furthermore the function $x \rightarrow g(x)(v) = f_{Sv}(x)$ lies in C_π^m and by (4.8) one has

$$(4.11) \quad (Sv)(\varphi) = \int_W f_{Sv}(x) \varphi(x) dx = \int_W g(x)(v) \varphi(x) dx.$$

C. For each pair $(x, y) \in W \times W$ we obtain

$$\begin{aligned}
 (4.12) \quad |(g(x) - g(y))(\varphi)| &= |f_{S\varphi}(x) - f_{S\varphi}(y)| = \\
 &= \lambda_n \left| \sum_{l \in \mathbb{Z}^n} (f_{S\varphi})_l (e^{i(l, x)} - e^{i(l, y)}) \right| \leq \\
 &\leq (\lambda_n \sum_{l \in \mathbb{Z}^n} |e^{i(l, x)} - e^{i(l, y)}|^2 / k_{\gamma+m}^2(l))^{1/2} (\lambda_n \sum_{l \in \mathbb{Z}^n} |(S\varphi)_l k_{\gamma+m}(l)|^2)^{1/2} \leq \\
 &\leq (\lambda_n \sum_{l \in \mathbb{Z}^n} |e^{i(l, x)} - e^{i(l, y)}|^2 / k_{\gamma+m}^2(l))^{1/2} C \|S\| \|\varphi\|_k
 \end{aligned}$$

for all $\varphi \in C_\pi^\infty$. This inequality yields us

$$(4.13) \quad \|g(x) - g(y)\| \leq C \|S\| (\lambda_n \sum_{l \in \mathbb{Z}^n} |e^{i(l, x)} - e^{i(l, y)}|^2 / k_{\gamma+m}^2(l))^{1/2}.$$

Since the series $\sum_{l \in \mathbb{Z}^n} |e^{i(l, x)} - e^{i(l, y)}|^2 / k_{\gamma+m}^2(l)$ is uniformly convergent (in $W \times W$) we see that the right hand side of (4.13) is tending to zero with $x \rightarrow y$. The boundedness of g follows immediately from (4.13). Hence the proof is complete. ■

4.2. For each $L := g(x) \in H_k^{\pi*}$ there exists $V := \tilde{g}(x) \in H_{1/k}^\pi$ such that

$$(4.14) \quad g(x)(\varphi) = \tilde{g}(x)(\varphi) \quad \text{for all } \varphi \in C_\pi^\infty$$

and $\|g(x)\| = \|g^\sim(x)\|_{1/k^v}$ for all $x \in W$ (cf. Lemma 2.2). In the sequel we denote the function $f_{S\varphi}$ by $S\varphi$.

Theorem 4.2. *Suppose that S is a continuous linear operator $H_k^\pi \rightarrow H_k^\pi$, where $k \in K_\pi$ satisfies the inequality (4.4). Then the operator S can be expressed as the extension of a Fourier series operator*

$$(4.15) \quad (S\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} S(x, l) \varphi_l e^{i(l, x)} \quad \text{for } \varphi \in C_\pi^\infty,$$

where the mapping $S(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ satisfies

- 1^o the mapping $x \rightarrow S(x, l)$ lies in C_π^m for each $l \in \mathbb{Z}^n$,
- 2^o for each $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$ there exists $C_\alpha > 0$ such that for all $x \in W$ and $l \in \mathbb{Z}^n$

$$(4.16) \quad |D_x^\alpha S(x, l)| \leq C_\alpha k_{\gamma+|\alpha|}(l).$$

PROOF. In virtue of Lemma 4.1 one has

$$(4.17) \quad \begin{aligned} (S\varphi)(x) &= g(x)(\varphi) = g^\sim(x)(\varphi) = \\ &= \sum_{l \in \mathbb{Z}^n} (g^\sim(x))_{-l} \varphi_l = \lambda_n \sum_{l \in \mathbb{Z}^n} \lambda_n^{-1} (g^\sim(x))_{-l} e^{-i(l, x)} \varphi_l e^{i(l, x)} \end{aligned}$$

for all $x \in W$ and $\varphi \in C_\pi^\infty$, where we used the fact that

$$(4.18) \quad T\varphi = \sum_{l \in \mathbb{Z}^n} T_{-l} \varphi_l \quad \text{for all } T \in D'_\pi \quad \text{and } \varphi \in C_\pi^\infty.$$

Define now a function $S(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ by

$$(4.19) \quad S(x, l) = \lambda_n^{-1} (g^\sim(x))_{-l} e^{-i(l, x)}.$$

Then for all $x \in W$ one obtains

$$(4.20) \quad S(x, l) = \lambda_n^{-1} g^\sim(x)(e^{i(l, \cdot)}) e^{-i(l, x)} = \lambda_n^{-1} S(e^{i(l, \cdot)})(x) e^{-i(l, x)}.$$

Hence the function $x \rightarrow S(x, l)$ lies in C_π^m and

$$(4.21) \quad \begin{aligned} |D_x^\alpha S(x, l)| &\leq \lambda_n^{-1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D_x^\beta (S(e^{i(l, \cdot)}))(x)| |l^{\alpha-\beta}| \leq \\ &\leq \lambda_n^{-1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C_\beta \|S(e^{i(l, \cdot)})\|_{k_{\gamma+|\beta|}} |l^{\alpha-\beta}| \leq \\ &\leq \lambda_n^{-1} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C_\beta \|S\| \|e^{i(l, \cdot)}\|_{k_{\gamma+|\beta|}} |l^{\alpha-\beta}| \leq C_\alpha k_{\gamma+|\alpha|}(l) \end{aligned}$$

where we applied the inequality

$$(4.22) \quad |(D_x^\alpha v)(x)| \leq C_\alpha \|v\|_{k_{\gamma+|\alpha|}}$$

for all $v \in H_k^\pi \subset H_{k_{\gamma+m}}^\pi \subset H_{k_{\gamma+|\alpha|}}^\pi$ (cf. (4.7)). This completes the proof. $\blacksquare$

Combining Theorems 3.5 and 4.2 we obtain

Corollary 4.3. Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ possesses a formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$ and that there exist constants $C_1 > 0$ and $C_2 \geq 0$ such that for all $\varphi \in C_\pi^\infty$

$$(3.17) \quad \|L' \varphi\|_{1/k^\vee} \cong C_1 \|\varphi\|_{k^\sim/k^\vee} - C_2 \|\varphi\|_{1/k^\vee},$$

where $k \in K_\pi$ satisfies (4.4) and $k^\sim \in K_\pi$ satisfies (3.2). Then there exists a continuous linear operator $K: R(L_k^{\prime\#}) \rightarrow H_k^\pi$ with the properties

$$(3.18) \quad \begin{aligned} &1^0 \text{ for all } f \in R(L_k^{\prime\#}) \\ &L_k^{\prime\#} (Kf) = f, \end{aligned}$$

$$2^0 \text{ for all } \varphi \in C_\pi^\infty \cap R(L_k^{\prime\#})$$

$$(4.23) \quad (K\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} K(x, l) \varphi_l e^{i(l, x)},$$

where the mapping $K(\cdot, \cdot)$ satisfies

$$\begin{aligned} &3^0 \text{ the function } x \rightarrow K(x, l) \text{ lies in } C_\pi^m, \\ &4^0 \text{ for each } \alpha \in \mathbb{N}_0^n, |\alpha| \leq m \end{aligned}$$

$$(4.24) \quad |D_x^\alpha K(x, l)| \leq C_\alpha k_{\gamma+|\alpha|}(l) \text{ for } x \in W \text{ and } l \in \mathbb{Z}^n.$$

PROOF. In virtue of Theorem 3.5 there exists a continuous operator $K: R(L_k^{\prime\#}) \rightarrow H_k^\pi$ satisfying (3.18). Let $\bar{K} = K \circ q$, where $q: H_k^\pi \rightarrow R(L_k^{\prime\#})$ is the continuous projection (of Corollary 3.4). Since k obeys the inequality (4.4) we obtain by Theorem 4.2 that

$$(K\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} K(x, l) \varphi_l e^{i(l, x)} \text{ for all } \varphi \in C_\pi^\infty,$$

where $K(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ satisfies 3^0 and 4^0 . Hence we have proved the assertion. ■

Similarly the combination of Corollary 3.6 and Theorem 4.2 yields

Corollary 4.4. Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ possesses a formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$ and that there exists a constant $\varkappa > 0$ such that

$$(4.25) \quad \|L' \varphi\|_{1/(kk^\sim)^\vee} \geq \varkappa \|\varphi\|_{1/k^\vee} \text{ for all } \varphi \in C_\pi^\infty,$$

where $k \in K_\pi$ satisfies (4.4) and where $k^\sim \geq 1$. Then there exists a continuous linear operator $K: H_k^\pi \rightarrow H_{kk^\sim}^\pi$ with the properties

$$\begin{aligned} &1^0 \text{ for all } f \in H_k^\pi \\ &L_k^{\prime\#} (Kf) = f, \end{aligned}$$

$$2^0 \text{ for all } \varphi \in C_\pi^\infty$$

$$(K\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} K(x, l) \varphi_l e^{i(l, x)},$$

where the mapping $K(\cdot, \cdot)$ satisfies 3^0 and 4^0 of Corollary 4.3.

5. On properties of the one-sided inverse when L is (globally) hypoelliptic

Suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ owns a formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$. Define a linear operator $L'_{-\infty}: D'_\pi \rightarrow D'_\pi$ such that

$$\begin{cases} D(L'_{-\infty}) = \{u \in D'_\pi \mid \text{there exists an element } f \in D'_\pi \text{ such that} \\ \quad u(L'\varphi) = f(\varphi) \text{ for all } \varphi \in C_\pi^\infty\}, \\ L'_{-\infty} u = f. \end{cases}$$

Then one sees easily that the domain $D(L'_{-\infty})$ is the whole space D'_π .

We say that the linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ possessing a formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$ is (D'_π, C_π^∞) -hypoelliptic when the solutions of the equation $L'_{-\infty} u = f$ with $f \in C_\pi^\infty$ lie in C_π^∞ .

It is well-known that the operator $L(D)$ defined by

$$(5.1) \quad (L(D)\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} L(l) \varphi_l e^{i(l, x)},$$

where $L(\cdot): \mathbb{R}^n \rightarrow \mathbb{C}$ is polynomial is (D'_π, C_π^∞) -hypoelliptic if and only if there exist constants $C > 0$, $R \geq 0$ and $\varkappa \in \mathbb{R}$ such that

$$(5.2) \quad |L(l)| \geq C k_\varkappa(l) \quad \text{for all } l \in \mathbb{Z}^n \text{ with } |l| \geq R$$

(cf. [6]). We show

Theorem 5.1. *Suppose that L is a linear operator $C_\pi^\infty \rightarrow C_\pi^\infty$ possessing a formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$. Furthermore we assume that L is (D'_π, C_π^∞) -hypoelliptic and that there exists a constant $\varkappa > 0$ such that for all $\varphi \in C_\pi^\infty$*

$$(3.16) \quad \|L'\varphi\|_{1/(kk^\sim)^v} \geq \varkappa \|\varphi\|_{1/k^v},$$

where $k^\sim \geq 1$. Then there exists a continuous linear operator $K: H_k^\pi \rightarrow H_{kk^\sim}^\pi$ with the properties

$$(3.21) \quad \begin{aligned} &1^0 \text{ for all } f \in H_k^\pi \\ &L'_k{}^\#(Kf) = f, \end{aligned}$$

2⁰ the restriction of K on C_π^∞ can be expressed as a Fourier series operator

$$(5.3) \quad (K\varphi)(x) = \lambda_n \sum_{l \in \mathbb{Z}^n} K(x, l) \varphi_l e^{i(l, x)} =: (K(x, D)\varphi)(x),$$

where

3⁰ the mapping $x \rightarrow K(x, l)$ lies in C_π^∞ for each $l \in \mathbb{Z}^n$
and

4⁰ for each $\alpha \in \mathbb{N}_0^n$ there exist $C_\alpha > 0$ and $\mu_\alpha \in \mathbb{R}$ such that

$$(5.4) \quad |D_x^\alpha K(x, l)| \leq C_\alpha k_{\mu_\alpha}(l) \quad \text{for all } x \in W \text{ and } l \in \mathbb{Z}^n.$$

PROOF. In virtue of Corollary 3.6 there exists a continuous linear operator $K: H_k^\pi \rightarrow H_{kk^\sim}^\pi$ satisfying (3.21). Because of (3.21) and (D'_π, C_π^∞) -hypoellipticity of L one sees that K maps C_π^∞ into itself (with $D(K) = C_\pi^\infty$).

Let $\{\psi_n\}$ be a sequence in C_π^∞ such that $\psi_n \rightarrow \psi$ and $K\psi_n \rightarrow f$ (in C_π^∞). Because the inclusion $C_\pi^\infty \subset H_k^\pi$ holds (also topologically) we get that $K\psi_n \rightarrow K\psi$ in H_{kk}^π . Hence $K\psi = f$, which shows that $K: C_\pi^\infty \rightarrow C_\pi^\infty$ is closed. Due to the Closed Graph Theorem K is continuous. Thus Lemma 2.4 completes the proof. ■

We remark that the relations (3.21) and (5.3) yield the validity of the equation

$$(5.5) \quad L(x, D)(K(x, D)\varphi) = \varphi \quad \text{for all } \varphi \in C_\pi^\infty.$$

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(Received November 5, 1984)