

On boundary value problems for nonlinear elliptic equations on unbounded domains

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Introduction

In [1] it has been proved the existence of variational solutions of boundary value problems for the equation

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, u, \dots, D^\beta u, \dots) + g(x, u) = F, \quad x \in \Omega$$

where Ω is a possibly unbounded domain in R^n , $|\beta| \leq m$. The terms $f_\alpha(x, \xi)$ are required to have polynomial growth in ξ , in the term $g(x, u)$, however, no growth restriction is imposed but it is supposed that g (essentially) satisfies the sign condition $g(x, u)u \geq 0$ and for all $t > 0$, $x \mapsto \sup_{|u| \leq t} |g(x, u)| \in L^1(\Omega)$.

In the present paper it will be proved the existence of variational solutions of boundary value problems for the elliptic equation

$$(0.1) \quad \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, u, \dots, D^\beta u, \dots) + \sum_{|\alpha| \leq l} (-1)^{|\alpha|} D^\alpha g_\alpha(x, u, \dots, D^\beta u, \dots) = F$$

where $|\beta| \leq m$, l is an integer with the property $l < m - \frac{n}{p}(1-p+q)$, p and q are real numbers such that $1 < p < \infty$, $p-1 < q \leq p$. Functions f_α satisfy the same conditions as in [1] and in [2], g_α are supposed to satisfy (essentially)

$$(0.2) \quad g_\alpha(x, \xi) \xi_\alpha \geq 0,$$

$$(0.3) \quad |g_\alpha(x, \xi)| \leq C(\xi') (K(x) + |\xi''|^q)$$

where $\xi = (\xi', \xi'')$ and ξ' contains those coordinates ξ_β of ξ for which $|\beta| < m - \frac{n}{p}$; $K \in L^{p/q}(\Omega)$.

In [3] and in [4] it is shown that there exist variational solutions of problems for (0.1) but g_α are supposed to satisfy other conditions instead of (0.3). In [5] the Dirichlet problem in bounded Ω for second order equations is considered with $l=0$ if $g_\alpha = g$ satisfies a condition of type (0.3).

§ 1. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a (possibly) unbounded domain, $p > 1$, m a positive integer. Assume that Ω has the weak cone property (see [6]) and for all sufficiently large μ , there exists a bounded $\Omega_\mu \subset \Omega$ with the weak cone property such that $\Omega_\mu \supset \{x \in \Omega: |x| < \mu\}$. Denote by $W_p^m(\Omega)$ the usual Sobolev space of real valued functions u whose distributional derivatives of order $\leq m$ belong to $L^p(\Omega)$. The norm on $W_p^m(\Omega)$ is

$$\|u\|_{W_p^m(\Omega)} = \left\{ \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u|^p dx \right\}^{1/p},$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $D_j = \frac{\partial}{\partial x_j}$. By $W_{p,0}^m(\Omega)$ will be denoted the closure in $\|\cdot\|_{W_p^m(\Omega)}$ of $C_0^\infty(\Omega)$, the set of infinitely differentiable functions with compact support contained in Ω .

Let N be the number of multiindices α satisfying $|\alpha| \leq m$. The vectors $\xi = (\xi_0, \dots, \xi_\beta, \dots) \in \mathbb{R}^N$ will be written in the form $\xi = (\eta, \zeta)$ where η consists of those ξ_β for which $|\beta| \leq m-1$. Assume that

I. Functions $f_\alpha: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy the Carathéodory conditions, i.e. they are measurable with respect to x for each fixed $\xi \in \mathbb{R}^N$ and continuous with respect to ξ for almost all $x \in \Omega$.

II. There exist a constant $c_1 > 0$ and a function $K_1 \in L^q(\Omega)$ (where $1/p + 1/q = 1$) such that

$$|f_\alpha(x, \xi)| \leq c_1 |\xi|^{p-1} + K_1(x)$$

for all $|\alpha| \leq m$, a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

III. For all $(\eta, \zeta), (\eta, \zeta') \in \mathbb{R}^N$ with $\zeta \neq \zeta'$ and a.e. $x \in \Omega$

$$\sum_{|\alpha|=m} [f_\alpha(x, \eta, \zeta) - f_\alpha(x, \eta, \zeta')] (\xi_\alpha - \xi'_\alpha) > 0.$$

IV. There exist a constant c_2 and a function $K_2 \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$

$$\sum_{|\alpha| \leq m} f_\alpha(x, \xi) \xi_\alpha \geq c_2 |\xi|^p - K_2(x).$$

V. Functions $p_\alpha, r_\alpha: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfy the Carathéodory conditions and

$$g_\alpha = p_\alpha + r_\alpha.$$

VI. $p_\alpha(x, \xi) \xi_\alpha \geq 0$ and $|r_\alpha(x, \xi)| \leq h_\alpha(x)$ for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$ where $h_\alpha \in L^{p/q}(\Omega)$.

VII. There exist a continuous function C and a function $K \in L^{p/q}(\Omega)$ such that

$$|p_\alpha(x, \xi)| \leq C(\xi') (K(x) + |\xi''|^q)$$

for all $|\alpha| \leq l$, $\xi = (\xi', \xi'') \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

VIII. V is a closed subspace of $W_p^m(\Omega)$ with the property: $v \in V$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ imply that $\varphi v \in V$. (V may be e.g. $W_p^m(\Omega)$ or $W_{p,0}^m(\Omega)$.)

Set

$$(1.1) \quad p_{\alpha, \mu}(x, \xi) = \begin{cases} p_{\alpha}(x, \xi) & \text{if } |x| \leq \mu, |p_{\alpha}(x, \xi)| \leq \mu, \\ \mu \frac{p_{\alpha}(x, \xi)}{|p_{\alpha}(x, \xi)|} & \text{if } |x| \leq \mu, |p_{\alpha}(x, \xi)| > \mu, \\ 0 & \text{if } |x| > \mu, \end{cases}$$

$$(1.2) \quad r_{\alpha, \mu}(x, \xi) = \begin{cases} r_{\alpha}(x, \xi) & \text{if } |x| \leq \mu, |r_{\alpha}(x, \xi)| \leq \mu, \\ \mu \frac{r_{\alpha}(x, \xi)}{|r_{\alpha}(x, \xi)|} & \text{if } |x| \leq \mu, |r_{\alpha}(x, \xi)| > \mu, \\ 0 & \text{if } |x| > \mu, \end{cases}$$

$$(1.3) \quad g_{\alpha, \mu}(x, \xi) = p_{\alpha, \mu}(x, \xi) + r_{\alpha, \mu}(x, \xi).$$

Assumptions I., II., V., VI. and (1.1)—(1.3) imply that formulas

$$\langle T(u), v \rangle = \sum_{|\alpha| \leq m} \int_{\Omega} f_{\alpha}(x, u, \dots, D^{\beta} u, \dots) D^{\alpha} v \, dx,$$

$$\langle S_{\mu}(u), v \rangle = \sum_{|\alpha| \leq l} \int_{\Omega} g_{\alpha, \mu}(x, u, \dots, D^{\beta} u, \dots) D^{\alpha} v \, dx$$

define linear continuous functionals $T(u)$ resp. $S_{\mu}(u)$ on V for any fixed (sufficiently large) μ .

By assumptions I.—V. and (1.1)—(1.3) operator $T + S_{\mu}$ satisfies the conditions of [2] for (sufficiently large) fixed μ and thus we have

Lemma 1. For any $F \in V^*$ there exists $u_{\mu} \in V$ such that

$$(T + S_{\mu})(u_{\mu}) = F.$$

Lemma 2. Assume that $(u_j) \rightarrow u$ weakly in V and for any bounded domain $\omega \subset \Omega$

$$(1.4) \quad \lim_{j \rightarrow \infty} \int_{\omega} h_j \, dx = 0$$

where

$$(1.5) \quad h_j(x) = \sum_{|\alpha|=m} [f_{\alpha}(x, u_j, \dots, D^{\gamma} u_j, \dots, D^{\beta} u_j, \dots) - f_{\alpha}(x, u_j, \dots, D^{\gamma} u_j, \dots, D^{\beta} u, \dots)] (D^{\alpha} u_j - D^{\alpha} u)$$

$|\gamma| < m$, $|\beta| = m$. Then there is a subsequence (u'_j) of (u_j) such that $D^{\beta} u'_j \rightarrow D^{\beta} u$ a.e. in Ω for all β with $|\beta| \leq m$.

PROOF. Since $(u_j) \rightarrow u$ weakly in V thus there is a subsequence $(\tilde{u}_j)$ of (u_j) such that

$$D^{\gamma} \tilde{u}_j \rightarrow D^{\gamma} u \quad \text{a.e. in } \Omega \quad \text{for } |\gamma| < m$$

(see e.g. [7]). Further, by assumption III. $h_j \geq 0$ and so (1.4) and Fatou's lemma imply that

$$h_j \rightarrow 0 \quad \text{a.e. in } \omega.$$

Thus there exists $\omega_0 \subset \omega$ of measure 0 such that for $x \in \omega \setminus \omega_0$

$$(1.6) \quad |D^\beta u(x)| < \infty, \quad |K_1(x)| < \infty, \quad |K_2(x)| < \infty$$

and

$$(1.7) \quad D^\gamma \tilde{u}_j(x) \rightarrow D^\gamma u(x) (|\gamma| < m), \quad \tilde{h}_j(x) \rightarrow 0.$$

Set

$$\xi^{(j)}(x) = (\dots, D^\beta \tilde{u}_j(x), \dots)$$

where $|\beta| = m$. By assumptions I., II., IV. and by (1.6), (1.7)

$$\begin{aligned} \tilde{h}_j(x) &\equiv \sum_{|\alpha|=m} f_\alpha(x, \tilde{u}_j, \dots, D^\gamma \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha \tilde{u}_j - \\ &\quad - \sum_{|\alpha|=m} |f_\alpha(x, \tilde{u}_j, \dots, D^\gamma \tilde{u}_j, \dots, D^\beta \tilde{u}_j, \dots) D^\alpha u| - \\ &\quad - \sum_{|\alpha|=m} |f_\alpha(x, \tilde{u}_j, \dots, D^\gamma \tilde{u}_j, \dots, D^\beta u, \dots) (D^\alpha \tilde{u}_j - D^\alpha u)| \equiv \\ &\equiv c_2 |\xi^{(j)}(x)|^p - c_3(x) [1 + |\xi^{(j)}(x)|^{p-1} + |\xi^{(j)}(x)|] \quad \text{if } x \in \omega \setminus \omega_0. \end{aligned}$$

($D^\gamma \tilde{u}_j(x)$ is bounded for a fixed $x \in \omega \setminus \omega_0$.) Since by (1.7) $\tilde{h}_j(x)$ is bounded for a fixed $x \in \omega \setminus \omega_0$ thus $\xi^{(j)}(x)$ is bounded, too. Consequently, $(\xi^{(j)}(x))$ contains a subsequence which converges to a vector $\xi^*(x)$.

Now we show that

$$(1.8) \quad \xi^*(x) = \xi(x) = (\dots, D^\beta u(x), \dots).$$

Indeed, applying (1.5) to the subsequence of $(\tilde{h}_j(x))$ and letting $j \rightarrow \infty$ (by (1.7)) we obtain

$$\begin{aligned} 0 &= \sum_{|\alpha|=m} [f_\alpha(x, u(x), \dots, D^\gamma u(x), \dots, \xi^*(x)) - \\ &\quad - f_\alpha(x, u(x), \dots, D^\gamma u(x), \dots, \xi(x)) [\xi_\alpha^*(x) - \xi_\alpha(x)] \end{aligned}$$

which implies (1.8) in virtue of assumption III.

So we have shown that all convergent subsequences of $(\xi^{(j)}(x))$ tend to $\xi(x)$. Therefore $\lim_{j \rightarrow \infty} \xi^{(j)}(x) = \xi(x)$ and thus by (1.7) $D^\beta \tilde{u}_j \rightarrow D^\beta u$ a.e. in ω for all β with $|\beta| \leq m$. Hence (by a "diagonal process") easily follows Lemma 2 since ω is an arbitrary bounded subset of Ω .

§ 2. The existence theorem

Theorem. Suppose that conditions I.—VIII. are fulfilled. Then for any $F \in V^*$ there exists $u \in V$ such that

$$\begin{aligned} (2.1) \quad &\sum_{|\alpha| \leq m} \int_{\Omega} f_\alpha(x, u, \dots, D^\beta u, \dots) D^\alpha v \, dx + \\ &+ \sum_{|\alpha| \leq l} \int_{\Omega} g_\alpha(x, u, \dots, D^\beta u, \dots) D^\alpha v \, dx = \langle F, v \rangle \end{aligned}$$

for all $v \in V$.

PROOF. By Lemma 1 for any $j=j_0, j_0+1, j_0+2, \dots$ there is $u_j \in V$ such that

$$(2.2) \quad \langle (T+S_j)(u_j), v \rangle = \langle F, v \rangle \quad \text{for all } v \in V.$$

From assumptions IV., VI. and (1.1)–(1.3) it follows that (u_j) is bounded in V . Thus there exist a subsequence (u_{j_k}) of (u_j) and $u \in V$ such that

$$(2.3) \quad (u_{j_k}) \rightarrow u \quad \text{weakly in } V,$$

$$(2.4) \quad D^\gamma u_{j_k} \rightarrow D^\gamma u \quad \text{a.e. in } \Omega \quad \text{for } |\gamma| \leq m-1$$

(see [7]).

Consider an arbitrary bounded domain $\omega \subset \Omega$ and take a function $\Theta \in C_0^\infty(R^n)$ such that $\Theta \geq 0$ and $\Theta(x) = 1$ for $x \in \omega$. By Sobolev's imbedding theorems (see e.g. [6]) it may be supposed that

$$(2.5) \quad D^\gamma u_{j_k} \rightarrow D^\gamma u \quad \text{in } L^p(\Omega \cap \text{supp } \Theta) \quad \text{for } |\gamma| \leq m-1$$

and

$$(2.6) \quad D^\gamma u_{j_k} \rightarrow D^\gamma u \quad \text{in } L^{q_1}(\Omega \cap \text{supp } \Theta) \quad \text{for } |\gamma| \leq l < m - \frac{n}{p}(1-p+q)$$

where q_1 is defined by $\frac{1}{p/q} + \frac{1}{q_1} = 1$.

By assumption VIII. $\Theta(u_{j_k} - u) \in V$ and so by (2.2)

$$(2.7) \quad \sum_{|\alpha| \leq m} \int_{\Omega} f_\alpha(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha [\Theta(u_{j_k} - u)] dx + \\ + \sum_{|\alpha| \leq l} \int_{\Omega} g_{\alpha, j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha [\Theta(u_{j_k} - u)] dx = \langle F, \Theta(u_{j_k} - u) \rangle.$$

Since $(u_{j_k} - u) \rightarrow 0$ weakly in V thus

$$(2.8) \quad \Theta(u_{j_k} - u) \rightarrow 0 \quad \text{weakly in } V.$$

In virtue of (2.7) we have

$$(2.9) \quad \sum_{|\alpha| = m} \int_{\Omega} [f_\alpha(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots) - \\ - f_\alpha(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u, \dots)] \Theta D^\alpha (u_{j_k} - u) dx = \\ = \sum_{|\alpha| = m} \int_{\Omega} f_\alpha(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u, \dots) \Theta D^\alpha (u - u_{j_k}) dx + \\ + \sum_{|\alpha| = m} \int_{\Omega} f_\alpha(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots) \sum_{|\gamma| \leq m-1} c_\gamma D^\gamma (u - u_{j_k}) D^{\alpha-\gamma} \Theta dx + \\ + \sum_{|\alpha| \leq m-1} \int_{\Omega} f_\alpha(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha [\Theta(u - u_{j_k})] dx + \\ + \sum_{|\alpha| \leq l} \int_{\Omega} g_{\alpha, j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha [\Theta(u - u_{j_k})] dx + \langle F, \Theta(u_{j_k} - u) \rangle$$

where $|\gamma| < m$, $|\beta| = m$.

Now we show that all the terms in the right of (2.9) converge to 0 as $k \rightarrow \infty$. By (2.3) $D^\alpha(u_{j_k} - u) \rightarrow 0$ weakly in $L^p(\Omega)$, further by (2.4), assumption I.

$$(2.10) \quad \Theta f_\alpha(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u, \dots) \rightarrow \Theta f_\alpha(x, u, \dots, D^\gamma u, \dots, D^\beta u, \dots)$$

a.e. in Ω and so by assumption II., (2.5) and Vitali's theorem (2.10) is valid in $L^q(\Omega)$ norm, too. Thus the first term in the right of (2.9) converges to 0.

Since by assumptions I., II.

$$f_\alpha(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots)$$

is bounded in $L^q(\Omega)$ thus (2.5) implies that the second and third terms in the right of (2.9) converge to 0 as $k \rightarrow \infty$.

From assumptions V.—VII. it follows that

$$g_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots)$$

is bounded in $L^{p/q}(\Omega \cap \text{supp } \Theta)$ thus (2.6) implies that the fourth term in the right of (2.9) converges to 0 as $k \rightarrow \infty$. Finally, from (2.8) it follows that the last term in the right of (2.9) converges to 0 as $k \rightarrow \infty$.

Thus we have shown that the term in the left of (2.9) converges to 0 as $k \rightarrow \infty$ and so by assumption III. and by $\Theta \equiv 0$ we find that (1.4) is valid for any bounded $\omega \subset \Omega$. Consequently, from Lemma 2 we obtain that (u_{j_k}) contains a subsequence $(u_{j'_k})$ such that

$$(2.11) \quad D^\beta u_{j'_k} \rightarrow D^\beta u \quad \text{a.e. in } \Omega \quad \text{if } |\beta| \leq m.$$

Thus assumption I. implies that

$$f_\alpha(x, u_{j'_k}, \dots, D^\beta u_{j'_k}, \dots) \rightarrow f_\alpha(x, u, \dots, D^\beta u, \dots)$$

a.e. in Ω and so by assumption II., Hölder's inequality Vitali's theorem shows that for any $v \in V$

$$(2.12) \quad \lim_{k \rightarrow \infty} \langle T(u_{j'_k}), v \rangle = \langle T(u), v \rangle.$$

By using (1.1)—(1.3), assumption V. and (2.11) we obtain

$$g_{\alpha, j'_k}(x, u_{j'_k}, \dots, D^\beta u_{j'_k}, \dots) \rightarrow g_\alpha(x, u, \dots, D^\beta u, \dots)$$

a.e. in Ω and so by assumptions VI., VII., Hölder's inequality Vitali's theorem shows that for any $v \in V$

$$(2.13) \quad \lim_{k \rightarrow \infty} \langle s_{j'_k}(u_{j'_k}), v \rangle = \sum_{|\alpha| \leq l} \int_{\Omega} g_\alpha(x, u, \dots, D^\beta u, \dots) D^\alpha v \, dx.$$

Thus from (2.2), (2.12), (2.13) one obtains (2.1) and the proof of the theorem is complete.

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